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Budden Tunnelling in Parallel Stratified Plasmas

D. B. Batchelor

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D. B. Batchelor

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CONTENTS

ABSTRACT

The process of Budden tunnelling of obliquely propagating extraordinary mode is investigated in plasmas whose parameters vary along the magnetic field (parallel stratification). The wave tunnels through the evanescent region separating the right-hand cutoff layer from the electron cyclotron resonance. Coupled mode equations describing both ordinary and extraordinary waves are derived for arbitrary angle of incidence with respect to the magnetic field. Under appropriate conditions (n_x and $d\ \ell n\ (B)/dx$ not too large) the coupling can be ignored, and the usual Whittaker equation is obtained for a linear magnetic field profile. It is shown that deviation from strictly parallel propagation ($n_x \neq 0$) has a very small effect on tunnelling for a wide range of angle of incidence. The analytical results are verified by numerical integration of the field equations. The theory is applied to the propagation of extraordinary mode waves in the surface plasma of the ELMO Bumpy Torus devices. Fractions of incident power absorbed and transmitted to the high field region in the range of 2-15% occur for a broad spectrum of n_x.

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I. INTRODUCTION

One of the most promising techniques for wave heating of plasmas makes use of the extraordinary mode at frequencies near the electron cyclotron frequency ($\omega \sim \Omega_e$). Electron cyclotron heating (ECH) has been successfully applied in bumpy tori [DANDL et al. (1975) and DANDL et al. (1976)], tokamaks [ALIKAEV et al. (1974) and ALIKAEV et al. (1976)], multipoles [SPROTT (1971) and KERST et al. (1971)], and other devices. One crucial aspect of ECH is the accessibility of the cyclotron resonance zone to waves propagating from outside the plasma. It is well-known that extraordinary mode waves propagating from a low magnetic field region (or in some cases from a low density region) are reflected at a wave cutoff before reaching the resonant zone. In particular, at the right-hand cutoff defined by

$$\omega = \omega_R = \frac{\Omega_e}{2} + \left(\frac{\Omega_e^2}{4} + \omega_{pe}^2\right)^{1/2} , \tag{1}$$

the extraordinary mode does not propagate at any angle. However, if the region between the cutoff and the resonance is thin, the wave energy can be partially absorbed at the resonance and partially transmitted into the high field region by the process of Budden tunnelling [BUDDEN (1961), STIX (1962), and WHITE and CHEN (1974)].

The details of the tunnelling process depend sensitively upon the plasma geometry. In some devices (e.g., tokamaks) the plasma is adequately modeled as a slab with straight magnetic field lines $\underset{\sim}{B}$ along the z direction and all gradients of density n_e and gradients of magnetic field strength being perpendicular to $\underset{\sim}{B}$, say in the x direction. In such a plasma, stratified perpendicular to $\underset{\sim}{B}$, the refractive indices in the z and y directions ($n_z = k_z c/\omega$, $n_y = k_y c/\omega$) are constant as the wave propagates because of Snell's law. Cold plasma theory then predicts that if $n_y = 0$, the extraordinary mode wave is cut off ($n_x \to 0$) at a density and magnetic field strength such that

$$(1 - \beta)n_z^4 - 2(1 - \alpha - \beta)n_z^2 + (1 - \alpha)^2 - \beta = 0 \tag{2}$$

where

$$\beta = \Omega_e^2/\omega^2$$

$$\alpha = \omega_{pe}^2/\omega^2$$

$$\Omega_e = \frac{eB_z}{m_e c}$$

$$\omega_{pe}^2 = \frac{4\pi n_e e^2}{m_e} ,$$

and the extraordinary mode wave has a resonance at the upper hybrid frequency $\omega^2 = \omega_{UH}^2 = \omega_{pe}^2 + \Omega_e^2$. The resonance frequency is independent of the angle of incidence, k_z. This case of stratification perpendicular to the magnetic field has received the most attention [BUDDEN (1961) and WHITE and CHEN (1974)].

In other devices such as bumpy tori, mirror machines, or multipoles, the variations in B and n_e along the magnetic field are very important and in some regions can dominate variations perpendicular to $\underset{\sim}{B}$. As an example, Fig. 1 shows a cross section in the equatorial plane of one cavity of the ELMO Bumpy Torus device (EBT). The magnetic field lines and cyclotron resonance surface are shown, as well as the location of the right-hand cutoff surface for a particular plasma density model [BATCHELOR (1978)]. Near the cyclotron resonance, particularly at the magnetic axis, the magnetic field varies strongly along the field lines (a magnetic beach). As a simple model of this type of geometry, we consider in this paper a plasma slab which is stratified parallel to the magnetic field. The magnetic field is taken along z, and the field strength is assumed to vary only with z [$\underset{\sim}{B} = B(z)\hat{z}$].

In this geometry the components of the refractive index in the x and y directions are fixed as a consequence of Snell's law, and we can without loss of generality choose $n_y = 0$. Using cold plasma wave theory, one finds that the parallel index of refraction n_z satisfies a dispersion relation of the form

$$An_z^4 + Bn_z^2 + C = 0 \tag{3}$$

where

$$A = (1 - \alpha)(1 - \beta)$$

$$B = -2(1 - \alpha)(1 - \alpha - \beta) - [(1 - \alpha)(1 - \beta) + 1 - \alpha - \beta]n_x^2$$

$$C = (1 - \alpha - \beta)n_x^4 - [(1 - \alpha)(1 - \alpha - \beta) + (1 - \alpha)^2 - \beta]n_x^2$$
$$+ (1 - \alpha)[(1 - \alpha)^2 - \beta].$$

Here the extraordinary mode resonance ($A \rightarrow 0$) occurs at $\beta = \Omega_e^2/\omega^2 = 1$ independent of n_x, and the extraordinary mode cutoff ($C \rightarrow 0$) occurs at

$$(1 - \alpha)^2 - \beta - (1 - \alpha - \beta)n_x^2 = 0 \; . \tag{4}$$

The specific application we have in mind is to the surface region of the EBT-I device. Here the density is low, and the resonance and cutoff zones are relatively close together (Fig. 1). Aside from the mechanism of tunnelling through the cutoff, the extraordinary mode microwaves injected near the mirror midplane cannot directly penetrate into the cyclotron resonance. The density and temperature of EBT-I are such that ordinary mode waves are only very weakly absorbed. Simple calculations for propagation along the magnetic field indicate absorption and transmission efficiencies in the range of 1-15% for parameters appropriate to the EBT-I surface plasma. The crucial question is how the absorption and transmission fraction vary as the wave propagation departs from parallel to $\underset{\sim}{B}$ (i.e., $n_x \neq 0$). If the transmission fraction is $\gtrsim 0.1$ for a significant spectrum of n_x, then tunnelling can play an important role in the ECH of EBT. In the final section of this paper, it is shown that for moderate values of n_x (e.g., $0 \lesssim n_x \lesssim 0.5$) the tunnelling and absorption efficiencies are nearly independent of n_x.

In section II we derive, from Maxwell's equations and the cold plasma dispersion tensor in this geometry, scaled equations for the electric field components E_x, E_y [Eq. (9)]. For propagation exactly along the magnetic field ($n_x = 0$) these equations can be combined to give simple, independent second-order equations for ordinary and extraordinary eigenmodes. However, if n_x is nonzero, the eigenmodes do not separate and one must deal with a fourth-order system. In section II we identify eigenvectors of the homogeneous plasma corresponding to forward and backward propagating waves of both ordinary and extraordinary mode and derive a set of four first-order coupled equations describing their behavior in an inhomogeneous plasma [Eq. (29)]. As might be expected, the actual coupling between ordinary and extraordinary modes is weak, even at the extraordinary mode resonance and cutoff, provided that the magnetic field scale length is small in comparison to the free space wavelength and that the n_x is not too large. We combine the coupled mode equations to show this coupling explicitly; then neglecting the ordinary mode we obtain a second-order equation for the extraordinary mode alone. Interestingly enough, the equation obtained is exactly what would have resulted by simply plugging the refractive index obtained from Eq. (3) with $n_x \neq 0$ into Eq. (10) which is valid for $n_x = 0$. The mode vector, however, is much more complicated than $E_+ = E_x + iE_y$ which one finds for $n_x = 0$.

In section IV we consider the problem of a linear magnetic field profile. The equation then reduces to the standard Budden tunnelling problem with modifications due to n_x of the effective parallel wave number k_0 and effective cutoff thickness x_0. It is found that the tunnelling and absorption coefficients are not significantly modified if n_x is small enough that coupling to the ordinary mode is unimportant. The analytic procedure is verified by comparing to a numerical integration of the field equations. Section V contains a brief summary. Details of the derivation of the mode coupling matrix are contained in an appendix.

II. DERIVATION OF THE FIELD EQUATIONS

For a cold magnetized plasma with the magnetic field oriented along the z axis, the wave equation takes the form

$$\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times \underset{\sim}{E} - \frac{\omega^2}{c^2} \underset{\sim}{E} = \frac{4\pi i\omega}{c^2} \underset{\approx}{\sigma} \cdot \underset{\sim}{E} , \tag{5}$$

where the conductivity tensor $\underset{\approx}{\sigma}$ is given by

$$\underset{\approx}{\sigma} = \frac{i\omega}{4\pi} \begin{bmatrix} \varepsilon_1 & i\varepsilon_2 & 0 \\ -i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} , \tag{6}$$

and

$$\varepsilon_1 = \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} , \quad \varepsilon_2 = \frac{\Omega_e}{\omega} \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} , \quad \varepsilon_3 = \frac{\omega_{pe}^2}{\omega^2} .$$

The plasma parameters are assumed to vary only with z so the electric field can be represented in the form $\underset{\sim}{E}(\underset{\sim}{x}) = \underset{\sim}{E}(z)e^{i(k_x x + k_y y)}$. Without loss of generality we choose $k_y = 0$. Equation (5) then becomes

$$- \frac{\partial^2 E_x}{\partial z^2} + ik_x \frac{\partial E_x}{\partial z} + \frac{\omega^2}{c^2} [(\varepsilon_1 - 1)E_x + i\varepsilon_2 E_y] = 0 \tag{7a}$$

$$- \frac{\partial^2 E_y}{\partial z^2} + k_x^2 E_y + \frac{\omega^2}{c^2} [-i\varepsilon_2 E_x + (\varepsilon_1 - 1)E_y] = 0 \tag{7b}$$

$$ik_x \frac{\partial E_x}{\partial z} + k_x^2 E_z + \frac{\omega^2}{c^2} (\varepsilon_3 - 1) E_z = 0 . \tag{7c}$$

These equations are simplified somewhat by using Eq. (7c) to eliminate E_z and noting that $\varepsilon_2 = \Omega_e/\omega\ \varepsilon_1$,

$$\left[(1 - \varepsilon_3) \frac{\partial^2}{\partial z^2} + k_0^2(1 - \varepsilon_1)(1 - \varepsilon_3 - n_x^2)\right] E_x - k_0^2\varepsilon_1(1 - \varepsilon_3 - n_x^2) \frac{i\Omega_e}{\omega} E_y$$

$$= - \frac{n_x^2 \varepsilon_3'}{1 - \varepsilon_3 - n_x^2} \frac{\partial E_x}{\partial z} , \tag{8a}$$

$$\left[\frac{\partial^2}{\partial z^2} + k_0^2(1 - \varepsilon_1) - k_x^2\right] E_y + k_0^2 \varepsilon_1\ i \frac{\Omega_e}{\omega} E_x = 0 \tag{8b}$$

where $k_0 = \omega/c$ is the vacuum wave number, $n_x^2 = k_x^2 c^2/\omega^2$ is the refractive index in the x direction, and $\varepsilon_3' = \frac{\partial \omega_{pe}^2}{\partial z} /\omega^2$. We now restrict consideration to systems in which only the magnetic field varies with z ($\varepsilon_3' = 0$). Introducing the dimensionless variable $\zeta = k_0 z$ and the notation $\alpha = \omega_{pe}^2/\omega^2$ and $\beta = \Omega_e^2/\omega^2$, Eqs. (8a) and (8b) become

$$\left[(1 - \alpha)(1 - \beta) \frac{\partial^2}{\partial \zeta^2} + (1 - \alpha - \beta)(1 - \alpha - n_x^2)\right] E_x$$

$$- i\alpha \sqrt{\beta}\ (1 - \alpha - n_x^2) E_y = 0 \tag{9a}$$

$$\left[(1 - \beta) \frac{\partial^2}{\partial \zeta^2} + (1 - \alpha - \beta) - (1 - \beta) n_x^2\right] E_y + i\alpha \sqrt{\beta}\ E_x = 0 . \tag{9b}$$

This set of coupled equations describes the propagation of electromagnetic waves at an arbitrary angle in a cold plasma where the magnetic field is a function of z (or ζ). In the limit $n_x \to 0$, Eqs. (9a) and (9b) can be combined to give independent equations for left and right circularly polarized waves $E_\pm = E_x \pm E_y$,

$$\frac{\partial^2 E}{\partial \zeta^2} + \left(1 - \frac{\alpha}{1 \mp \sqrt{\beta}}\right) E_\pm = 0 \; . \tag{10}$$

Here the upper sign corresponds to the extraordinary mode and exhibits the resonance/cutoff pair, whereas the lower sign corresponds to the ordinary mode. Taking the upper sign and assuming a linear profile for $B_z(\zeta)$ near the resonance (i.e., $\sqrt{\beta} = 1 + \kappa\zeta$) gives a form of the Whittaker equation,

$$\frac{\partial^2 E_+}{\partial \zeta^2} + \left(1 - \frac{z_0}{\zeta}\right) E_+ = 0 \tag{11}$$

where $\zeta_0 = \alpha/\kappa$. This is the standard form of the Budden tunnelling problem. For nonzero n_x, however, the two modes do not separate unless the magnetic field is constant. The problem must therefore be treated by means of the coupled mode equations.

III. DERIVATION OF THE COUPLED MODE EQUATIONS

The field equations (9a) and (9b) can be solved in nonuniform geometry by considering the characteristic modes of the infinite uniform plasma and developing equations describing the coupling between these characteristic modes [see for example Chapter 18, BUDDEN (1961)]. To accomplish this, it is most convenient to work with a set of four first-order equations. Introducing the field vector,

$$\underset{\sim}{u}(\zeta) = (u_1, u_2, u_3, u_4) \equiv \left(E_x , \frac{\partial E_x}{\partial \zeta} , E_y , \frac{\partial E_y}{\partial \zeta}\right) ,$$

Eqs. (9a) and (9b) can be written in the form

$$\frac{\partial \underset{\sim}{u}}{\partial \zeta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c_{21} & 0 & c_{23} & 0 \\ 0 & 0 & 0 & 1 \\ c_{41} & 0 & c_{43} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \underset{\approx}{c} \cdot \underset{\sim}{u} , \tag{12}$$

where

$$c_{21} = - \frac{(1 - \alpha - \beta)(1 - \alpha - n_x^2)}{(1 - \alpha)(1 - \beta)} , \qquad c_{23} = \frac{i\alpha \sqrt{\beta}\, (1 - \alpha - n_x^2)}{(1 - \alpha)(1 - \beta)}$$

$$c_{41} = - \frac{i\alpha \sqrt{\beta}}{1 - \beta} , \qquad c_{43} = - \frac{(1 - \alpha - \beta) - (1 - \beta) n_x^2}{1 - \beta} .$$

For a uniform plasma $\frac{\partial \beta}{\partial \zeta} = 0$, we find solutions of the form $\underset{\sim}{u}(\zeta) = \underset{\sim}{u} e^{i\lambda\zeta}$ where the eigenvector $\underset{\sim}{u}$ satisfies

$$(i\lambda \underset{\approx}{I} - \underset{\approx}{c}) \cdot \underset{\sim}{u} = 0 , \tag{13}$$

and λ satisfies the secular equation

$$\Delta = \det\,(i\lambda \underset{\approx}{I} - \underset{\approx}{c}) = 0 \,. \tag{14}$$

Expansion of the determinant above yields the quartic dispersion relation for λ, given previously in Eq. (3). We now enumerate the roots of the dispersion relation as follows

$$\lambda_{\substack{1\\2}} = \pm\left(\frac{-B + \sqrt{D}}{2A}\right)^{1/2} \qquad \lambda_{\substack{3\\4}} = \pm\left(\frac{-B - \sqrt{D}}{2A}\right)^{1/2} \tag{15}$$

where $D \doteq B^2 - 4AC$ and A, B, and C are defined after Eq. (3). Roots 1 and 2 correspond to ordinary mode waves propagating in the plus and minus z direction respectively, while roots 3 and 4 correspond to extraordinary mode waves propagating in the plus and minus z direction. For each eigenvalue λ_j, there is an eigenvector $\underset{\sim}{u}^j$ given by Eq. (13). A simple choice for these eigenvectors is

$$u_1^j = i\alpha\sqrt{\beta}\,(1 - \alpha - n_x^2)\;, \qquad u_2^j = i\lambda_j u_1^j\;,$$

$$u_3^j = -(1-\alpha)(1-\beta)\lambda_j^2 + (1-\alpha-\beta)(1-\alpha-n_x^2)\;, \qquad u_4^j = i\lambda_j u_3^j\;. \tag{16}$$

To proceed, it is also necessary to solve the adjoint eigenvalue problem

$$(-i\lambda^* \underset{\approx}{I} - \underset{\approx}{c}^\dagger) \cdot \underset{\sim}{v} = 0 \tag{17}$$

where $\underset{\approx}{c}^\dagger$ is the adjoint of the matrix $\underset{\approx}{c}$ defined in Eq. (12). A convenient choice for adjoint eigenvectors $\underset{\sim}{v}^j$ is

$$v_1^j = -i\lambda_j^* v_2^j$$

$$v_2^j = (1 - \alpha)(1 - \beta)\lambda_j^{*2} - (1 - \alpha)[(1 - \alpha - \beta) - (1 - \beta)n_x^2]$$

$$v_3^j = -i\lambda_j^* v_4^j$$

$$v_4^j = i\alpha\sqrt{\beta}\,(1 - \alpha - n_x^2) \ . \tag{18}$$

A direct calculation shows that, with this choice, $\underset{\sim}{u}^j$ and $\underset{\sim}{v}^j$ satisfy an inner product relation of the form

$$\underset{\sim}{v}^{i*} \cdot \underset{\sim}{u}^j = -\alpha\sqrt{\beta}\,(1 - \alpha - n_x^2)(\lambda^i + \lambda^j)[A(\lambda^{i^2} + \lambda^{j^2}) + B] \ . \tag{19}$$

Using the dispersion relation this can be reduced to an orthogonality relation

$$\underset{\sim}{v}^{i*} \cdot \underset{\sim}{u}^j = -2\alpha\sqrt{\beta}\,(1 - \alpha - n_x^2)\lambda_i\sigma_i\sqrt{D}\,\delta_{ij} = w_i\delta_{ij} \ , \tag{20}$$

where w_i is a weight factor, δ_{ij} is the Kroeneker delta, and σ_i is the sign with which the discriminant D appears in the eigenvalue, i.e.,

$$\sigma_i = \begin{cases} +1 & \text{for} \quad i = 1,\ 2\ \text{(ordinary mode)} \\ -1 & \text{for} \quad i = 3,\ 4\ \text{(extraordinary mode)}\ . \end{cases} \tag{21}$$

We now introduce a 4×4 matrix $\underset{\approx}{S}$ whose column vectors are the eigenvectors $\underset{\sim}{u}^1$, $\underset{\sim}{u}^2$, $\underset{\sim}{u}^3$, and $\underset{\sim}{u}^4$ and a matrix $\underset{\approx}{T}$ whose row vectors are complex conjugates of the adjoint eigenvectors $\underset{\sim}{v}^1$, $\underset{\sim}{v}^2$, $\underset{\sim}{v}^3$, and $\underset{\sim}{v}^4$. The product matrix TS is diagonal

$$[TS]_{ij} = \underset{\sim}{v}^{i*} \cdot \underset{\sim}{u}^{j} = w_i \delta_{ij} \ . \tag{22}$$

Also, since the columns of $\underset{\approx}{S}$ are eigenvectors of $\underset{\approx}{c}$ we have

$$[TCS]_{ij} = i\lambda_j w_i \delta_{ij} \ . \tag{23}$$

The field vector $\underset{\sim}{u}(\zeta)$ is resolved at each point into a linear combination of the four uniform plasma eigenvectors. Let $\underset{\sim}{\hat{u}}(\zeta)$ be a vector whose components $\hat{u}_j(\zeta)$ are the local amplitudes of the <u>jth</u> eigenmode in $\underset{\sim}{u}(\zeta)$, then

$$\underset{\sim}{u}(\zeta) = \underset{\approx}{S} \cdot \underset{\sim}{\hat{u}}(\zeta) \ . \tag{24}$$

Using this in Eq. (12) gives

$$\frac{\partial \underset{\sim}{u}}{\partial \zeta} = \frac{\partial \underset{\approx}{S}}{\partial \zeta} \cdot \underset{\sim}{\hat{u}} + \underset{\approx}{S} \cdot \frac{\partial \underset{\sim}{\hat{u}}}{\partial \zeta} = \underset{\approx}{c} \cdot \underset{\sim}{u} = \underset{\approx}{c} \cdot \underset{\approx}{S} \cdot \underset{\sim}{\hat{u}} \ . \tag{25}$$

Multiplying on the left by $\underset{\approx}{T}$ and eliminating $\underset{\sim}{u}$ gives the equation for $\hat{u}(\zeta)$

$$\underset{\approx}{T} \cdot \underset{\approx}{S} \cdot \frac{\partial \underset{\sim}{\hat{u}}}{\partial \zeta} - \underset{\approx}{T} \cdot \underset{\approx}{c} \cdot \underset{\approx}{S} \cdot \underset{\approx}{\hat{u}} = -\underset{\approx}{T} \cdot \frac{\partial \underset{\approx}{S}}{\partial \zeta} \cdot \underset{\sim}{\hat{u}} \ . \tag{26}$$

The left side of Eq. (26) is diagonal and describes the evolution of the separate modes in the absence of coupling, while the right side of Eq. (26) describes the coupling between modes due to the inhomogeneity of the plasma.

The evaluation of the coupling matrix TS′ is quite tedious and is therefore outlined in the appendix. The result is

$$[TS']_{ij} = -\frac{1}{2}\alpha(1-\alpha-n_x^2)\{\sqrt{\beta}\sqrt{D}\,(\lambda^j)'(\sigma_i+\sigma_j)$$

$$+ (\lambda^i+\lambda^j)[(\sigma_i+\sigma_j)\sqrt{D}\,\kappa + \sqrt{\beta}\,\sigma_j D_0' - \alpha\beta n_x^2\kappa]\} \quad (27)$$

where prime denotes differentiation with respect to ζ, and λ^j and σ_j are given by Eqs. (15) and (21) respectively. We have also defined a dimensionless inverse scale length for magnetic field variations $\kappa(\zeta) \equiv \partial\sqrt{\beta}/\partial\zeta$ and introduced the quantity D_0 where

$$D = \sqrt{\beta}\,\alpha\{[2(1-\alpha) - n_x^2]^2 - (1-\beta)n_x^4\}^{1/2} \equiv \sqrt{\beta}\,D_0 \ . \quad (28)$$

It is clear from Eq. (27) that ordinary and extraordinary mode waves ($\sigma_i = -\sigma_j$) are coupled only by the final two terms of TS′. Furthermore, this coupling disappears completely as $n_x \to 0$ [note $D_0 \to 2\alpha(1-\alpha)$ which is independent of ζ]. Also, oppositely propagating waves of the same type ($\lambda^i = -\lambda^j$) are coupled only by the first term of Eq. (27). Using this equation in Eq. (26) gives a set of four coupled equations for the mode amplitudes $\hat{u}_j(\zeta)$,

$$\left[2\left(\frac{\partial}{\partial\zeta} - i\lambda_1\right) + \frac{\lambda_1'}{\lambda_1} + \frac{2\kappa}{\sqrt{\beta}} + \frac{q_+}{\sqrt{D}}\right]\hat{u}_1 - \frac{\lambda_1'}{\lambda_1}\,\hat{u}_2$$

$$= -\frac{q_-}{2\sqrt{D}}\left[\frac{\lambda_1+\lambda_3}{\lambda_1}\,\hat{u}_3 + \frac{\lambda_1-\lambda_3}{\lambda_1}\,\hat{u}_4\right] \quad (29a)$$

$$\left[2\left(\frac{\partial}{\partial\zeta} + i\lambda_1\right) + \frac{\lambda_1'}{\lambda_1} + 2\frac{\kappa}{\sqrt{\beta}} + \frac{q_+}{\sqrt{D}}\right]\hat{u}_2 - \frac{\lambda_1'}{\lambda_1}\hat{u}_1$$

$$= -\frac{q_-}{2\sqrt{D}}\left[\frac{\lambda_1 - \lambda_3}{\lambda_1}\hat{u}_3 - \frac{\lambda_1 + \lambda_3}{\lambda_1}\hat{u}_4\right] \quad (29b)$$

$$\left[2\left(\frac{\partial}{\partial\zeta} - i\lambda_3\right) + \frac{\lambda_3'}{\lambda_3} + 2\frac{\kappa}{\sqrt{\beta}} - \frac{q_-}{\sqrt{D}}\right]\hat{u}_3 - \frac{\lambda_3'}{\lambda_3}\hat{u}_4$$

$$= \frac{q_+}{2\sqrt{D}}\left[\frac{\lambda_1 + \lambda_3}{\lambda_3}\hat{u}_1 - \frac{\lambda_1 - \lambda_3}{\lambda_3}\hat{u}_2\right] \quad (29c)$$

$$\left[2\left(\frac{\partial}{\partial\zeta} + i\lambda_3\right) + \frac{\lambda_3'}{\lambda_3} + 2\frac{\kappa}{\sqrt{\beta}} - \frac{q_-}{\sqrt{D}}\right]\hat{u}_4 - \frac{\lambda_3'}{\lambda_3}\hat{u}_3$$

$$= \frac{q_+}{2\sqrt{D}}\left[\frac{-\lambda_1 + \lambda_3}{\lambda_3}\hat{u}_1 + \frac{\lambda_1 + \lambda_3}{\lambda_3}\hat{u}_2\right] \quad (29d)$$

where

$$q_\pm \equiv \pm D_0' - \alpha\sqrt{\beta}\, n_x^2\kappa \; .$$

These equations are quite general in that no assumption has been made concerning the plasma parameters α and β, the angle of incidence n_x, or the shape of the magnetic field profile $\sqrt{\beta\;(\zeta)}$. Neither is there any restriction on the wavelength compared to the magnetic field scale length. A variety of coupling and reflection problems could be attacked by direct solution of the coupled mode equations. However, for the present application, it is instructive to make contact with previous work on the Budden tunnelling problem based on second-order field equations. To this end we take the sum and difference of Eqs. (29a) and (29b)

$$\frac{\partial}{\partial\zeta}(\hat{u}_1+\hat{u}_2) - i\lambda_1(\hat{u}_1-\hat{u}_2) + X_+(\hat{u}_1+\hat{u}_2) = -\frac{q_-}{2\sqrt{D}}(\hat{u}_3+\hat{u}_4) \tag{30a}$$

$$\frac{\partial}{\partial\zeta}(\hat{u}_1-\hat{u}_2) + \left[\frac{\lambda_1'}{\lambda_1}+X_+\right](\hat{u}_1-\hat{u}_2) - i\lambda_1(\hat{u}_1+\hat{u}_2)$$

$$= -\frac{q_-}{2\sqrt{D}}\frac{\lambda_3}{\lambda_1}(\hat{u}_3-\hat{u}_4) \tag{30b}$$

where

$$X_+ = \frac{\kappa}{\sqrt{\beta}} + \frac{q_+}{2\sqrt{D}} .$$

Similarly, using Eqs. (29c) and (29d) gives

$$\left[\frac{\partial}{\partial\zeta}+X_-\right](\hat{u}_3+\hat{u}_4) - i\lambda_3(\hat{u}_3-\hat{u}_4) = \frac{q_+}{2\sqrt{D}}(\hat{u}_1+\hat{u}_2) \tag{31a}$$

$$\left[\frac{\partial}{\partial\zeta}+\frac{\lambda_3'}{\lambda_3}+X_-\right](\hat{u}_3-\hat{u}_4) - i\lambda_3(\hat{u}_3+\hat{u}_4) = \frac{\lambda_1}{\lambda_3}\frac{q_+}{2\sqrt{D}}(\hat{u}_1+\hat{u}_2) \tag{31b}$$

where

$$X_- = \frac{\kappa}{\sqrt{\beta}} - \frac{q_-}{2\sqrt{D}} .$$

Equation (30a) is solved for $(\hat{u}_1 - \hat{u}_2)$ and the result substituted in Eq. (30b); also, Eq. (31a) is solved for $(\hat{u}_3 - \hat{u}_4)$ which is substituted in Eq. (31b). This yields a set of second-order coupled equations involving only the sum of the amplitudes for the forward and backward propagating ordinary modes $\hat{u}_1 + \hat{u}_2$ and the sum of the amplitudes for the forward and backward propagating extraordinary modes $\hat{u}_3 + \hat{u}_4$,

$$\left[\frac{\partial^2}{\partial\zeta^2} + \lambda_1^2 + \frac{\partial X_+}{\partial\zeta} + 2X_+\frac{\partial}{\partial\zeta} + X_+^2 + \frac{q_+q_-}{4D}\right](\hat{u}_1 + \hat{u}_2)$$

$$= -\left[\frac{\partial}{\partial\zeta}\left(\frac{q_-}{2\sqrt{D}}\right) + \frac{q_-(q_+ + q_-)}{4D}\right](\hat{u}_3 + \hat{u}_4) \quad (32)$$

$$\left[\frac{\partial^2}{\partial\zeta^2} + \lambda_3^2 + \frac{\partial X_-}{\partial\zeta} + 2X_-\frac{\partial}{\partial\zeta} + X_-^2 + \frac{q_+q_-}{4D}\right](\hat{u}_3 + \hat{u}_4)$$

$$= \left[\frac{\partial}{\partial\zeta}\left(\frac{q_+}{2\sqrt{D}}\right) - \frac{q_+(q_+ + q_-)}{4D}\right](\hat{u}_1 + \hat{u}_2) \,. \quad (33)$$

These equations are still exact. However, the terms involving $q_\pm$ are second-order in the magnetic field gradient. Furthermore, these terms are not influenced by the singularity in λ_3 which occurs at the electron cyclotron resonance. Assuming that the magnetic field gradient is weak ($\kappa \ll 1$), we can neglect the coupling terms on the right and the last term on the left of Eqs. (32) and (33) unless the discriminant D vanishes. Reference to Eq. (28) shows that this does not occur unless $\alpha \cong 1$ or $n_x^2 \cong 1$ in which case the tunnelling is exponentially small anyway. The terms involving $X_\pm$ can also be eliminated by means of the transformations

$$\hat{u}_1 + \hat{u}_2 = U \exp\left[-\int^{\zeta} d\zeta' X_+(\zeta')\right] ,$$

$$\hat{u}_3 + \hat{u}_4 = V \exp\left[-\int^{\zeta} d\zeta' X_-(\zeta')\right] . \tag{34}$$

The newly defined field variables U representing ordinary mode and V representing extraordinary mode satisfy simple uncoupled differential equations of the form

$$\left(\frac{\partial^2}{\partial\zeta^2} + \lambda_1^2\right) U = 0 \tag{35}$$

$$\left(\frac{\partial^2}{\partial\zeta^2} + \lambda_3^2\right) V = 0 . \tag{36}$$

These equations are precisely what would be obtained if one simply substituted the refractive index for $n_x \neq 0$ given by Eq. (3) into Eq. (10) replacing $1 - \alpha/(1 \pm \sqrt{\beta})$. The field variables U and V are, of course, much more complicated than the $E_\pm$ in Eq. (10).

IV. SOLUTION FOR LINEAR PROFILE AND DISCUSSION

Budden tunnelling of the extraordinary mode in parallel stratified plasmas is described by Eq. (36) with λ_3^2 given by Eq. (15). In order that coupling to the ordinary mode be weak, we must restrict to $\kappa = \partial \sqrt{\beta}/\partial\zeta \ll 1$ where $\zeta = k_0 z$ and $n_x \ll 1$. To proceed, λ_3^2 is expanded, keeping only terms of order n_x^2. Expanding the discriminant gives

$$\sqrt{D} = 2\alpha \sqrt{\beta} \left(1 - \alpha - \frac{n_x^2}{2}\right) + O(n_x^4) \ , \tag{37}$$

so that

$$\lambda_3^2 = 1 - n_x^2 - \frac{\alpha}{1 - \sqrt{\beta}} \left[1 - \frac{\sqrt{\beta}\ n_x^2}{2(1 - \alpha)}\right] + O(n_x^4) \ . \tag{38}$$

We now assume a linear profile for the magnetic field and measure ζ from the cyclotron resonance,

$$\Omega_e(\zeta)/\omega = \sqrt{\beta} = 1 + \kappa\zeta = 1 + \frac{z}{L} \ , \tag{39}$$

where κ is now constant. Equation (36) then reduces to the form

$$\left[\frac{d^2}{d\zeta^2} - K_0^2 \left(1 + \frac{X_0}{\zeta}\right)\right] V = 0 \ , \tag{40}$$

where

$$K_0^2 = 1 - \frac{(2 - \alpha)n_x^2}{2(1 - \alpha)}$$

$$X_0 = \frac{\alpha/\kappa}{1 - \dfrac{(1 - \alpha)n_x^2}{2(1 - \alpha) - n_x^2}} .$$

Equation (40) is a Whittaker equation in the standard form of the Budden tunnelling problem. The Stokes parameters for this equation are well-known, and the usual analysis gives reflection coefficient $|R|^2$, transmission coefficient $|T|^2$, and absorption coefficient $|A|^2$ of the form [BUDDEN (1961), WHITE and CHEN (1974), and ABRAMOWITZ and STEGUN (1964)]

$$|R|^2 = \left(1 - e^{-\pi K_0 X_0}\right)^2 \tag{41}$$

$$|T|^2 = e^{-\pi K_0 X_0} \tag{42}$$

$$|A|^2 = 1 - |R|^2 - |T|^2 = e^{-\pi K_0 X_0}\left(1 - e^{-\pi K_0 X_0}\right) . \tag{43}$$

Insight into the behavior of these coefficients can be obtained by expanding K_0X_0 for small density ($\alpha \ll 1$) and small n_x^2,

$$K_0X_0 \cong \frac{\alpha}{\kappa}\left[1 - \frac{\alpha n_x^2}{4(1 - \alpha)} + 0(n_x^4)\right] , \tag{44}$$

where we have assumed

$$n_x^2 \frac{(2 - \alpha)}{2(1 - \alpha)} \ll 1 \ .$$

The first term of Eq. (44) is the correct limiting value for $n_x \to 0$. For low density and $n_x < 1$, the second term is indeed a small correction. Within the range of validity of the expansion, the transmission and absorption coefficients actually increase ($K_0 X_0$ decreases) as n_x^2 departs from zero. Although finite n_x^2 increases the separation between resonance and cutoff, X_0, it also tends to increase the effective parallel wavelength $(K_0/2\pi)^{-1}$. The total effect is to increase transmission and absorption for small values of n_x^2.

For the application to EBT-I, the wave frequency is 18 GHz; the density in the outer region of the plasma is $n_e \sim 1 \times 10^{11}/\text{cm}^3$ to $2 \times 10^{11}/\text{cm}^3$, $\alpha = 0.025$ to 0.05, and the magnetic field scale length is typically $L \cong 6.5$ cm $[\kappa = (k_0 L)^{-1} \sim 0.04]$. With these parameters the $n_x = 0$ transmission and absorption coefficients are in the range $|T|^2 \sim$ 0.15 to 0.02 and $|A|^2 \sim 0.125$ to 0.02. The correction due to nonzero n_x is quite small since

$$\exp\left[\frac{\alpha}{n} \frac{\alpha n_x^2}{4(1 - \alpha)}\right] \cong e^{0.01\, n_x^2} \cong 1 \text{ for } n_x^2 \leqslant 1 \ .$$

We conclude therefore that fractional energy loss from Budden tunnelling and absorption of the order 20% to 30% should occur for a wide spectrum of incident wave angles.

As n_x increases, the assumptions under which the ordinary and extraordinary mode equations were separated must eventually break down.

To estimate this we could go back to the coupled mode equations Eqs. (32) and (33) and treat the coupling as a perturbation. Instead we have chosen to verify the entire analysis by solving the field Eq. (12) numerically. Using a linearly increasing magnetic field profile, $\underset{\sim}{u}(\zeta)$ is initialized to be an outgoing extraordinary mode wave for $\zeta = \zeta^0 \gg 0$, i.e., $\underset{\sim}{u}(\zeta^0) = \underset{\sim}{u}^3$ as defined by Eqs. (15) and (16). Equation (12) is then numerically integrated backward through the resonance-cutoff to a point $\zeta = \zeta^F \ll 0$. To avoid the mathematical singularity at $\zeta = 0$, a small collision frequency ($\nu/\omega \sim 4 \times 10^{-6}$) is included in the conductivity tensor Eq. (6). After the integration, $\underset{\sim}{u}(\zeta^F)$ is resolved into a linear combination of the eigenmodes $\underset{\sim}{u}^j(\zeta^F)$. In general, for $n_x \neq 0$ the final wave on the left $\underset{\sim}{u}^j(\zeta^F)$ is found to contain a nonzero component of incoming ordinary mode $\underset{\sim}{u}'(\zeta^F)$ due to the coupling. Since the problem we seek to solve is that of purely extraordinary incoming wave, we add a small component of outgoing ordinary mode on the right $\underset{\sim}{u}'(\zeta^0)$ and adjust its amplitude and phase iteratively until the component of ordinary mode incoming from the left is $<10^{-5}$.

Figure 2a shows the reflection coefficient $|R|^2$ calculated analytically using Eq. (41) and numerically (circles) as well as the fraction of extraordinary mode energy converted to ordinary mode (dots) as a function of n_x. For this calculation the parameters were $\alpha = 0.04$ and $\kappa = 0.0398$ (i.e., $L = 8\pi/k_0 = 4$ free space wavelengths). It can be seen that the agreement is almost exact for $n_x \lesssim 0.3$. There is virtually no disagreement until significant coupling to the ordinary mode occurs at $n_x \gtrsim 0.5$. Examination of the solutions reveals that the ordinary mode energy is generated in the vicinity of the resonance-cutoff and that it is roughly equally divided between the left and right going waves. Figure 2b shows analytic and numerical (circles) values of the transmission $|T|^2$ versus n_x. The absorption coefficients $|A|^2$, both analytic and numerical, are nearly equal to corresponding values of $|T|^2$. Again the agreement is excellent until ordinary mode coupling becomes important. The transmission coefficient does increase slightly with n_x as suggested by Eq. (44) although this is not evident from the figure.

V. SUMMARY

We have investigated the tunnelling of obliquely propagating extraordinary mode energy through the right-hand cutoff to the electron cyclotron resonance in plasmas whose parameters vary along the magnetic field. Starting with Maxwell's equations in a cold magnetized plasma, a set of four coupled mode equations [Eq. (29)] were derived which describe the propagation of ordinary and extraordinary mode waves in an inhomogeneous plasma. These equations are quite general; no assumptions are made concerning the plasma density, angle of incidence, magnetic field, or shape of magnetic field profile. It was shown that for sufficiently weak magnetic field gradients and a small deviation from parallel propagation, the ordinary and extraordinary mode equations can be decoupled [Eqs. (35) and (36)] even at the resonance-cutoff. Under these restrictions we have found simple analytic expressions for the Budden tunnelling transmission, absorption, and reflection coefficients. By numerical integration of the original field equations, the results were verified for $n_x \lesssim 0.5$, and the breakdown of the approximations and coupling to ordinary mode were demonstrated for larger n_x (Fig. 2). An application was made to the surface plasma in the ELMO Bumpy Torus device.

APPENDIX. DERIVATION OF THE COUPLING MATRIX TS′

The coupling of the four eigenmodes in Eq. (26) is defined by $\underset{\approx}{T} \cdot \partial\underset{\approx}{S}/\partial\zeta$ where $\underset{\approx}{T}$ and $\underset{\approx}{S}$ are defined above by Eq. (22). The i,j<u>th</u> component of TS′ is given by

$$[TS']_{ij} = \underset{\sim}{v}^{i*} \cdot \frac{\partial \underset{\sim}{u}^j}{\partial\zeta} , \tag{A1}$$

where $\underset{\sim}{u}^i$ and $\underset{\sim}{v}^i$ are defined in Eqs. (16) and (18). Differentiating Eq. (16) gives

$$\begin{aligned}
(u_1^j)' &= i\alpha(1 - \alpha - n_x^2)\kappa \\
(u_2^j)' &= i\lambda_j(u_1^j)' + i\lambda_j' u_1^j \\
(u_3^j)' &= 2\sqrt{\beta}\,\kappa[(1 - \alpha)\lambda_j^2 - (1 - \alpha - n_x^2)] - 2(1 - \alpha)(1 - \beta)\lambda_j\lambda_j' \\
(u_4^j)' &= i\lambda_j(u_3^j)' + i\lambda_j' u_3^j ,
\end{aligned} \tag{A2}$$

where prime denotes $\partial/\partial\zeta$, and we have defined $\kappa = \partial\sqrt{\beta}/\partial\zeta$. Using these expressions in Eq. (A1) gives

$$[TS']_{ij} = i(\lambda_j)'(v_2^{i*} u_1^j + v_4^{i*} u_3^j) + i(\lambda_i' + \lambda_j)[v_2^{i*}(u_1^j)' + v_4^{i*}(u_3^j)'] . \tag{A3}$$

Using Eqs. (16) and (18), the first term in brackets can be written as

$$v_2^{i*}u_1^j + v_4^{i*}u_3^j = i\alpha\,\sqrt{\beta}(1 - \alpha - n_x^2)\Big\{(1 - \alpha)(1 - \beta)(\lambda_i^2 + \lambda_j^2)$$

$$- 2(1 - \alpha)(1 - \alpha - \beta) + \Big[(1 - \alpha - \beta) + (1 - \alpha)(1 - \beta)n_x^2\Big]n_x^2\Big\}$$

$$= i\alpha\,\sqrt{\beta}(1 - \alpha - n_x)\Big[A(\lambda_i^2 + \lambda_j^2) + B\Big] = \frac{i}{2}\,\alpha\,\sqrt{\beta}(1 - \alpha - n_x^2)\,\sqrt{D}\,(\sigma_i + \sigma_j) \tag{A4}$$

where A and B are given in Eq. (3), and the last form is obtained by using the expressions for λ_j in Eq. (15). Using Eqs. (18) and (A2) the second square bracket in Eq. (A3) can be written as

$$v_2^{i*}(u_1^j)' + v_4^{i*}(u_3^j)' = i\alpha(1 - \alpha - n_x^2)\Big\{\Big[(1 - \alpha)(1 - \beta)\lambda_i^2$$

$$- (1 - \alpha)(1 - \alpha - \beta) + (1 - \alpha)(1 - \beta)n_x^2\Big]\kappa + 2\,\sqrt{\beta}(1 - \alpha)(1 - \beta)\lambda_j(\lambda_j)'$$

$$- \Big[2\beta(1 - \alpha)\lambda_j^2 - 2\beta(1 - \alpha - n_x^2)\Big]\kappa\Big\} = i\alpha(1 - \alpha - n_x^2)\Big[- \frac{1}{2}\,\alpha\beta n_x^2\kappa$$

$$+ \frac{\sigma_i}{2}\sqrt{D}\,\kappa + \sqrt{\beta}\,\frac{\sigma_i}{2}\,(\sqrt{D})'\Big] \tag{A5}$$

where the last form is obtained using the dispersion relation and the expression for $\lambda_j\lambda_j'$

$$2\lambda_j(\lambda_j)' = \frac{\partial\lambda_j^2}{\partial\zeta} = \frac{\partial}{\partial\zeta}\left(\frac{-B + \sigma_j\,\sqrt{D}}{2A}\right) = \frac{1}{2(1 - \alpha)(1 - \beta)}\Big\{- \sqrt{\beta}\,\kappa\Big[4(1 - \alpha)$$

$$- 2(2 - \alpha)n_x^2\Big] + 4(1 - \alpha)\,\sqrt{\beta}\,\kappa\lambda_j^2 + \sigma_j(\sqrt{D})'\Big\}\ .$$

Using Eqs. (A4) and (A5) and defining D_0 by $\sqrt{D} \equiv \sqrt{\beta}\,D_0$ gives Eq. (27).

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FIGURE CAPTIONS

FIG. 1. Cross section in the equatorial plane of an EBT sector.

FIG. 2. (a) Fraction of incident extraordinary mode power reflected or converted to ordinary mode; (b) transmitted to the high field region. Parameters are $\alpha = 0.04$, $\kappa = 8\pi/k_0$.

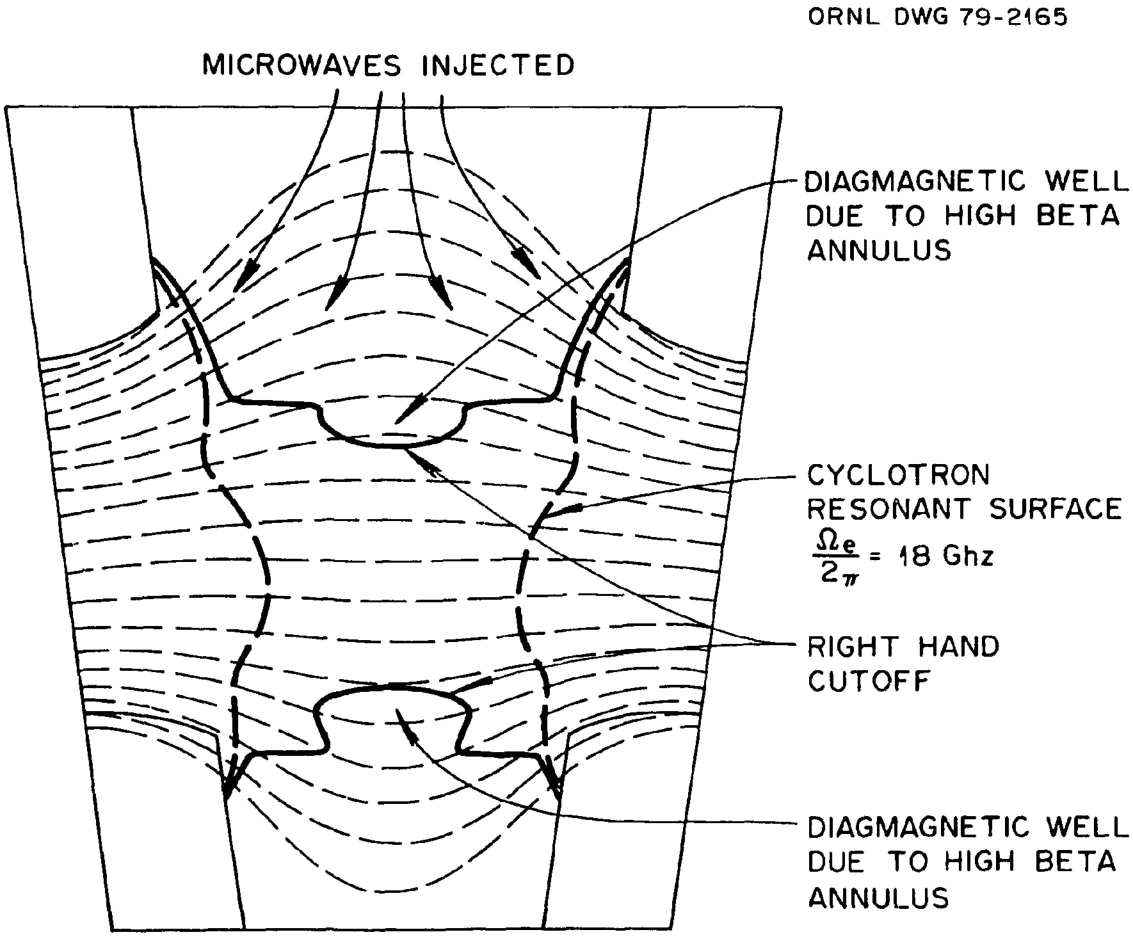
ORNL DWG 79-2165
MICROWAVES INJECTED
DIAGMAGNETIC WELL DUE TO HIGH BETA ANNULUS
CYCLOTRON RESONANT SURFACE
$\frac{\Omega_e}{2\pi}$ = 18 Ghz
RIGHT HAND CUTOFF
DIAGMAGNETIC WELL DUE TO HIGH BETA ANNULUS

Fig. 1.

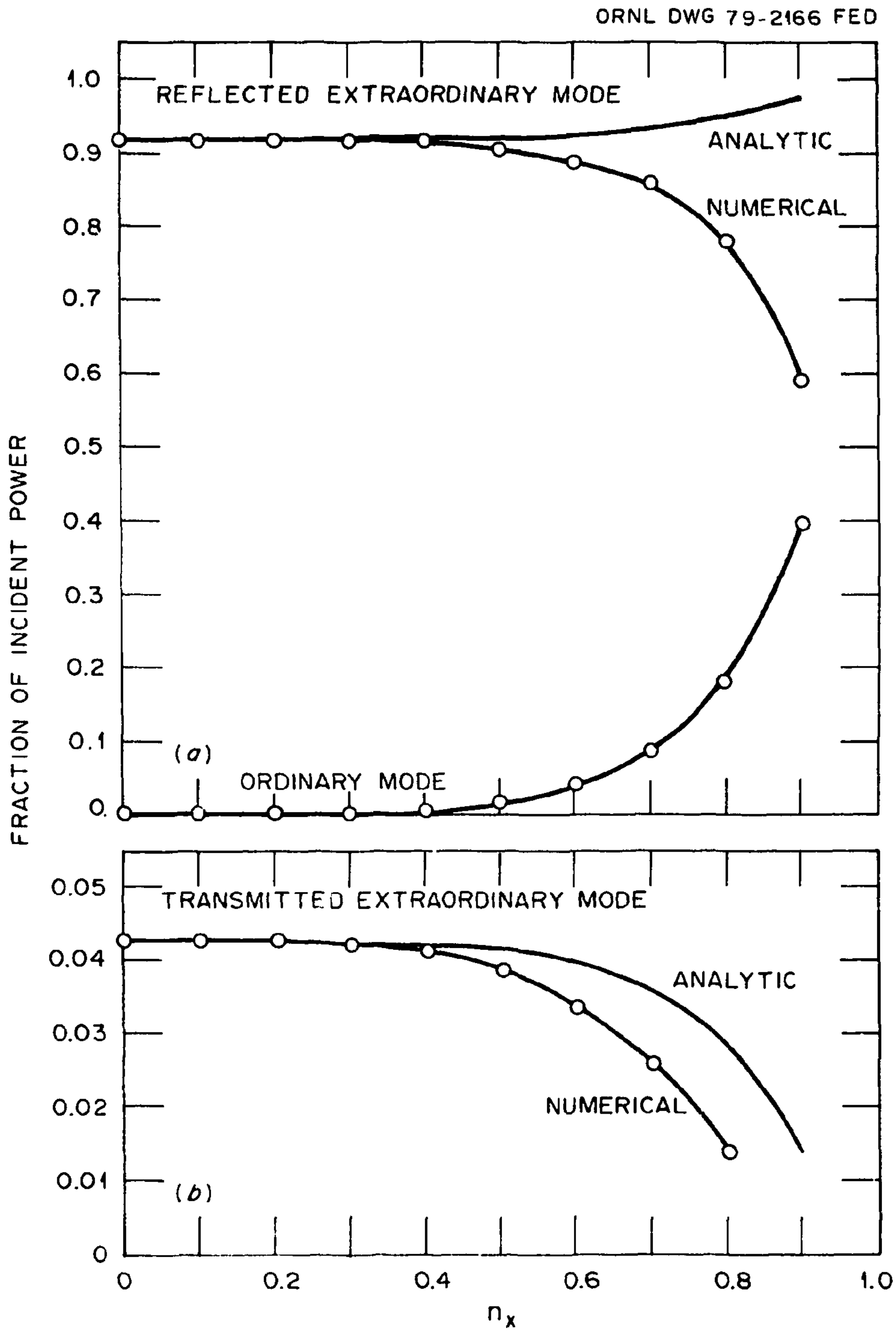
ORNL DWG 79-2166 FED
REFLECTED EXTRAORDINARY MODE
ANALYTIC
NUMERICAL
(a)
ORDINARY MODE
TRANSMITTED EXTRAORDINARY MODE
ANALYTIC
NUMERICAL
(b)
FRACTION OF INCIDENT POWER
n_x
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.
0.05
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1.0

Fig. 2.

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