COMITETUL DE STAT PENTRU ENERGIA NUCLEARA
INSTITUTUL DE FIZICA SI INGINERIE NUCLEARA

FT - 140 - 1977
Septembrie

EXTREMAL PROBLEMS IN THE PHENOMENOLOGY OF THE PION ELECTROMAGNETIC FORM FACTOR

I.RASZILLIER

BUCURESTI - ROMANIA

EXTREMAL PROBLEMS IN THE PHENOMENOLOGY

OF THE PION ELECTROMAGNETIC FORM FACTOR +

I.Raszillier

Institute for Nuclear Physics and Engineering

P.O.Box 5206, Magurele, Bucharest, Romania

Abstract

The sets in Euclidean spaces are determined, which are the images of the mappings, performed by certain systems of functionals, from a set Ω in the Hilbert space H^2 of functions analytic in the unit disk. These sets express the correlations established by the elements of Ω between the functionals of the systems. Physically they give, for the functionals chosen, the correlation by experimental data between the pion charge radius, the pionic contribution to the muon magnetic moment and an Euclidean (or equivalent Chebyshev) measure of errors.

1. INTRODUCTION

A number of problems in the (rigorous) phenomenology of the pion electromagnetic form factor, referring to the determination of the implications of experimental data for physical quantities related to the pion form factor, may be conveniently formulated mathematically with the following elements :

a) a (normed) space of analytic functions, to which the pion form factor is assumed to belong,

b) a set Ω in this space, which includes all the elements obeying (theoretical and) experimental requirements imposed on the pion form factor,

c) a system $(\ell_1, \ldots, \ell_n)$ of functionals ℓ_i $(i=1, \ldots, n)$ in this space, representing physical magnitudes or quantities expressing information contained in experimental data.

The system $(\ell_1, \ldots, \ell_n)$ performs a mapping from the space of analytic functions into R^n. The object of the type of problems we refer to is then the determination of the image through this mapping of the set Ω. It displays the whole picture of correlation between the physical quantities of the system $(\ell_1, \ldots, \ell_n)$ imposed by experimental information (expressed in the characterisation of Ω and in values of some of the functionals $\ell_1, \ldots, \ell_n$).

In this paper we present the solutions to two (closely related) problems of this type. They express essentially the (joint) information one obtains on the pion charge radius and the pionic contribution to the muon magnetic moment from experimental data for the pion form factor, obtained in the processes $e^- p \to e^- \pi^+ n$ and $e^+ e^- \to \pi^+ \pi^-$.

It turns out that the natural space for the formulation of these problems is the Hilbert space H^2 of functions $h(z)$ analytic in the unit disk ($|z|<1$) : the pionic contribution to the muon magnetic moment can be brought to the form of the (squared) norm of an element of this space. The pion charge radius is related to a functional in H^2, the derivative $h'(0)$. As to experimental data we use those from $e^- p \to e^- \pi^+ n$ in order to construct an Euclidean (Chebyshev) error functional $\lambda(h)$ ($\pi(h)$) in terms of the values $h(x_1), \ldots, h(x_n)$. The data from $e^+e^- \to \pi^+\pi^-$ define, together with general theoretical conditions, the set Ω . For a detailed description of this correspondence between the physical aspects of the problems and the mathematical framework we refer, however, to Refs. 1-4 .

The structure of the paper is as follows : In Sec.2 we formulate the problems. In order to give, in Sec.4, the solution of one of them, we make in Sec.3 the necessary preparations in the form of an (auxiliary) extremal problem. These sections are in fact, together with Sec.5, necessary elements for the solution of the second problem (Sec.6). The remaining part of the paper refers to qualitative properties of the solutions (Sec.7), to elements of importance in its numerical computation (Sec.8), and to the (slight) modification implied by the substitution, in the problems, of the Euclidean error functional $\lambda(h)$ by the Chebyshev functional $\pi(h)$ (Sec.9).

2. FORMULATION OF THE PROBLEMS

In the (Hilbert) space H^2 of functions $h(z)$ analytic in the unit disk $|z|<1$ and obeying

$$\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)|^2 d\theta \quad , \quad z = \exp(i\theta) \quad , \tag{2.1}$$

we consider the set $\Omega(s,a)$ defined by :

(a) $\quad f^*(z) = f(z^*) \quad ,$

(b) $\quad f(0) = a > 0 \quad ,$ (2.2)

(c) $\quad |f(z)| = s(\theta) \quad , \quad \theta \in \Gamma \quad (\Gamma : 0 < \theta_1 \leq \theta \leq \theta_2 < \pi)$,

with $s(\theta) \geq 0$ and $s(\theta) \in L^2$, $\ln s(\theta) \in L^1$ on Γ . Further, with given sequences of real numbers $a_1, \ldots, a_n$, $\gamma_1, \ldots, \gamma_n$ ($\gamma_i > 0$), and $x_1, \ldots, x_n$ ($0 < x_i^2 < 1$, $x_i \neq x_j$, $i \neq j$) and with the values $f(x_i)$ we define

$$\chi(f) = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\left(\frac{f(x_i) - a_i}{\gamma_i}\right)^2\right)^{\frac{1}{2}} \tag{2.3}$$

Now we may formulate our two (extremal) problems :

<u>PROBLEM $\Pi_\varepsilon(2)$</u>: To determine for the set $\Omega(s,a)$ in H^2 the image $\Delta_\varepsilon(2)$ in the 2-dimensional Euclidean space R^2 , given by the mapping $f(z) \rightarrow$ ($\|f\|^2$, $\chi(f)$),

<u>PROBLEM $\Pi_\varepsilon(3)$</u>: To determine for the set $\Omega(s,a)$ in H^2 the image $\Delta_\varepsilon(3)$ in the 3-dimensional Euclidean space R^3 , given by the mapping $f(z) \rightarrow$ ($f'(0)$, $\|f\|^2$, $\chi(f)$).

The sets $\Delta_\varepsilon(2)$ and $\Delta_\varepsilon(3)$ express the correlation between the quantities $\|f\|^2$, $\chi(f)$ and $f'(0)$, $\|f\|^2$, $\chi(f)$, respectively, for functions $f(z) \in \Omega(s,a)$.

3. AN AUXILIARY EXTREMAL PROBLEM

We consider the positive cone K^+ of functions $\tau(\theta)$ (≥ 0 a.e.) in the real Hilbert space L^2 over the arc $C\Gamma$, the complement, with positive Lebesgue measure, of Γ with respect to $[0,\pi]$. In this cone we define the sets $\mathcal{U}_i(k_i)$, $i = 0, \dots, n$ by $\mathcal{U}_i(k_i) = \{ \tau(\theta) \geq 0 /$ $\mathcal{J}_i(\tau; k_i) \leq 0 \}$, where the $\mathcal{J}_i(\tau; k_i)$ are given by

$$\mathcal{J}_i(\tau; k_i) = 2\ln|k_i| - \frac{1}{\pi}\int_{\Gamma} p_i(\theta)\ln s^2(\theta)\,d\theta - \frac{1}{\pi}\int_{C\Gamma} p_i(\theta)\ln \tau^2(\theta)\,d\theta\,, \quad (3.1)$$

with

$$p_0(\theta) = 1$$
$$p_i(\theta) = \frac{1 - x_i^2}{1 + x_i^2 - 2x_i\cos\theta}\,, \quad i = 1, \dots, n\,, \quad (3.2)$$

$k_0 = a$, and fixed (but arbitrary) real (nonzero) numbers k_i ($i = 1, \dots, n$). These sets are convex and closed since they may be considered as level sets of (proper) convex lower semicontinuous functionals in L^2 .[5,6] Therefore, also their intersection $\mathcal{U}(a, k_1, \dots, k_n) = \bigcap_{i=0}^{n} \mathcal{U}_i(k_i)$ ($\mathcal{U} \subset K^+$) is convex and closed.

Our auxiliary extremal problem for $\Pi(k)$ is : to establish the existence of the minimum of the squared norm in L^2

$$\|\tau\|^2 = \frac{1}{\pi}\int_{C\Gamma} \tau^2(\theta)\,d\theta \quad (3.3)$$

subject to $\tau(\theta) \in K^+$ and the constraints $\mathcal{J}_i(\tau; k_i) \leq 0$ ($i = 0, \dots, n$), and to determine the form of the extremal function.

The existence of the minimum and the uniqueness of the minimizing function $r_*(\theta)$ follow from the fact that the problem is a minimum norm problem in Hilbert space : the determination of $\inf_{r \in CL(\varkappa, k_0, \ldots, k_n)} \|r\|^2$ .7

In order to determine the form of the solution we make use of the (global) Lagrange multiplier technique for inequality constraints. This is justified by the convexity of K^+, of the mapping $\{ J_i(r, k_i) \}$ from K^+ into R^{n+1}, by the existence of an interior point in the positive cone of R^{n+1}, and by the existence of functions $\bar{r}(\theta)$ obeying $J_i(\bar{r}, k_i) < 0$ .7 For our situation this technique assures the existence of n+1 numbers $\lambda_i \geq 0$ (Lagrange multipliers) with the property ($\sum_{i=0}^{n} \lambda_i J_i(r_*, k_i) = 0$ and)

$$\|r_*\|^2 + \sum_{i=0}^{n} \lambda_i J_i(r_*, k_i) \leq \|r\|^2 + \sum_{i=0}^{n} \lambda_i J_i(r, k_i) , \tag{3.4}$$

valid for any $r(\theta) \in K^+$.

The last inequality, written explicitly as

$$\int r_*^2(\theta)\, d\theta - \int r^2(\theta)\, d\theta \leq$$

$$\int p_\lambda(\theta) \ln r_*^2(\theta)\, d\theta - \int p_\lambda(\theta) \ln r^2(\theta)\, d\theta \quad , \quad r \in K^+ , \tag{3.5}$$

with

$$p_\lambda(\theta) = \sum_{i=0}^{n} \lambda_i p_i(\theta) \qquad (\geq 0) \tag{3.6}$$

(and the integration understood over $C\Gamma$), we compare with the inequality

$$\int f(\theta)\,d\theta - \int g(\theta)\,d\theta \le \int f(\theta)\ln f(\theta)\,d\theta - \int f(\theta)\ln g(\theta)\,d\theta \, , \tag{3.7}$$

which is valid for any pair of positive functions $f(\theta)$, $g(\theta)$ and satisfies the equality sign only for $f(\theta) = g(\theta)$. The validity of both inequalities, (3.5) and (3.7), leads to

$$r_*^2(\theta) = p_\lambda(\theta) \tag{3.8}$$

since if $r_*^2(\theta)$ were not of the form (3.8), then (3.5) should be valid with strict inequality for $r^2(\theta) = p_\lambda(\theta)$, i.e. we had

$$\int p_\lambda(\theta)\,d\theta - \int r_*^2(\theta)\,d\theta > \int p_\lambda(\theta)\ln p_\lambda(\theta)\,d\theta - \int p_\lambda(\theta)\ln r_*^2(\theta)\,d\theta \, , \tag{3.9}$$

in contradiction to (3.7). The squared norm of the extremal function is

$$\|r_*\|^2 = \sum_{i=0}^{n} \lambda_i \omega_i \quad , \quad \omega_i = \frac{1}{\pi}\int_{C\Pi} p_i(\theta)\,d\theta \quad (>0) \, . \tag{3.10}$$

The values of the Lagrange multipliers depend, of course, on those of h_i , but we are not interested in the determination of this dependence.

We now remove the limitation $h_i \neq 0$ imposed so far for $i = 1, \dots, n$; Since $h_0 = a > 0$, we have in $\mathfrak{A}_0(a)$ $\ln r(\theta) \in L^1$. Then, if e.g. $h_k = 0$, we have $J_k(r,0) \equiv -\infty$ on the intersection $\mathfrak{A}' = \bigcap \mathfrak{A}_i(h_i)$ of all $\mathfrak{A}_i(h_i)$, for which $h_i \neq 0$, and the condition $J_k(r,c) \le 0$ is (trivially) obeyed on $\mathfrak{A}'$, i.e. the set $\mathfrak{A}$ is identical with $\mathfrak{A}'$. This fact may be conveniently formulated as the disappearance of the corresponding Lagrange multiplier λ_k in (3.8) and (3.10) ($\lambda_k = 0$). Thus, for all values of h_i the form of the extremal function is that given by (3.8).

If $n=0$, then one can easily compute the extremal function (3.8) and the minimum (3.10) by using the Jensen inequality [1], with the result

$$t_{*,0}^2(\theta) = a^{\frac{2}{\omega_0}} \exp\left(-\frac{1}{\omega_0 \pi}\int_\Gamma \ln s^2(\theta)\, d\theta\right), \tag{3.11}$$

$$\|t_{*,0}\|^2 = \omega_0\, a^{\frac{2}{\omega_0}} \exp\left(-\frac{1}{\omega_0 \pi}\int_\Gamma \ln s^2(\theta)\, d\theta\right) \equiv k_0^2(\varepsilon r) \tag{3.12}$$

Since $\mathfrak{A} < \mathfrak{A}_0(a)$ for $n \geq 1$, we always have $\|t_*\|^2 \geq k_0^2(\varepsilon r)$ in this general situation.

Starting now from the solution of our auxiliary (extremal) problem we try to determine (in R^n) the set $H_n(k^2(\varepsilon r))$ of those values ($k_1, \ldots, k_n$) for which there exists an $t(\theta) \in \mathfrak{A}(a, k_1, \ldots, k_n)$ of squared norm $\|t\|^2 = k^2(\varepsilon r)$, with a given number $k^2(\varepsilon r)$ ($\geq k_0^2(\varepsilon r)$). If such an $t(\theta)$ exists, then the numbers ($k_1, \ldots, k_n$) are such that the minimum of $\|t\|^2$ over $\mathfrak{A}(a, k_1, \ldots, k_n)$ is smaller than $k^2(\varepsilon r)$, i.e. there exist Lagrange multipliers $\lambda_0, \ldots, \lambda_n$ such that $\partial_i(t_*, k_i) \leq 0$ ($i = 0, \ldots, n$) with these k_i and $\sum_{i=0}^{n} \lambda_i \omega_i = k^2(\varepsilon r)$ are obeyed. Since the values of ∂_i decrease with the increase of λ_i, there also exist parameters $\mu_0 \geq \lambda_0, \ldots, \mu_n \geq \lambda_n$ obeying

$$\sum_{i=0}^{n} \mu_i \omega_i = k^2(\varepsilon r) \tag{3.13}$$

and $\partial_i(t_\mu, k_i) \leq 0$, with $t_\mu^2(\theta) = \sum_{i=0}^{n} \mu_i p_i(\theta)$ and these k_i. This leads to the inclusion $H_n(k^2(\varepsilon r)) \subset D_n(k^2)$, where $D_n(k^2)$ is the set of those ($k_1, \ldots, k_n$), for which there exist numbers $\mu_i \geq 0$, obeying (3.13) and $J_0(t_\mu, a) \le 0$, such that $J_i(t_\mu, k_i) \le 0$ ($i = 1, \dots, n$) is satisfied ; k^2 is defined by

$$k^2 = k^2(er) + \frac{1}{\pi}\int_{\Gamma} s^2(\theta)\, d\theta \tag{3.14}$$

This set may be written as

$$\mathcal{D}_n(k^2) = \bigcup_{\mu \in \Lambda(k^2)} \mathcal{D}_n(k^2, \mu) \quad , \mu \equiv (\mu_0, \dots, \mu_n) , \tag{3.15}$$

where the set $\Lambda(k^2)$ (in R^{n+1}) is defined by

$$\Lambda(k^2) = \left\{ (\mu_0, \dots, \mu_n) \,|\, \mu_i \ge 0 , \sum_{i=0}^{n} \mu_i \omega_i = k^2(er) , J_0(t_\mu, a) \le 0 \right\} , \tag{3.16}$$

and $\mathcal{D}_n(k^2, \mu)$ by

$$\mathcal{D}_n(k^2, \mu) = \tag{3.17}$$
$$= \left\{ (k_1, \dots, k_n) \,|\, J_1(t_\mu, k_1) \le 0 , \dots , J_n(t_\mu, k_n) \le 0 , \mu \in \Lambda(k^2) \right\} .$$

The inverse inclusion $\mathcal{D}_n(k^2) \subset H_n(k^2(er))$ is also readily established, and thereby the equality

$$H_n(k^2(er)) = \mathcal{D}_n(k^2) . \tag{3.18}$$

Through a subset of $\mathcal{D}_n(k^2)$ we establish in the next section the connection with our problem $\Pi_\varepsilon(k)$.

4. SOLUTION OF THE PROBLEM $\Pi_\varepsilon(k)$

Coming back to the space H^2 we recall that any real function $k(z) \in H^2$ can be factorised into an inner function $w(z)$, $|w(z)| \le 1$,

$|w(z)| = 1$ ($w^*(z) = w(z^*)$) and an outer function

$$E(z) = \exp\left(\frac{1}{\pi} \int_0^{\pi} \frac{1-z^2}{1+z^2-2z\cos\theta} \ln |k(z)| \, d\theta \right) \tag{4.1}$$

according to

$$k(z) = w(z) \, E(z) \quad . \tag{4.2}$$

A restriction $k(0) = a$ on $k(z) \in H^2$ has the implication that

$$E(0) \geq a \quad . \tag{4.3}$$

In the same way, a restriction $k(x_i) = k_i$ would imply

$$E(x_i) \geq |k_i| \tag{4.4}$$

for the outer function. The conditions (4.3) and (4.4) are in close relation to the sets $\mathcal{U}_i(k_i)$: The conditions $J_i(r, k_i) \leq 0$ defining $\mathcal{U}_i(k_i)$ are, in fact, (4.3) and (4.4) for functions $k(z)$ obeying $|k(z)| = s(\theta)$ for $\theta \in \Gamma$, with the notation $|k(z)| = r(\theta)$, $\theta \in C\Gamma$. Therefore $\mathcal{U}_i(k_i)$ ($i = 0, \ldots, n$) may be considered as the set of those functions $r(\theta) \in K^+$ for which the set $\mathcal{C}_i(r, k_i)$ of (real) functions $k(z) \in H^2$, obeying $k(x_i) = k_i$ ($x_0 = 0$) and

$$|k(z)| \leq \begin{cases} s(\theta) & , \ \theta \in \Gamma \ , \\ r(\theta) & , \ \theta \in C\Gamma \ , \end{cases} \tag{4.5}$$

is not empty. All n+1 conditions $J_i(r, k_i) \leq 0$ give the set ($\mathcal{U}(a, k_1, \ldots, k_n)$) of those $r(\theta) \in K^+$, for which $\mathcal{C}_0(r, a) \neq 0, \ldots, \mathcal{C}_n(r, k_n) \neq 0$.

These considerations allow us to write the sets $\mathcal{D}_n(k^2, \mu)$ of the preceding section as

$$\mathcal{D}_n(k^2,\mu) =$$

$$= \{(k_1, \ldots, k_n) \,/\, E_\mu(k_1) \geq |k_1|, \ldots, E_\mu(k_n) \geq |k_n|, \mu \in \Lambda(k^2)\} \tag{4.6}$$

with $E_\mu(z)$ defined by (4.1) and

$$|k(z)| = \begin{cases} s(\theta) &, \theta \in \Gamma \\ t_\mu(\theta) &, \theta \in C\Gamma \end{cases} \tag{4.7}$$

and interpret them as the sets of points $(k_1, \ldots, k_n)$ in H^2, for which $\mathcal{C}_1(t_\mu, k_1) \neq 0, \ldots, \mathcal{C}_n(t_\mu, k_n) \neq 0$, with $\mu \in \Lambda(k^2)$:

$$\mathcal{D}_n(k^2,\mu) =$$

$$= \{(k_1, \ldots, k_n) \,/\, \mathcal{C}_1(t_\mu, k_1) \neq 0, \ldots, \mathcal{C}_n(t_\mu, k_n) \neq 0, \mu \in \Lambda(k^2)\}. \tag{4.8}$$

Also $\Lambda(k^2)$ may be brought to a slightly modified form,

$$\Lambda(k^2) = \{(\mu_0, \ldots, \mu_n) \,/\, \mu_i \geq 0, \sum_{i=0}^{n} \mu_i \omega_i = k^2(c\Gamma), E_\mu(0) \geq a\}. \tag{4.9}$$

We now take the subset $\mathcal{V}_n(k^2,\mu)$ of $\mathcal{D}_n(k^2,\mu)$, defined by

$$\mathcal{V}_n(k^2,\mu) =$$

$$= \{(k_1, \ldots, k_n) \,/\, \mathcal{C}_0(t_\mu, a) \cap \mathcal{C}_1(t_\mu, k_1) \cap \ldots \cap \mathcal{C}_n(t_\mu, k_n) \neq 0; \mu \in \Lambda(k^2)\}. \tag{4.10}$$

It has the significance of (the set of) those points $(k_1, \ldots, k_n)$ for which the set of interpolating functions associated with a, $k_1, \ldots, k_n$, $(\mu_0, \ldots, \mu_n)$, and $k^2(c\Gamma)$, i.e. the set of (real) functions $k(z) \in H^2$ obeying

$$|h(z)| \le \begin{cases} s(\theta) & , \ \theta \in \Gamma \\ t_\mu(\theta) & , \ \theta \in C\Gamma \quad , \quad \mu \in \Lambda(h^2) \end{cases} \tag{4.11}$$

$$h(c) = a \ , \ h(x_1) = h_1 \ , \ \dots , \ h(x_n) = h_n \tag{4.12}$$

is not empty. From this it follows [8] that it is the set of values $(h_\mu(x_1) ,\dots, h_\mu(x_n))$ the functions

$$h_\mu'(z) = \frac{a E_\mu^{-1}(0) + z \tilde{w}(z)}{1 + a \bar{E}_\mu^{-1}(0) z \tilde{w}(z)} E_\mu(z) \qquad , \quad \mu \in \Lambda(h^2) \tag{4.13}$$

may take for all (real) functions $\tilde{w}(z)$, $|\tilde{w}(z)| \le 1$. $\mathcal{V}_n(h^2, \mu)$ is thus in fact parametrized by the image B^n (in R^n) of the mapping : $\tilde{w}(z) \longrightarrow (\tilde{w}(x_1) ,\dots, \tilde{w}(x_n))$ of the (real subset) of the unit ball in H^∞ .

The set

$$\mathcal{V}_n(h^2) = \bigcup_{\mu \in \Lambda(h^2)} \mathcal{V}_n(h^2; \mu) \tag{4.14}$$

then is that subset of $\mathcal{D}_n(h^2)$, which characterizes the values $(h(x_1) ,\dots, h(x_n))$ of the functions $h(z) \in \Omega_{(\cdot;a)}$ with (squared) norm

$$\|h\|^2 = \|t\|^2 + \frac{1}{\pi} \int_\Gamma s^2(\theta)\, d\theta \tag{4.15}$$

($t(\theta) = |h(z)|$, $\theta \in C\Gamma$) equal to h^2 (as given by (3.14)).

The solution of problem $\Pi_\varepsilon(2)$ is given by the determination of the values of $\chi(h)$ over the set $\mathcal{V}_n(h^2)$. This set is bounded, convex and closed. The first of these properties needs no proof. The second follows from the fact that if $(\ell_\mu(x_1), \dots, \ell_\mu(x_n)) \in \vartheta_n(\ell^2, \mu)$, $\mu \in \Lambda(\ell^2)$, and $(\ell_\lambda(x_1), \dots, \ell_\lambda(x_n)) \in \vartheta_n(\ell^2, \lambda)$, $\lambda \in \Lambda(\ell^2)$, then for any $0 \le \alpha \le 1$ $(\alpha \ell_\mu(x_1) + (1-\alpha)\ell_\lambda(x_1), \dots, \alpha \ell_\mu(x_n) + (1-\alpha)\ell_\lambda(x_n)) \in \vartheta_n(\ell^2, \alpha\mu + (1-\alpha)\lambda)$, $\alpha\mu + (1-\alpha)\lambda \in \Lambda(\ell^2)$ as a result of the conditions (4.11), (4.12), the linear dependence of $t_\mu^2(\theta)$ on $(\mu_0, \dots, \mu_n)$ and the convexity of $\Lambda(\ell^2)$. Closedness follows from the validity of the compactness principle for the union (over $\Lambda(\ell^2)$) of the sets of analytic functions $\ell_\mu(z)$ defined by (4.13). Over the (bounded, closed and convex) set $\vartheta_n(\ell^2)$ the function $\chi(\ell)$ takes the values situated between the (global) minimum $\chi_m(\ell^2)$,

$$\chi_m(\ell^2) = \min_{(\ell(x_1), \dots, \ell(x_n)) \in \vartheta_n(\ell^2)} \chi(\ell) \tag{4.16}$$

and the (local) maximum $\chi_M(\ell^2)$,

$$\chi_M(\ell^2) = \max_{(\ell(x_1), \dots, \ell(x_n)) \in \vartheta_n(\ell^2)} \chi(\ell) \tag{4.17}$$

The set $\Delta_\varepsilon(z)$ is, therefore, given by the inequalities

$$\|\ell\|^2 \ge \ell_0^2 = \ell_0^2(c\Gamma) + \frac{1}{\pi}\int_\Gamma s^2(\theta)\,d\theta \quad , \tag{4.18}$$

$$\chi_m(\|\ell\|^2) \le \chi(\ell) \le \chi_M(\|\ell\|^2) \quad . \tag{4.19}$$

5. EXTENSION OF THE AUXILIARY PROBLEM

We come back again to the positive cone K^+ in the space L^2. In it we define for arbitrary but fixed numbers ℓ, $k_1, \ldots, k_n$ in addition to the sets $\mathcal{U}_i(k_i)$ ($i = 1, \ldots, n$), which we characterize through the inequalities

$$\mathcal{U}_i(k_i) \qquad |k_i| - E(x_i) \leq c \tag{5.1}$$

for functions $E(z)$ with $|k(z)| = s(\theta)$, $\theta \in \Gamma$, $|k(z)| = r(\theta)$, $\theta \in C\Gamma$ in (4.1), two new sets $\mathcal{U}_\pm(a, \ell)$ (instead of $\mathcal{U}_0(a)$) by

$$\mathcal{U}_+(a,\ell): \quad -\frac{E'(0)}{E(0)} + \frac{\ell}{a} - \frac{E(0)}{a} + \frac{a}{E(0)} \leq c$$

$$\mathcal{U}_-(a,\ell): \quad \frac{E'(0)}{E(0)} - \frac{\ell}{a} - \frac{E(0)}{a} + \frac{a}{E(0)} \leq c \tag{5.2}$$

The intersection $\mathcal{U}(a,\ell) = \mathcal{U}_+(a,\ell) \cap \mathcal{U}_-(a,\ell)$, a subset of $\mathcal{U}_0(a) = \{r(\theta) \geq 0 \,/\, E(0) \geq a\}$, is the set of those functions $r(\theta) \in K^+$ for which the set $\mathcal{E}(r,a,\ell)$ of (real) functions $k(z) \in H^2$ obeying $k(0) = a$, $k'(0) = \ell$ and (4.5) is not empty. The set $\mathcal{U}(a,\ell)$ is also convex [9] and closed (as intersection of level sets of (proper) convex, lower semicontinuous functionals) and so is, therefore, $\mathcal{U}(a,\ell,k_1, \ldots, k_n) = \left(\bigcap_{i=1}^{n} \mathcal{U}_i(k_i)\right) \cap \mathcal{U}(a,\ell)$. This already assures the existence (and uniqueness) of the solution of the auxiliary extremal problem for $\Pi\omega\beta$ to determine the minimum of $\|r\|^2$ in $\mathcal{U}(a,\ell,k_1, \ldots, k_n)$ and the form of the extremal function $r_a(\theta)$. The direct determination of $r_a(\theta)$ by the method of Sec.3 is, however, no more possible since the functionals defining the sets $\mathcal{U}_\pm(a,\ell)$ have not the suitable form. Therefore we shall apply it in an indirect way which avoids this inconvenience. Namely we make use of the fact that the closed, convex set $\mathcal{U}(a,\ell,k_1, \ldots, k_n)$ is the union, characterized by the coordinates of points (in a set) in E^{n+2}, of other closed and convex sets in L^2.

The derivation of the structure of $\mathcal{U}(a, b, \ell_1, \ldots, \ell_n)$ makes use of another form of the factorization of (real) functions $f(z) \in H^2$, than (4.2) :

$$f(z) = B(z)\, S(z) \tag{5.3}$$

where $B(z)$ is a (real) Blaschke product, completely specified by the number and position of zeros of $f(z)$ in $|z|<1$, $|B(z)|=1$, and $S(z) \neq 0$ in $|z|<1$. We concentrate on the (real) functions $S(z) \in H^2$ without zeros in $|z|<1$. Those functions $S(z)$ for which

$$|S(z)| \leq \begin{cases} \Delta(\theta) & , \ \theta \in \Gamma \\ \gamma(\theta) & , \ \theta \in C\Gamma \end{cases} \tag{5.4}$$

(with $\gamma(\theta) \in E^+$, $\ln \gamma(\theta) \in L^1$) have to obey [10] the restrictions

$$\frac{E'(0)}{E(0)} - 2h\,\frac{E(0)}{S(0)} \leq \frac{S'(0)}{S(0)} \leq \frac{E'(0)}{E(0)} + 2h\,\frac{E(0)}{S(0)} \tag{5.5}$$

and

$$S(x_i) \leq E(x_i) \tag{5.6}$$

with $E(z)$ defined by $\Delta(\theta)$ and $\gamma(\theta)$. In analogy to Sec.3 we now consider fixed values $S(0) = a_s > 0$, $S'(0) = b_s$, and $S(x_i) = \ell_i > 0$, and define by

$$\widetilde{\mathcal{U}}_+(a_s, b_s) : \quad -\frac{E'(0)}{E(0)} + \frac{b_s}{a_s} - 2h\,\frac{E(0)}{a_s} \leq 0 \quad , \tag{5.7}$$

$$\widetilde{\mathcal{U}}_-(a_s, b_s) : \quad \frac{E'(0)}{E(0)} - \frac{b_s}{a_s} - 2h\,\frac{E(0)}{a_s} \leq 0 \quad ,$$

and

$$\mathcal{U}_i(\ell_i) \quad : \quad \ell_i - E(x_i) \leq 0 \tag{5.8}$$

the closed, convex sets $\widetilde{\mathcal{O}\!\mathcal{L}}_{\pm}(a_\mathcal{S}, b_\mathcal{S})$, $\mathcal{O}\!\mathcal{L}_i(s_i)$ $(i = 1, \dots, n)$ in K^+ . The interpretation of these sets in connection with functions $\ell(z) \neq 0$ in H^2 is completely analogous to that given for $\mathcal{O}\!\mathcal{L}_{\pm}(a, b)$, $\mathcal{O}\!\mathcal{L}_i(\ell_i)$ (since $\widetilde{\mathcal{O}\!\mathcal{L}}(a_\mathcal{S}, b_\mathcal{S}) = \widetilde{\mathcal{O}\!\mathcal{L}}_+(a_\mathcal{S}, b_\mathcal{S}) \cap \widetilde{\mathcal{O}\!\mathcal{L}}_-(a_\mathcal{S}, b_\mathcal{S}) \subset \mathcal{O}\!\mathcal{L}_c(a_\mathcal{S}) = \{\mu(\theta) \geq 0 \,/\, \bar{\varepsilon}(0) \geq a_\mathcal{S}\}$).

From the conditions

$$-1 \leq B(0) \leq 1 \tag{5.9}$$

$$-|B(0)|^{-1} + |B(0)| \leq \frac{B'(0)}{B(0)} \leq |B(0)|^{-1} - |B(0)|$$

and

$$-1 \leq B(x_i) \leq 1 \quad , \quad i = 1, \dots, n \quad , \tag{5.10}$$

obeyed by the Blaschke products and from (5.3) we derive

$$|\ell(0)| \leq S(0) \quad ,$$

$$\frac{S'(0)}{S(0)} - \frac{S(0)}{|\ell(0)|} + \frac{|\ell(0)|}{S(0)} \leq \frac{\ell'(0)}{\ell(0)} \leq \frac{S'(0)}{S(0)} + \frac{S(0)}{|\ell(0)|} - \frac{|\ell(0)|}{S(0)} \tag{5.11}$$

and

$$|\ell(x_i)| \leq S(x_i) \tag{5.12}$$

These inequalities ($|\ell(0)| \leq S(0)$ is implied by the second line of (5.11)) define in R^{n+2} the set of allowed values $S(0) = a_\mathcal{S}$, $S'(0) = b_\mathcal{S}$, $S(x_i) = s_i$ for given values of $\ell(0) = a$, $\ell'(0) = b$, $\ell(x_i) = \ell_i$:

$$\frac{b}{a} - \frac{a_\mathcal{S}}{a} + \frac{a}{a_\mathcal{S}} \leq \frac{b_\mathcal{S}}{a_\mathcal{S}} \leq \frac{b}{a} + \frac{a_\mathcal{S}}{a} - \frac{a}{a_\mathcal{S}} \tag{5.13}$$

(which implies $a_\mathcal{S} \geq a$), and

$$s_i \geq |\ell_i| \tag{5.14}$$

this set we denote by $\mathcal{L}(a,b,k_1,\ ,k_n) \equiv \mathcal{L}$. From the fact that in R^2 (with coordinates $x = E(0)$, $y = \frac{E'(0)}{E(0)}$) the domain defined by (5.2) is the envelope of the domains defined by (5.7) over the values a_1 , b_1 allowed by (5.13) and in R^1 (with coordinates $x = E(\kappa_1)$) the domain defined by (5.1) is the envelope of the domains defined by (5.8) over the values s_1 allowed by (5.14) it follows that

$$\mathcal{O}\!\mathcal{L}(a,b,k_1,\ ,k_n) = \bigcup_{(a_1,b_1,s_1,\ ,s_n)\in\mathcal{L}} \widetilde{\mathcal{O}\!\mathcal{L}}(a_1,b_1,s_1,\ ,s_n) \quad , \tag{5.15}$$

where $\widetilde{\mathcal{O}\!\mathcal{L}}(a_1,b_1,s_1,\ ,s_n)$ is the closed, convex set defined by the intersection

$$\widetilde{\mathcal{O}\!\mathcal{L}}(a_1,b_1,s_1,\ ,s_n) = \Big(\bigcap_{i=1}^{n} \mathcal{O}\!\mathcal{L}_i(s_i)\Big) \cap \widetilde{\mathcal{O}\!\mathcal{L}}(a_1,b_1) \tag{5.16}$$

The structure of $\mathcal{O}\!\mathcal{L}(a,b,k_1,\ ,k_n)$, expressed by (5.15), allows to derive the equality

$$\min_{r(0)\in\mathcal{O}\!\mathcal{L}(a,b,k_1,\dots,k_n)} \|r\|^2 = \min_{(a_1,b_1,s_1,\ ,s_n)\in\mathcal{L}} \Big(\min_{r(0)\in\widetilde{\mathcal{O}\!\mathcal{L}}(a_1,b_1,s_1,\ ,s_n)} \|r\|^2 \Big), \tag{5.17}$$

by which the auxiliary extremum problem of $\Pi(3)$ is transferred, as far as the infinity of its dimensionality is concerned, from $\mathcal{O}\!\mathcal{L}(a,b,k_1,\ ,k_n)$ to $\widetilde{\mathcal{O}\!\mathcal{L}}(a_1,b_1,s_1,\ ,s_n)$. Here it may be easily solved by the method of Sec.3 . The possibility of solution is due to the convenient form of the functionals

$$\tilde{J}_\pm(r,a_1,b_1) = 2\ln a_1 \pm \frac{b_1}{a_1}$$
$$-\frac{1}{\pi}\int_{\Gamma} p_\pm(\theta)\ln s^2(\theta)\,d\theta - \frac{1}{\pi}\int_{C\Gamma} p_\pm(\theta)\ln r^2(\theta)\,d\theta \quad , \tag{5.18}$$

$$p_\pm(\theta) = 1 \pm \cos\theta$$

which define by $\widetilde{\mathcal{U}}_\pm(a_\pm, b_\pm) = \{r(\theta) \geq 0 \,/\, \widetilde{J}_\pm(r, a_\pm, b_\pm) \leq c\}$ the sets $\widetilde{\mathcal{U}}_\pm(a_\pm, b_\pm)$. The form of the extremal function $r_*(\theta)$ in $\widetilde{\mathcal{U}}(a_\pm, b_\pm, b_1, \ldots, b_n)$ is thus given by

$$r_*^2(\theta) = \sum_j \lambda_j p_j(\theta) \quad , \quad \lambda_j \geq 0 \quad , \quad j = +, -, 1, \ldots, n \, . \tag{5.19}$$

The second step of the minimisation in (5.17), over $\mathcal{L}(a, b, k_1, \ldots, k_n)$, is just the selection of one of the extremal functions of the form (5.19). We therefore may conclude that the extremal function in the auxiliary problem of $\Pi(3)$ for $\mathcal{U}(a, b, k_1, \ldots, k_n)$ is of the form (5.19).

When $n = 0$, the extremal function $r_{a,b}^2(\theta)$ is of the form $r_{a,b}^2(\theta) = \lambda_+(b)\, p_+(\theta) + \lambda_-(b)\, p_-(\theta)$. The values of $\lambda_\pm(b)$ can, however, no more be easily determined; we denote the minimum $\|r_{a,b}\|^2$ by $k_b^2(cr)$:

$$\|r_{a,b}\|^2 = \lambda_+(b)\,\omega_+ + \lambda_-(b)\,\omega_- = k_b^2(cr) \, , \quad \omega_\pm = \frac{1}{\pi} \int_{cr} p_\pm(\theta)\, d\theta \, . \tag{5.20}$$

Proceeding now in analogy to Sec.3, we first define for $\mathcal{U}(a, b, k_1, \ldots, k_n)$ and fixed (a and) b the set $H_n(b, k^2(cr))$ (with $k^2(cr) \geq k_b^2(cr)$). For the description of its structure we define

$$\begin{aligned} J_\pm(r_\nu, a, b) = \pm \frac{b}{a} \mp \frac{1}{\pi} \int_0^\pi \cos\theta \ln s_\nu^2(\theta)\, d\theta \, - \\ - \frac{1}{a} \exp\left(\frac{1}{\pi}\int_0^\pi \ln s_\nu(\theta)\, d\theta\right) + a \exp\left(-\frac{1}{\pi}\int_0^\pi \ln s_\nu(\theta)\, d\theta\right) , \end{aligned} \tag{5.21}$$

with

$$s_\nu(\theta) = \begin{cases} s(\theta) \, , & \theta \in \Gamma \, , \\ r_\nu(\theta) \, , & \theta \in c\Gamma \, , \end{cases} \qquad \nu = (\nu_+, \nu_-, \nu_1, \ldots, \nu_n) \, , \tag{5.22}$$

and $t_\nu^2(\theta) = \sum_j \nu_j\, p_j(\theta)$, $\nu_j \geq 0$, and further the set

$$\Lambda(b,k^2) = \Big\{ (\nu_1,\nu_2,\nu_3,\ \dots,\nu_k) \,\Big|\, \nu_j \geq 0 ,\ \sum_j \nu_j \omega_j = k^2(e,\gamma) ,\ J_+(t_\nu,a,b) \leq c ,\ J_-(t_\nu,a,b) \leq c \Big\} . \tag{5.23}$$

The description of $H_n(b,k^2(e,\gamma))$ is given by

$$H_n(b,k^2(e,\gamma)) = \mathcal{D}_n(b,k^2) \tag{5.24}$$

and

$$\mathcal{D}_n(b,k^2) = \bigcup_{\nu \in \Lambda(b,k^2)} \mathcal{D}_n(b,k^2,\nu) \tag{5.25}$$

with

$$\mathcal{D}_n(b,k^2,\nu) = \Big\{ (k_1,\ \dots,k_n) \,\Big|\, J_1(t_\nu,k_1) \leq c,\ \dots,J_n(t_\nu,k_n) \leq c ,\ \nu \in \Lambda(b,k^2) \Big\} \tag{5.26}$$

The solution of the problem $\Pi_\varepsilon(\delta)$ will be given in the next section with the help of a subset of $\mathcal{D}_n(b,k^2)$.

6. SOLUTION OF THE PROBLEM $\Pi_\varepsilon(3)$

We observe that we may take over directly from Sec.4 the function theoretic interpretation for the set $\mathcal{D}_n(b,k^2,\nu)$:

$$\mathcal{D}_n(b,k^2,\nu) = \Big\{ (k_1,\ \dots,k_n) \,\Big|\, \mathcal{C}_1(t_\nu,k_1) \neq 0,\ \dots,\mathcal{C}_n(t_\nu,k_n) \neq 0 ,\ \nu \in \Lambda(b,k^2) \Big\} \tag{6.1}$$

In order to construct its subset we are interested in, we use in addition the set $\mathcal{C}(t_\nu,a,b)$ introduced in Sec.5, which is not empty if $t_\nu(\theta) \in \mathcal{U}_-(a,b)$. This subset is

$$\vartheta_n(b,k^2,\nu) = \Big\{ (k_1,\ \dots,k_n) \,\Big|\, \mathcal{C}(t_\nu,a,b) \cap \mathcal{C}_1(t_\nu,k_1) \cap\ \dots \cap \mathcal{C}_n(t_\nu,k_n) \neq 0 ,\ \nu \in \Lambda(b,k^2) \Big\} \tag{6.2}$$

It is the set of those (f_1 ,...., f_n), for which the set of (real) functions $f(z)\in H^2$ obeying

$$|f(z)| \le \lambda_\nu(0) \qquad , \qquad \nu \in \Lambda(b,\ell^2) \tag{6.3}$$

and

$$f(0)=a \ , \ f'(0)=b \ , \ f(x_1)=f_1 \ , \ \ldots \ , f(x_n)=f_n \ , \tag{6.4}$$

is not empty. Therefore it is given [8] by the values ($f_\nu(x_1)$,...., $f_\nu(x_n)$) of the functions

$$f_\nu(z) = \frac{a\,E_\nu^{-1}(0) + z\,w_\nu(z)}{1 + a\,E_\nu^{-1}(0)\,z\,w_\nu(z)}\,E_\nu(z) \qquad , \qquad \nu\in\Lambda(b,\ell^2), \tag{6.5}$$

where $E_\nu(z)$ is defined by (4.1) with $|f(z)| = \lambda_\nu(0)$, and $w_\nu(z)$ by

$$w_\nu(z) = \frac{w_\nu(0) + z\,\tilde{w}(z)}{1 + w_\nu(0)\,z\,\tilde{w}(z)} \qquad , \tag{6.6}$$

with

$$w_\nu(0) = \left(\frac{b}{a} - \frac{E_\nu'(0)}{E_\nu(0)}\right)\left(\frac{E_\nu(0)}{a} - \frac{a}{E_\nu(0)}\right)^{-1} \tag{6.7}$$

and a (real) function $\tilde{w}(z)$, which is arbitrary except $|\tilde{w}(z)| \le 1$. Otherwise stated : $\vartheta_n(b,\ell^2,\nu)$ is parametrised, like $\vartheta_n(\ell^2_\nu)$, by the set B^n , only in a different manner.

The set which characterizes the values ($f(x_1)$,...., $f(x_n)$) of the functions $f(z)\in\Omega(\lambda;a)$, obeying $f'(0)=b$, with squared norm (4.15) equal to ℓ^2 (given by (3.14)), is

$$\vartheta_n(b,\ell^2) = \bigcup_{\nu\in\Lambda(b,\ell^2)} \vartheta_n(b,\ell^2,\nu) \tag{6.8}$$

This set is, again, bounded, closed and convex. The arguments of the proofs are the same as those for $\vartheta_n(\ell^2)$, supplemented by the convexity of the set $\Lambda(\ell,\ell^2)$.[11]

With the quantities

$$\chi_m(\ell,\ell^2) = \min_{(\ell(x_1),\ \ldots,\ \ell(x_n)) \in \vartheta_n(\ell,\ell^2)} \chi(\ell) \tag{6.9}$$

$$\chi_M(\ell,\ell^2) = \max_{(\ell(x_1),\ \ldots,\ \ell(x_n)) \in \vartheta_n(\ell,\ell^2)} \chi(\ell) \tag{6.10}$$

we describe in R^3 (of coordinates $\ell'(0)$, $\|\ell\|^2$, $\chi(\ell)$) a set $\Delta_\varepsilon^\ell(3)$ by

$$\ell'(0) = \ell \quad , \tag{6.11}$$

$$\|\ell\|^2 \geq \ell_\ell^2(C\Gamma) + \frac{1}{\pi}\int_\Gamma \Lambda^2(\theta)\,d\theta = \ell^2(\ell) \quad , \tag{6.12}$$

$$\chi_m(\ell,\|\ell\|^2) \leq \chi(\ell) \leq \chi_M(\ell,\|\ell\|^2) \quad . \tag{6.13}$$

The solution of the problem $\Pi_\varepsilon(3)$, the set $\Delta_\varepsilon(3)$, is then given by the union

$$\Delta_\varepsilon(3) = \bigcup_\ell \Delta_\varepsilon^\ell(3) \tag{6.14}$$

over all real values of ℓ .

7. CONVEXITY PROPERTIES OF THE SOLUTIONS

We derive in this section a few qualitative results, which give an insight into the structure of the sets $\Delta_\varepsilon(2)$ and $\Delta_\varepsilon(3)$. Since $\Delta_\varepsilon(2)$ is just the projection of $\Delta_\varepsilon(3)$ on the subspace $R^2(\|\ell\|^2, \chi(\ell))$

of $R^3(\,k'(0)\,,\,\|k\|^2,\,\chi(k)\,)$, we shall derive only the properties of $\Delta_\varepsilon(3)$.

First we prove convexity of the projection ϑ of $\Delta_\varepsilon(3)$ on the subspace $R^2(\,k'(0)\,,\,\|k\|^2)$. The set ϑ consists of the points $(\,k'(0)=b,\ \|k\|^2=k^2)$ for which $\vartheta_n(b,k^2)$ is not empty. If $(\,b_0\,,\,k_0^2\,)$ and $(\,b_1\,,\,k_1^2\,)$ belong to it, then there exist functions $k_0(z)$, $k_1(z)$ of the form (6.5) belonging (through $\vartheta_n(b_0,k_0^2)$ and $\vartheta_n(b_1,k_1^2)$) to them. The values $(\,k_\alpha(x_1)\,,\ldots,\,k_\alpha(x_n))$ of the function $k_\alpha(z)=\alpha\,k_1(z)+(1-\alpha)\,k_0(z)$, $0\le\alpha\le 1$, belong due to $k_\alpha(0)=a$, $k_\alpha'(0)=\alpha\,b_1+(1-\alpha)\,b_0$ and $|k_\alpha(\tau)|\le\left(\alpha|k_1(\tau)|^2+(1-\alpha)|k_0(\tau)|^2\right)^{\frac{1}{2}}$ to $\vartheta_n(\alpha\,b_1+(1-\alpha)\,b_0\,,\,\alpha\,k_1^2+(1-\alpha)\,k_0^2)$, which is thus proved to be not empty. Therefore $(\alpha\,b_1+(1-\alpha)\,b_0\,,\,\alpha\,k_1^2+(1-\alpha)\,k_0^2\,)\in\vartheta$ and ϑ is convex. The set ϑ is closed : to any boundary point of it there corresponds only one function $k_0(z)$ of the form (6.5).

If we let $k_0(z)$, $k_1(z)$ go through the whole classes defining by (6.5) the sets $\vartheta_n(b_0,k_0^2)$ and $\vartheta_n(b_1,k_1^2)$, then all values $(\,k_\alpha(x_1)\,,\ldots,\,k_\alpha(x_n)\,)$, $k_\alpha(x_i)=\alpha\,k_1(x_i)+(1-\alpha)\,k_0(x_i)$ define the convex combination $\alpha\,\vartheta_n(b_1,k_1^2)+(1-\alpha)\,\vartheta_n(b_0,k_0^2)$ of these two sets. The arguments presented above in the convexity proof of ϑ also show that $\alpha\,\vartheta_n(b_1,k_1^2)+(1-\alpha)\,\vartheta_n(b_0,k_0^2)$ is (strictly) included (i.e. with no common boundary points) in $\vartheta_n(\alpha\,b_1+(1-\alpha)\,b_0\,,\,\alpha\,k_1^2+(1-\alpha)\,k_0^2\,)$.

For a fixed value of b the set $\vartheta_n(b,k^2)$ increases strictly with k^2 (the boundary points become interior points since they are of norm $\|k\|^2=k^2$). Therefore $\chi_M(b,k^2)$ is a strictly increasing function of k^2, whereas $\chi_m(b,k^2)$ is a strictly decreasing function of k^2 as long as $(a_1\,,\ldots,a_n)\notin\vartheta_n(b,k^2)$ i.e. for $k^2\le k_a^2(b)$, where $k_a^2(b)$ is the value of k^2 at which $(\,a_1\,,\ldots,\,a_n\,)$ lies on the boundary of $\vartheta_n(b,k^2)$. For $k^2\ge k_a^2(b)$ we have $\chi_m(b,k^2)=0$.

The function $\chi_m(b,k^2)$ is convex, i.e. it satisfies over ϑ the inequality

$$\alpha\,\chi_m(b_1,\ell_1^2)+(1-\alpha)\,\chi_m(b_0,\ell_0^2)\geq \chi_m(\alpha b_1+(1-\alpha)b_0,\ \alpha\ell_1^2+(1-\alpha)\ell_0^2)\ ,\quad 0\leq\alpha\leq 1\ . \tag{7.1}$$

The proof of it makes use of the convexity of the function $\chi(\ell)$ and of the fact (just proved) that the convex combination of $\vartheta_n(b_0,\ell_0^2)$ and $\vartheta_n(b_1,\ell_1^2)$ is strictly included in the set $\vartheta_n(\alpha b_1+(1-\alpha)b_0,\ \alpha\ell_1^2+(1-\alpha)\ell_0^2)$.

From (7.1) it follows (since $\chi_m(b,\ell^2)\geq 0$) that the subset ϑ_0 of ϑ, where $\chi_m(b,\ell^2)=0$, is convex. If not both points (b_0,ℓ_0^2) (b_1,ℓ_1^2) belong to ϑ_0, then there is strict inequality in (7.1) for $0<\alpha<1$.

Since intersections and projections of sets preserve convexity, we may derive in this manner further interesting and useful convexity properties of partial correlations between $\ell'(0)$, $\|\ell\|^2$, and $\chi(\ell)$.

By projection of $\Delta_\varepsilon(3)$ on the plane $E^2(\|\ell\|^2,\chi(\ell))$ we obtain for the increase (decrease) of $\chi_M(\|\ell\|^2)$ $(\chi_m(\|\ell\|^2))$ with $\|\ell\|^2$ and the convexity of the curve $\chi_m(\|\ell\|^2)$ in the region $\|\ell\|^2\geq\ell_0^2$ of their definition (the projection of ϑ on the $\|\ell\|^2$-axis). These properties are all strict (for $\chi_m(\|\ell\|^2)$ in the domain where it is not zero).

A somewhat stronger correlation between $\|\ell\|^2$ and $\chi(\ell)$, than given by $\Delta_\varepsilon(2)$, is obtained by the projection of the intersection of $\Delta_\varepsilon(3)$ with the (convex) set $b_m\leq\ell'(0)\leq b_M$ (with b_m, b_M fixed, $b_m\leq b_M$). The functions $\chi_m(\|\ell\|^2;b_m,b_M)=\inf\limits_{\ell'(0)\in\vartheta(b_m,b_M,\|\ell\|^2)}\chi_m(\ell'(0),\|\ell\|^2)$ and $\chi_M(\|\ell\|^2;b_m,b_M)=\sup\limits_{\ell'(0)\in\vartheta(b_m,b_M,\|\ell\|^2)}\chi_M(\ell'(0),\|\ell\|^2)$, $\vartheta(b_m,b_M,\|\ell\|^2)=\{\ell'(0)\mid(\ell'(0),\|\ell\|^2)\in\vartheta,\ b_m\leq\ell'(0)\leq b_M\}$, defining by $\Delta_\varepsilon(2;b_m,b_M)=\{(\|\ell\|^2,\chi(\ell))\mid \|\ell\|^2\geq\ell^2(b_m,b_M),\ \chi_m(\|\ell\|^2;b_m,b_M)\leq\chi(\ell)\leq\chi_M(\|\ell\|^2;b_m,b_M)\}$, $\ell^2(b_m,b_M)=$

$= \inf_{\ell_m \le \ell \le \ell_M} \ell^2(\ell)$, the set $\Delta_\varepsilon(2; \ell_m, \ell_M)$ expressing this correlation, have the same qualitative properties as $\chi_m(\|\ell\|^2), \chi_M(\|\ell\|^2)$.

By intersection of the convex set $\Delta_\varepsilon(3,m) = \{(\ell'(0), \|\ell\|^2, \chi(\ell)) \mid (\ell'(0), \|\ell\|^2) \in \vartheta, \chi(\ell) \ge \chi_m(\ell'(0), \|\ell\|^2)\}$ with the halfspace $\|\ell\|^2 \le \ell^2$ (with a fixed value $\ell^2 \ge \ell_0^2$) we get a convex set, which we project on the plane $E^2(\ell'(0), \chi(\ell))$. The convex set $\Delta_\varepsilon(\ell^2) = \{(\ell'(0), \chi(\ell)) \mid (\ell'(0), \ell^2) \in \vartheta, \chi(\ell) \ge \chi_m(\ell'(0), \ell^2)\}$ thus obtained expresses the correlation (as far as χ_m is implied) between $\ell'(0)$, $\chi(\ell)$ under the condition $\|\ell\|^2 \le \ell^2$.

By intersection of $\Delta_\varepsilon(3,m)$ with the halfspace $\chi(\ell) \le \varkappa$ (with a fixed value $\varkappa \ge 0$) we get another convex set, of significance through its (convex) projection $\Delta_\varepsilon(\varkappa) = \{(\ell'(0), \|\ell\|^2) \mid (\ell'(0), \|\ell\|^2) \in \vartheta, \chi_m(\ell'(0), \|\ell\|^2) \le \varkappa\}$ on the plane $E^2(\ell'(0), \|\ell\|^2)$, which gives the correlation (as far as χ_m is concerned) between $\ell'(0)$ and $\|\ell\|^2$ under the condition $\chi(\ell) \le \varkappa$.

8. PARAMETRIZATION OF THE BOUNDARIES OF $\vartheta_n(\ell^2)$ AND $\vartheta_n(t, \ell^2)$

In the preceding sections we have reduced the determination of the sets $\Delta_\varepsilon(2)$ and $\Delta_\varepsilon(3)$ to finite-dimensional optimization problems. The (computational) solution of these problems is much facilitated by a convenient parametrization of the sets $\vartheta_n(\ell^2)$, $\vartheta_n(t, \ell^2)$ which express their constraints. An important part of the parametrization of $\vartheta_n(\ell^2)$, $\vartheta_n(t, \ell^2)$ is solved if one for $\Lambda(\ell^2)$, $\Lambda(t, \ell^2)$ is found.

The definition of $\Lambda(\ell^2)$, $\Lambda(t, \ell^2)$ shows that they are subsets of simplexes in Euclidean spaces. Concentrating on $\Lambda(\ell^2)$ in order to be specific (the discussion for $\Lambda(t, \ell^2)$ is identical, only the dimension of the space is larger by one unit), we define by $\mu_i = \ell^2(er)\frac{\alpha_i}{\omega_i}$ ($\ell^2(er) \geq \ell_0^2(er) > 0$) a mapping of the standard n-dimensional simplex in R^{n+1} (α): $\alpha_i \geq 0$, $\sum_{i=0}^{n} \alpha_i = 1$ on the simplex : $\mu_i \geq 0$, $\sum_{i=0}^{n} \mu_i \omega_i = \ell^2(er)$ associated with $\Lambda(\ell^2)$. The standard simplex (α) may be considered as the image of the unit cube $0 \leq t_i \leq 1$ ($i = 1, \ldots, n$) in R^n through the (singular) transformation of Serre :

$$
\begin{aligned}
\alpha_0 &= 1 - t_1 \\
\alpha_1 &= t_1(1-t_2) \\
&\vdots \\
\alpha_i &= t_1 \ldots t_i(1-t_{i+1}) \\
&\vdots \\
\alpha_n &= t_1 \ldots t_n
\end{aligned}
\qquad (8.1)
$$

This transformation is, geometrically, the projection on the simplex (α), by $\alpha_i = y_i^2$, of the intersection of the unit sphere with the nonnegative orthant in R^{n+1} ($y_i \geq 0$, $\sum_{i=0}^{n} y_i^2 = 1$), parametrized in terms of spherical coordinates θ_i ($i = 1, \ldots, n$) : $t_i = \sin^2\theta_i$, $0 \leq \theta_i \leq \frac{\pi}{2}$. The nature of the singularities of the transformation (8.1) is indicated by its Jacobi determinant,

$$
\frac{D(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)}{D(t_1, \ldots\ldots\ldots\ldots, t_n)} = (-1)^i\, t_1^{n-1} \ldots t_i^{n-i} \ldots t_{n-1} , \qquad (8.2)
$$

$$(i = 0, \ldots, n)$$

Their presence is, however, harmless in computation.

We have thus parametrized the sets $\Lambda(\ell^2)$, $\Lambda(k,\ell^2)$ as subsets of the unit cubes in R^n and R^{n+1}, respectively.

Since the (nontrivial) values of χ_n , χ_{nj} are realized by points on the boundaries $\partial\vartheta_n(\ell^2)$, $\partial\vartheta_n(k,\ell^2)$ of $\vartheta_n(\ell^2)$, $\vartheta_n(k,\ell^2)$ and these are included in the unions of the boundaries $\partial\vartheta_n(\ell^2;\mu)$, $\partial\vartheta_n(\ell,\ell^2;\nu)$ of $\vartheta_n(\ell^2;\mu)$ and $\vartheta_n(\ell,\ell^2;\nu)$,

$$\partial\vartheta_n(\ell^2) \subset \bigcup_{\mu\in\Lambda(\ell^2)} \partial\vartheta_n(\ell^2;\mu) \quad , \tag{8.3}$$

$$\partial\vartheta_n(\ell,\ell^2) \subset \bigcup_{\nu\in\Lambda(\ell,\ell^2)} \partial\vartheta_n(\ell,\ell^2;\nu) \quad . \tag{8.4}$$

we shall seek convenient parametrizations of $\partial\vartheta_n(\ell^2;\mu)$, $\partial\vartheta_n(\ell,\ell^2;\nu)$. It is known [8] that the boundary ∂B^n of B^n is the image of (real) Blaschke products $B_{n-1}(z)$ with at most n-1 zeros. Therefore $\partial\vartheta_n(\ell^2;\mu)$, $\partial\vartheta_n(\ell,\ell^2;\nu)$ are parametrized by the zeros of Blaschke products, by the substitution $\tilde{w}(z) = B_{n-1}(z)$ in (4.13) and (6.6). But there is a pairwise correlation between complex zeros, which causes a difference of status between real and complex zeros. In order to eliminate this difference we observe that the zeros of the real second order polynomial $p(z) = z^2 + A_1 z + A_0$ lie in the unit disk $|z| \leq 1$ if (and only if) the coefficients (A_0 , A_1) are confined to the closed triangle (Δ) with corners of coordinates (-1,o), (1,-2) and (1,2).[8] This triangle (2-dimensional simplex) may be parametrized by the barycentric coordinates [12] $\alpha_0, \alpha_1, \alpha_2$ ($\alpha_i \geq 0$, $\alpha_0+\alpha_1+\alpha_2 = 1$) as

$$\begin{aligned} A_0 &= -\alpha_0 + \alpha_1 + \alpha_2 \quad , \\ A_1 &= -2\alpha_1 + 2\alpha_2 \quad , \end{aligned} \tag{8.5}$$

and further, by the Serre transformation (8.1) (with n=2), as

$$\begin{aligned} A_0 &= 2t_1 - 1 \\ A_1 &= 2t_1(2t_2-1) \quad , \qquad 0 \leq t_1, t_2 \leq 1 \end{aligned} \tag{8.6}$$

This parametrization of Δ is simpler than the one given earlier in Ref.8, which turns out to be essentially the result of the present procedure, applied separately to the two triangles, of corners (1,-2), (1,0), (-1,0) and (1,2), (1,0), (-1,0), respectively, of a subdivision of Δ.

Any pair of (real or complex conjugated) zeros thus leads in the Blaschke product to a factor of the form $\frac{z^2+A_1 z+A_0}{1+A_1 z+A_0 z^2}$, with (A_0, A_1) corresponding to some values t_1, t_2, according to (8.6). If n-1 is even, then any $\mathcal{B}_{n-1}(z)$ may be written, up to a sign, as a product of (n-1)/2 factors of this type. For n-1 odd it has up to a sign the form of a product of n/2-1 such factors and a factor $\frac{z+\beta}{1+\beta z}$, $-1 \le \beta \le 1$.

The sign ambiguity may also be considered as coming from a factor, ξ, with $\xi = \pm 1$. If we extend this factor to all values $-1 \le \xi \le 1$, we get (real) functions

$$\mathcal{B}_{n-1}(z;\xi) = \begin{cases} \xi \displaystyle\prod_{k=1}^{\frac{n-1}{2}} \frac{z^2+A_1^{(k)}z+A_0^{(k)}}{1+A_1^{(k)}z+A_0^{(k)}z^2}, & n=\text{odd} \\ \xi \dfrac{z+\beta}{1+\beta z} \displaystyle\prod_{k=1}^{\frac{n}{2}-1} \frac{z^2+A_1^{(k)}z+A_0^{(k)}}{1+A_1^{(k)}z+A_0^{(k)}z^2}, & n=\text{even} \end{cases} \tag{8.7}$$

with $(A_0^{(k)}, A_1^{(k)})$ given by formulas of the form (8.6) in terms of $t_1^{(k)}, t_2^{(k)}$, and with $-1 \le \xi, \beta \le 1$. The substitution $\bar{\psi}(z) = \mathcal{B}_{n-1}(z;\xi)$ in (4.13) and (6.6) then leads to a parametrization in etrms of the unit cube in R^n of the whole sets $\vartheta_n(L^2,\mu)$, $\vartheta_n(L, L^2, \nu)$, but this is computationally equivalent with the parametrization of the boundaries as long as $(a_1, \dots, a_n) \notin \vartheta_n^0(\ell^2, \mu), \vartheta_n^0(b, \ell^2, \nu)$ and even the parametrization needed when $(a_1, \dots, a_n) \in \vartheta_n^0(\ell^2, \mu), \vartheta_n^0(b, \ell^2, \nu)$.

9. ALTERNATIVE FORMULATION OF THE PROBLEMS

The function $\chi(\ell)$, defined by (2.3), has the significance of an Euclidean norm in R^n. Its use is computationally very convenient but physically less well motivated since its value does not keep the quantities $\frac{\ell(x_i)-a_i}{\gamma_i}$ individually under control. The opposite situation appears if one uses instead the Chebyshev norm (in R^n)

$$\pi(\ell) = \max_{1 \le i \le n} \left| \frac{\ell(x_i)-a_i}{\gamma_i} \right| \tag{9.1}$$

Geometrically $\chi(\ell) \le \varkappa$ ($\varkappa > 0$) represents in R^n an ellipsoid, whereas $\pi(\ell) \le \varkappa$ is the parallelotope inscribed in it.

The use of the Chebyshev norm $\pi(\ell)$ instead of $\chi(\ell)$ modifies the formulation of our problems into :

PROBLEM $\Pi_p(2)$: To determine for the set $\Omega(\Lambda; a)$ in H^2 the image $\Delta_p(2)$ in the 2-dimensional Euclidean space R^2, given by the mapping $\ell(x) \to (\|\ell\|^2, \pi(\ell))$,

PROBLEM $\Pi_p(3)$: To determine for the set $\Omega(\Lambda; a)$ in H^2 the image $\Delta_p(3)$ in the 3-dimensional Euclidean space R^3, given by the mapping $\ell(x) \to (\ell'(0), \|\ell\|^2, \pi(\ell))$.

Since the construction of the sets $\vartheta_n(\ell^2), \vartheta_n(b, \ell^2)$ in Secs. 4 and 6 is still independent of $\chi(\ell)$, and $\pi(\ell)$ is a continuous and convex function of $\ell(x_1), \dots, \ell(x_n)$, we may write directly their solutions with the functions

$$\begin{aligned}\bar{\pi}_m(\ell^2) &= \min_{(\ell(u_1),\ldots,\ell(u_n))\in \vartheta_n(\ell^2)} \bar{\pi}(\ell)\\ \bar{\pi}_M(\ell^2) &= \max_{(\ell(u_1),\ldots,\ell(u_n))\in \vartheta_n(\ell^2)} \bar{\pi}(\ell)\end{aligned} \tag{9.2}$$

(defined, as $\chi_m(\ell^2)$, $\chi_M(\ell^2)$, for $\ell^2 \geq \ell_o^2$) and

$$\begin{aligned}\bar{\pi}_m(t,\ell^2) &= \min_{(\ell(u_1),\ldots,\ell(u_n))\in \vartheta_n(t,\ell^2)} \bar{\pi}(\ell)\\ \bar{\pi}_M(t,\ell^2) &= \max_{(\ell(u_1),\ldots,\ell(u_n))\in \vartheta_n(t,\ell^2)} \bar{\pi}(\ell)\end{aligned} \tag{9.3}$$

(defined over the same set ϑ as $\chi_m(t,\ell^2)$ and $\chi_M(t,\ell^2)$), as

$$\Delta_p(2) = \left\{ (\|\ell\|^2, \bar{\pi}(\ell)) \,\middle|\, \|\ell\|^2 \geq \ell_o^2 \,,\ \bar{\pi}_m(\|\ell\|^2) \leq \bar{\pi}(\ell) \leq \bar{\pi}_M(\|\ell\|^2) \right\}, \tag{9.4}$$

$$\Delta_p(3) = \bigcup_t \Delta_p^t(3) \qquad , \tag{9.5}$$

$$\Delta_p^t(3) = \left\{ (\ell'(0), \|\ell\|^2, \bar{\pi}(\ell)) \,\middle|\, \ell'(0) = t \,,\ \|\ell\|^2 \geq \ell^2(t) \,,\ \bar{\pi}_m(t,\ell^2) \leq \bar{\pi}(\ell) \leq \bar{\pi}_M(t,\ell^2) \right\}.$$

Thereby, all convexity properties derived in Sec.7 can be extended to the analogous quantities constructed with $\bar{\pi}(\ell)$.

The relative position of the sets Δ_E and Δ_p can be determined qualitatively from the (topological equivalence) relation

$$\chi(\ell) \leq \bar{\pi}(\ell) \leq \sqrt{n}\,\chi(\ell) \tag{9.6}$$

FOOTNOTES AND REFERENCES

+ Work performed under contract with the Romanian Nuclear Energy Committee.

1 G.Nenciu and I.Raszillier, Nuovo Cimento A 11, 319 (1972).

2 I.Raszillier, W.Schmidt and I.S.Stefanescu, Nucl. Phys. B 109, 452 (1976).

3 I.Raszillier, Paralipomena to the rigorous phenomenology of the pion electromagnetic form factor, Institute of Physics Bucharest, Preprint (1976).

4 I.Raszillier, On correlation between the pion charge radius and the pionic contribution to the muon magnetic moment, Institute for Nuclear Physics and Engineering Bucharest, Preprint FT-133-77 (1977).

5 P.-J.Laurent, Approximation et optimisation (Hermann, Paris, 1972).

6 A.D.Ioffe and V.M.Tikhomirov, Theory of extremal problems (in Russian) (Nauka, Moscow, 1974).

7 D.G.Luenberger, Optimization by vector space methods (Wiley, New York, 1969).

8 G.Nenciu, I.Raszillier, W.Schmidt and H.Schneider, Nucl. Phys. B 63, 285 (1973).

9 A simple proof of the convexity of $\mathcal{U}_f(\alpha,\ell)$ may be given by function theoretic methods: If we have two functions $f_0(\theta)$, $f_1(\theta) \in K^+$ with $\ln f_i(\theta) \in L^1$, then we may construct the sets $\Omega(f_0)$, $\Omega(f_1)$ of (real) functions $h_0(z), h_1(z) \in H^2$ obeying $|h_i(z)| \le \mu(\theta)$, $\theta \in \Gamma$, and $|h_i(z)| \le f_i(\theta)$, $\theta \in C\Gamma$. Their convex combination $\alpha\,\Omega(f_1) + (1-\alpha)\,\Omega(f_0)$

consisting of the functions $h_\alpha(z) = \alpha h_1(z) + (1-\alpha) h_0(z)$, $0 \le \alpha \le 1$, obeys $|h_\alpha(z)| \le \alpha |h_1(z)| + (1-\alpha)|h_0(z)| \le \left(\alpha |h_1(z)|^2 + (1-\alpha)|h_0(z)|^2\right)^{\frac{1}{2}}$ and is therefore included in the sets $\Omega(\alpha t_1 + (1-\alpha) t_0)$ and $\Omega\left((\alpha t_1^2 + (1-\alpha) t_0^2)^{\frac{1}{2}}\right)$. The maximum modulus (Szegö) theorem then leads to

$$\alpha \exp\left(\frac{1}{\pi}\int_0^\pi \ln \lambda_1(\theta)\, d\theta\right) + (1-\alpha)\exp\left(\frac{1}{\pi}\int_0^\pi \ln\lambda_0(\theta)\,d\theta\right) \le \exp\left(\frac{1}{\pi}\int_0^\pi \ln\left(\alpha\lambda_1(\theta)+(1-\alpha)\lambda_0(\theta)\right)d\theta\right) \qquad (+)$$

and to

$$\alpha \exp\left(\frac{1}{2\pi}\int_0^\pi \ln \lambda_1^2(\theta)\, d\theta\right) + (1-\alpha)\exp\left(\frac{1}{2\pi}\int_0^\pi \ln\lambda_0^2(\theta)\,d\theta\right) \le \exp\left(\frac{1}{2\pi}\int_0^\pi \ln\left(\alpha\lambda_1^2(\theta)+(1-\alpha)\lambda_0^2(\theta)\right)d\theta\right) , \qquad (++)$$

with $\lambda_i(\theta) = \lambda(\theta)$, $\theta \in \Gamma$, $\lambda_i(\theta) = t_i(\theta)$, $\theta \in C\Gamma$. From (+) follows the convexity of $\mathcal{U}_0(a)$ and from (++) one may derive the convexity of $\Lambda(k^2)$.- If $\lambda_i(\theta)$ are such that $\exp\left(\frac{1}{\pi}\int_0^\pi \ln \lambda_i(\theta)\, d\theta\right) \ge a$, then we define with them the sets $\Omega_i(t_i, a)$ of real functions $h_i(z) \in H^2$ obeying $|h_i(\theta)| \le \lambda_i(\theta)$ and $h_i(0) = a$. Their convex combination $\alpha\Omega(t_1,a) + (1-\alpha)\Omega(t_0,a)$ is again included, due to the same inequalities as before, in $\Omega(\alpha t_1 + (1-\alpha)t_0, a)$ and in $\Omega\left((\alpha t_1^2+(1-\alpha)t_0^2)^{\frac{1}{2}}, a\right)$. From the combination of these inclusions with the range of $h'(0)$ in a set $\Omega(t,a)$ we get (over $\mathcal{U}_0(a)$) the convexity in $t(\theta)$ and in $t^2(\theta)$ of the functionals defining the sets $\mathcal{U}_t(a,b)$.- Convexity of the sets $\mathcal{U}_0(a) \cap \mathcal{U}_t(a,b)$ (and therefore of $\mathcal{U}(a,b)$) is implied by the convexity in $t(\theta)$ of these functionals.

10 I. Raszillier, Lett. Nuovo Cimento 4, 690 (1972).

11 Convexity of $\Lambda(\ell,\ell^2)$ is implied (through the convexity of $\mathcal{J}_t(t_v, a, b)$ in $\overset{2}{t_v}(e)$ over $E_v(e) \geq a$) by the convexity in $\overset{2}{\tau}(e)$ (over $\mathcal{U}_t(a)$) of the functionals defining $\mathcal{U}_t(a,b)$.

12 R.T.Rockafellar, Convex analysis (University Press, Princeton, 1970).