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STOCHASTIC ACCELERATION BY HYDROMAGNETIC TURBULENCE

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ABSTRACT

A general theory for particle acceleration by weak hydromagnetic turbulence with a given spectrum of waves is described. Various limiting cases, corresponding to Fermi acceleration and magnetic pumping, are discussed and two numerical examples illustrating them are given. An attempt is made to show that the expression for the rate of Fermi acceleration is valid for finite amplitudes.

INTRODUCTION

In astrophysics one finds that whenever one detects turbulent motion one finds evidence for energetic particles. It is as though some engine exists for turning violence into turbulence and then employing some of the turbulent energy to create a population of energetic particles. This mechanism becomes more plausible when one appreciates that almost all astrophysical plasmas are embedded in magnetic fields. The turbulent plasma motions then transport the magnetic field lines back and forth producing random electric fields. These electric fields can then change particles energies accelerating them and decelerating them but in the course of these random processes increasing the energy of at least some to a large extent.

As an example: Consider the interstellar medium. It is filled with magnetic field lines and we detect fast motion on the largest scale-cloud motions. These motions give large scale fast moving magnetic fields, which can accelerate either the cosmic rays already present or any new particles injected into the interstellar medium. An actual description of accelera in this framework was given by Fermi in 1949. He pointed out that particles would be reflected off the moving magnetic clouds gaining or losing energy on each encounter. He demonstrated that there is a systematic effect, each particle tending to gain more energy than it loses because it encounters more head-on collisions than collision with receding clouds. Thus, to second order in the cloud velocities u the particles systematically gain energy.

A number of other mechanisms have been proposed for converting turbulent energy to energetic particle energy. One is a method akin to the betatron acceleration process, proposed by Swan[2] for the sun in 1933, and subsequently developed by Falthammer[3] in 1963 in relation to the process of magnetic pumping. The idea is that a particle in a rising magnetic field increases its energy perpendicular to the field in such a way as to keep all the flux enclosed by its orbit conserved. The trouble is that when the field decreases again, as it usually does in a turbulent situation, its energy decreases again and one is back where one started.

This difficulty is inherent in all methods of acceleration by large scale turbulence. The underlying difficulty lies with the adiabatic invariant

$$m = \frac{p_\perp^2}{B} \tag{1}$$

which remains constant for any particle seeing changing magnetic fields which only change slowly on the scale of its basic cyclotron motion. Thus, as long as this invariant is preserved, no matter what the fields do the "perpendicular energy" will not change systematically, since B doesn't. For example, Fermi's mechanism tends to increase p_{11} but $p_\perp$ does not change, so that gradually the pitch angle, $\Theta = \cos^{-1}(p_z/p)$, becomes too small for the particle to be reflected off magnetic clouds. Thus, stochastic acceleration does not exist without some additional mechanism which operates on the scale of the gyration radius of the <u>energetic particles</u> and changes the pitch angle Θ. Fermi, of course, appreciated the point and invoked thin shocks as a mechanism for scattering the particles.

Fortunately, there is a simple and natural mechanism to produce such scattering.[4] Whenever any momentum space anisotropies are present they lead to a build up of Alfvén waves on the scale of the cyclotron radius. The build up mechanism is an inverse Cherenkov effect acting on the cyclotron resonance between the energetic particles and the doppler shifted frequency in these Alfvén waves. This resonance occurs between the energetic particles and waves on <u>their</u> cyclotron radius scale, thus changing m. Since the waves are hardly moving, the primary effect of this resonance interaction is a random scattering in pitch angle with very small change in energy. This instability will develop in a natural way in any acceleration situation because, first, the loss of particles out of the turbulent region is a streaming which builds up the waves and, second, the acceleration mechanisms themselves act on the distribution function of the particles in an anisotropic way leading to build up of the small scale waves.

It is to be emphasized that, from this point on, in discussing acceleration by hydromagnetic turbulence we deal with turbulence on two different scales: <u>one</u> the original <u>given</u> large scale turbulence and <u>two</u> the small scale turbulence (on the gyration radius scale) which must arise and provides us with the necessary amount of pitch angle scattering to complete the acceleration processes.

THE EFFICIENCY OF ACCELERATION

Since in general we are dealing with many different energetic particles, each being chaotically accelerated in a random way, it is best to employ a statistical description of them. It turns out that because each particle changes its momentum p by a large number of small steps it is possible to write down a Fokker Planck diffusion equation for the time evolution of the particles in momentum space f. Let $f(\underline{p})d^3p$ be the number of particles in the momentum box d^3p about $\underline{p}$. We may take f as nearly isotropic because of the remarks above. Then neglecting space dependence we may write

$$\frac{\partial f}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p}\left(p^2 D \frac{\partial f}{\partial p}\right) \qquad (2)$$

where D is the diffusion coefficient $(\Delta p)^2/t$, and must be determined in terms of the properties of the turbulence.

Ferrari and the author[5] have obtained a fairly complete solution for D in the case where the hydromagnetic turbulence is of small amplitude $\underline{\delta B} \ll \underline{B}_0$. We employ a double expansion first in the guiding center limit according to the technique of Chew, Goldberger and Low, and then a quasilinear expansion. The result is as follows.

We start with the full equation

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f + e\left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c}\right)\cdot\nabla_p f \approx \frac{\partial}{\partial \mu}\left(\nu \frac{1-\mu^2}{2} \frac{\partial f}{\partial \mu}\right) \qquad (3)$$

where $\underline{E}$ and $\underline{B}$ are the electromagnetic fields of the turbulence, $\mu = \cos\theta = p_z/p$, ν is the scattering rate due to turbulence on the microscale and it is assumed that in the lab frame this scattering conserves energy. (Acceleration due to this process has been considered by Skilling and will be discussed later.)

The electromagnetic fields are assumed to consist of a uniform magnetic field $\underline{B}_o$, a plasma filling a certain region, and superimposed on this uniform field and plasma a large number of waves gathered in randomly disposed wave packets. Such a situation is best described by the random phase approximation. It turns out we are only interested in variations in the magnetic field strength B.

Writing

$$B = |\underline{B}| = B_o + B_1 \tag{4}$$

and Fourier analyzing B_1

$$B_1 = \int d\underline{k}\, d\omega\, e^{i(\underline{k}\cdot\underline{r} - \omega t)}\, B_1(\underline{k},\omega) \tag{5}$$

we find that $B_1(\underline{k},\omega)$ has an exceedingly wild dependence on $\underline{k}$ and ω since it depends on the position of the wave packets. Ensemble averaging over the positions of these wave packets we have

$$\left\langle B_1^*(\underline{k}',\omega')\, B_1(\underline{k},\omega)\right\rangle = B_o^2\, I(\underline{k},\omega)\cdot\delta(k'-k)\delta(\omega'-\omega) \tag{6}$$

where I $(\underline{k},\omega)$ describes the distribution of relative magnetic energy in $\underline{k}$, and ω. For fixed $\underline{k}$ it will be generally peaked in ω near the frequencies of the natural wave modes ω_k with width comparable to the lifetime of the corresponding wave packets. The integral of $I(\underline{k},\omega)$ over ω, $I(\underline{k})$, will have a smooth dependence on $\underline{k}$ and by the Wiener theorem can be considered to be the square of the transform of a typical wave packet. Thus, if each wave packet is many wave lengths, N, long the function $I(\underline{k})$ will be peaked about the principal wave number k_o in the wave packet and of width $\Delta k \sim k_o/N$. The normalization of I is such that

$$\left\langle B_1^2 \right\rangle = B_o^2 \int d\underline{k}\, d\omega\, I\, (\underline{k},\omega) \quad . \tag{7}$$

Each wave packet will interact with the particle by some acceleration mechanism, Fermi, etc., and the particle will random walk in energy ε or the magnitude of momentum p as a result of these frequent encounters. The sum of all these interactions leads to an expression for D

$$D(p) = p^2 \int d\underline{k}\, d\omega\, \gamma\, (k_z,\omega,\mu)\, I\, (\underline{k},\omega) \quad . \tag{8}$$

Here γ represents a certain rate of diffusion in p per unit relative amplitude of B for each wave packet, characterized by k_z, ω, and v the velocity of the particle of momentum p. It can be found from the double expansion-guiding center and quasilinear—mentioned above, and its evaluation is thus reduced to evaluation of the integral

$$\gamma = -\frac{\omega^2}{2} \, \mathrm{Re} \int_{-1}^{1} d\mu \, \frac{1-\mu^2}{2} z(\mu) \tag{9}$$

where $z(\mu)$ satisfies the differential equation in μ

$$\frac{\partial}{\partial\mu} \left(\frac{1-\mu^2}{2} \nu \frac{\partial z}{\partial \mu} \right) + i\left(\omega - k_z v\mu\right) z = \frac{1-\mu^2}{2} \quad . \tag{10}$$

ν, the microscopic scattering rate, will in general depend upon μ, but for simplicity let us consider it to be a constant. In this case γ can be evaluated in various limiting cases. Let $\ell \equiv v/\nu$ denote the mean free path.

Then in the strongly collisional limit we have

$$\gamma = \frac{1}{15} \frac{\omega^2 \nu}{\omega^2 + 9\nu^2} + \frac{1}{3} \frac{\omega^2 \omega_D}{\omega_D^2 + 9\omega^2} \, , \quad k_z \ell << 1 \tag{11}$$

$$\omega_D \equiv k_z^2 \; v^2/\nu \quad , \tag{11a}$$

This result is valid for waves with wave length much longer than the mean free path. The first term is the familiar result valid for $k_z = 0$, ω finite.

It arises as follows: The field compresses, $p_\perp$ increases, and then a fraction of the perpendicular energy is removed by collisions (at least if $\nu \ll \omega$). Then on decompression of the field a smaller amount of $p_\perp$ energy is removed and there is a net gain. A similar argument applies if $\omega \ll \nu$ and a better argument gives the formula for ω/ν any magnitude.

The second term applies to a different process of avoiding full decompression, which we denote as inhomogeneous magnetic pumping. Here, energy is removed from $p_\perp$ by diffusion to a part of the field line which is not expanding.

It turns out that the second term always predominates if v is larger than the phase velocity of the wave ω/k_z. This is always the case for really energetic particles, so in calculating the amount of magnetic pumping care must be taken to employ the second more important term. Otherwise the actual efficiency of magnetic pumping will be seriously underestimated.

Next let us consider the collisionless limit $k\ell \gg 1$. Here there are two cases $v > \omega/k$ or $v < \omega/k$. For the first case we have

$$\gamma = \frac{\pi}{8} \frac{\omega^2}{|k_z| \mu} \left(1 - \frac{\omega^2}{k_z^2 \mu^2}\right)^2 , \quad \frac{\omega}{k} < v, \quad k\ell > 1 \quad . \tag{12}$$

This is the low amplitude limit for Fermi acceleration as shown by Sturrock and Hall.[6] Since the waves propagate at slower speed (in cases of interest much slower speed) than that of the particle, there exists a certain pitch angle at which the particle travels with the same speed as the wave along B_o. For this pitch angle it suffers a resonance interaction with the wave exchanging energy with the wave. Particles travelling at other pitch angles are occasionally scattered into this resonant region of pitch angle space also getting accelerated or decelerated. γ thus represents the mean efficiency for acceleration. It is of interest that γ is independent of ν to lowest order. The reason for this will be discussed later when we discuss nonlinear effects but Sturrock and Hall[6] derived the identical result from a collisionless theory. (There are essential differences between the collisional and collisionless theory when higher order non-linear effects are included.)

We can see the regions of applicability of our results from the log $k\ell$ - log ω/ν diagram of Figure 1.

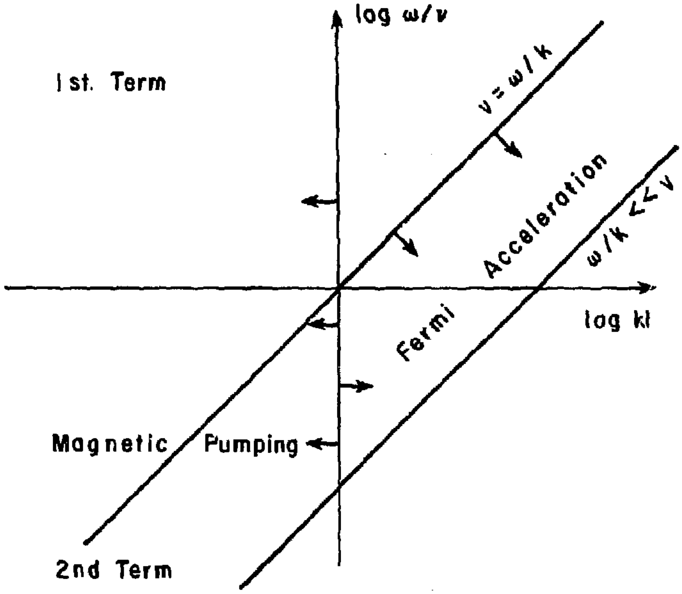

Figure 1

The right hand lower side is the region of collisionless theory where Fermi acceleration is applicable. The left hand side is the region of magnetic pumping. For the upper left hand side the first term for homogeneous magnetic pumping predominates. On the left hand lower side the second term, inhomogeneous magnetic pumping, applies. As one increases k assuming $v_\phi \equiv \omega/k$ is a constant one proceeds along a line with 45° slope directed toward the upper right hand corner of the diagram. One can show that if $\omega/k < v$ and fixed, γ increases monotonically with k.

For very small k (very long wave lengths) the diffusion time ω_D is small compared to ω and one has

$$\gamma \approx \frac{\omega_D}{27} = \frac{k^2 \ell^2}{27} \nu , \qquad k\ell < \frac{v_\phi}{v} < 1 . \tag{13}$$

At intermediate values $\omega_D > \omega$ and

$$\gamma \approx \frac{\omega^2}{3\omega_D} = \frac{v_\phi{}^2}{3v^2} \nu , \qquad \frac{v_\phi}{v} < k\ell < 1 . \tag{14}$$

At large values of k we have

$$\gamma \approx \frac{\Pi}{8} \frac{\omega^2}{kv} = \frac{\Pi}{8} k\ell \frac{v_\phi{}^2}{v^2} \nu , \qquad 1 < k\ell . \tag{15}$$

Thus, Fermi acceleration is the most efficient type of acceleration for the same amplitude. However, it applies only to shorter wave length waves where in general the energy density I_k is smaller. For a Kolmogoroff spectrum

$$kI_k \sim k^{-2/3}, \tag{16}$$

so the effective acceleration of each region goes as $k^{4/3}$, $k^{-2/3}$, $k^{1/3}$. Thus for such a spectrum a peak occurs in the effective acceleration rate when $\omega_D \sim \omega$ and then another near the cutoff point for the inertial range of the Kolomogoroff spectrum. Hence it is not possible to say in general that Fermi acceleration dominates magnetic damping but my impression is that it usually does.

Now consider the energy diffusion by the microscopic turbulence on the scale of the gyration radius of the energetic particles. Skilling[7] has shown that if the Alfvén waves propagate with equal energy back and forth then

$$\gamma = \nu \frac{v_\phi{}^2}{v^2} . \tag{17}$$

It is unlikely that such waves would be set up by the cosmic rays themselves as this would imply energy flow to the waves. It is clear that for these waves $k\ell >> 1$, so it is appropriate to compare this with Fermi acceleration the ratio being $k\ell$ in favor of Fermi. Thus, one can probably disregard acceleration in this regime unless for some reason the wave amplitude is very large.

TWO NUMERICAL EXAMPLES

To put these results into perspective let us consider two numerical examples, with a rather hypothetical choice of parameters:

First, let us examine acceleration of cosmic rays $v \approx c$, by hydromagnetic turbulence in the interstellar medium (ISM). As a rough value for the pitch angle scattering let us choose a mean free path of 1 pc., a mean intercloud density of .2H atoms per cm^3 and a magnetic field B of 3×10^{-6} gauss. All the waves will propagate at the Alfvén speed

$$v_A = v_\phi = 1.5 \times 10^6 \text{ cm/s} \tag{18}$$

so v_ϕ is very small compared to c. Expressing k in units of pc^{-1}, k_{pc} we find

$$\omega = 5 \times 10^{-13}\, k_{pc} \quad s^{-1} \tag{19}$$

$$\nu = 10^{-8} \quad s^{-1} \tag{20}$$

$$\omega_D = (k\ell)^2\, \nu = k_{pc}{}^2\, 10^{-8} \quad s^{-1} \tag{21}$$

Thus, for $k_{pc} \lesssim 1$, $\omega_D > \omega$, $\nu > \omega$
and,

$$\gamma = \frac{\omega^2}{135\nu} + \frac{1}{3}\frac{\omega^2}{\omega_D{}^2}$$

$$= 1.8 \times 10^{-19}\, k_{pc}{}^2 + 8 \times 10^{-18} \quad s^{-1} \tag{22}$$

so the second term is clearly predominant. Thus

$$\gamma \approx \frac{2.4}{T_H} \qquad (23)$$

where T_H is the Hubble time. Since γ must be multiplied by $kI_k << 1$, acceleration by magnetic pumping is clearly negligible.

For Fermi acceleration we have

$$\gamma = \frac{\Pi}{8} \frac{\omega^2}{kv} = 6 \times 10^{-18}\ k_{pc}\ \ s^{-1} \quad . \qquad (24)$$

However, when $k\ell >> 1$ the charged part of the ISM separates from the neutral part and only this part enters into the Alfvén speed. Assuming the ISM to be 10% ionized, $v_A^{\ 2}$ is about ten times larger and we have

$$\gamma = 6 \times 10^{-17}\ k_{pc}\ \ s^{-1} \quad . \qquad (25)$$

For $k \approx 10^{-16}\ cm^{-1}$, say, $k_{pc} = 300$ and

$$\gamma = 2.8 \times 10^{-14}\ s^{-1} = \frac{1}{10^6\ yr} \quad . \qquad (26)$$

Thus, if kI_k were of order unity in this range then Fermi acceleration could sustain the cosmic rays and provide a good origin theory. However $kI_k \sim k^{\cdot 2/3}$ so $kI_k < .02$ if $kI_k \sim 1$ at $k = (1pc)^{-1}$. Thus, it is unlikely that Fermi acceleration in the ISM produces cosmic rays.

SOLAR FLARES

Let us consider acceleration of MeV protons in the solar chromosphere produced by turbulence in a solar flare.

We take rather arbitrarily, $\ell = 10^9$ cm

$$B = 10g$$
$$n = 10^{10}\ \ cm^{-3}$$
$$\mu_A = 2.2 \times 10^7\ cm/s \quad . \qquad (27)$$

Let us take $v = 3 \times 10^9$ cm/s, corresponding to 5 MeV proton. Again $v_A << v$. Now choose k in units of 10^{-9} cm^{-1}, k_9.

$$\omega = 2.2 \times 10^{-2}\, k_9 \quad s^{-1} \tag{28}$$

$$\omega_D = 3\, k_9^{\,2} \quad s^{-1} \tag{29}$$

$$\nu = 3 \quad s^{-1} \tag{30}$$

and if $k_9 < 1$,

$$\gamma = \frac{\omega^2}{135\nu} + \frac{1}{3}\frac{\omega^2}{\omega_D} = 1.2 \times 10^{-6} k_9^{\,2} + 1.6 \times 10^{-4} \quad s^{-1}$$

$$= \frac{1}{6200\ s} \quad . \tag{31}$$

Again only inhomogeneous magnetic pumping is important.

For Fermi acceleration $k_9 > 1$

$$\gamma = \frac{\Pi}{8}\ \frac{\omega^2}{\kappa\mu} = \frac{v_A^{\,2}}{v^2} k\mu = 1.6 \times 10^{-4}\ k_9 \quad s^{-1} \tag{32}$$

For $k = 10^{-8} cm^{-1}$, $k_9 = 10$ and

$$\gamma = \frac{1}{600\ s} \quad . \tag{33}$$

It is not unreasonable to expect $kI_k \approx .1$ at $k = 10^{-8}\ cm^{-1}$ so that we expect an acceleration time of 6000s $\approx$ 1-1/2 hrs.

NONLINEAR ESTIMATES

The results described so far are based on a quasilinear theory, whose applicability is only guaranteed when the wave amplitudes in the turbulence are sufficiently small. Let us consider the most important acceleration process, Fermi acceleration, more closely to see at what point the small amplitude theory breaks down. In order to do this let us first give a qualitative derivation for the effective frequency γ in the small amplitude limit, Eq. (12). From guiding center theory the equation for the component of the momentum parallel to p_z is:

$$\frac{dp_z}{dt} = -\frac{v_\perp p_\perp}{2 B_o} \frac{\partial}{\partial z} |B| \quad . \tag{34}$$

Also $\Delta\varepsilon = v_\phi \Delta p_z$ from a simple frame transformation argument. If t is the time for the particle to cross a wave packet, and if the perturbed force does not average to zero (some resonance effect), then

$$\Delta p_z = t \overline{\frac{dp_z}{dt}} \approx t\left(\frac{dp_z}{dt}\right)_{max} \quad . \tag{35}$$

But Δp_z and $\Delta\varepsilon$ are of random sign, and t is also roughly the time between encounters of wave packets so we have

$$\begin{aligned} \frac{(\Delta\varepsilon)^2}{t\varepsilon^2} &\approx \frac{v_\phi{}^2(\Delta p_z)^2}{v^2\, p^2 t} \approx \frac{v_\phi{}^2(\mu p)^2\ k^2 B_1{}^2 t}{(\mu p)^2} \\ &\approx k^2 v_\phi{}^2 t \frac{B_1{}^2}{B_o{}^2} = \omega^2 t \frac{B_1{}^2}{B_o{}^2} \end{aligned} \tag{36}$$

for quasiresonant particles.

If collisions are neglected, we have $t \sim 1/\omega$. Also the parallel velocity v_z must be comparable to v_ϕ in order to avoid cancellation of the force so only v_ϕ/v particles are being accelerated at one time. Thus,

$$D = \gamma \frac{B_1{}^2}{B_o{}^2} \approx \frac{\omega v_\phi}{v} \frac{B_1{}^2}{B_o{}^2} \tag{37}$$

in agreement (up to a numerical factor) with Eq. (12).

On the other hand, if collisions are included, the velocity with which a particle traverses the wave packet is spread out stochastically by small angle collisions. The spread of velocities in a time t is

$\sqrt{\nu t}\, v$

small compared to v_ϕ for collisions to be neglected. his condition is violated when

$$> \frac{v_\phi}{v} \qquad \text{or} \qquad \frac{1}{k\ell} > \left(\frac{v_\phi}{v}\right)^3 \quad . \qquad (38)$$

r numerical example for the ISM we see this implies $k\ell >$ effect all Fermi interaction in the ISM should be colli-

lisional limit the condition that a particle in crossing a s not see F_z average to zero, is clearly that it changes bstantially during the time t in which it manages to cross a hen it spends different times in regions with positive and the cancellation must be incomplete. There is a velocity v_c article with v_z less the v_c diffuses up this velocity v_c in

$$(kv_c)^{-1} \qquad (39)$$

to cross the wave packet at velocity v_c. From this and uation,

$$/(k\ell)^{1/3} \quad . \qquad (40)$$

the particles are being accelerated at any one time. Sub- xpression for t in our expression for $(\Delta\varepsilon)^2/t\varepsilon^2$ and multi- we again get our same expression for Fermi acceleration,

he above derivation we tacitly assume the particle crosses t at a rate unaffected by the wave packet itself, and merely add up the perturbed force it sees. When the wave amplitude becomes large enough this will no longer be the case and the limit of the small amplitude theory will be reached. We would expect this to happen when the wave amplitude is big enough that the particle changes its parallel momentum by a substantial amount, for example, when it is "mirrored."

We take as the breakdown in linear theory, the wave amplitude for which a particle with parallel velocity

$$v_c = v/(k\ell)^{1/3} \tag{41}$$

mirrors, or when

$$\left(\frac{B_1}{B_o}\right)^{1/2} = \frac{1}{(k\ell)^{1/3}} \quad . \tag{42}$$

For smaller amplitudes, particles in the mirror region $\mu < (B_1/B)^{1/2}$ will diffuse to velocities v_c, and μ of order of $1/(k\ell)^{1/3}$, and thus not be mirrored. For larger amplitudes particles throughout the mirroring region can only diffuse through an angle smaller than or equal to $1/(k\ell)^{1/3}$ and will thus remain in the mirroring region.

For amplitudes larger than Eq. (42) the idea of particles being near resonance or collisional resonance and passing through essentially undisturbed by the wave no longer holds. Such particles are easily mirrored being reflected back. Thus, in this limit we are closer to the idea of Fermi of particles being scattered by moving magnetic mirrors. The principal differences are: when $B_1 \ll B$ only a fraction of order $(B_1/B)^{1/2}$ of all particles are mirrored and accelerated at any one time. Also these particles have very small parallel velocities of order $(B_1/B)^{1/2}\, v$, so the rate at which they encounter magnetic mirrors is considerably reduced.

Let us attempt to estimate the rate of diffusion of energy in this limit. Consider only broad band turbulence so each wave packet is approximately one wave length long. Then as before

$$\Delta\varepsilon = v_\phi \Delta p_z = v_\phi p(B_1/B_o)^{1/2} \quad . \tag{43}$$

The time between encounters is

$$t = \frac{1}{k(B_1/B_o)^{1/2} v} \tag{44}$$

and so

$$D \approx \left(\frac{B_1}{B_o}\right)^{1/2} \frac{(\Delta\varepsilon)^2}{t\varepsilon^2} = \left(\frac{B_1}{B_o}\right)^{1/2} \frac{v_\phi{}^2 p^2}{v^2\ p^2} \left(\frac{B_1}{B_o}\right) kv \left(\frac{B_1}{B_o}\right)^{1/2}$$

$$= \frac{v_\phi{}^2}{v^2} kv \left(\frac{B_1}{B_o}\right)^2 \quad .$$

This agrees with the linear result in Eq. (12) up to a numerical factor. Thus, although the mechanism is quite different the actual formula for acceleration is nearly identical with the linear one.

Finally, when B_1/B is of order unity the factor B_1/B can be dropped and we recover Fermi's original result. Thus, although breakdown in linear theory occurs at relatively small amplitudes, the linear formulas still seem to be applicable.

CONCLUSION

I have described a theory which gives the stochastic behavior of energetic particles in a turbulent medium. The diffusion in energy depends on the turbulent spectrum I_k of the fluctuating magnitude of the magnetic field. The diffusion coefficient is proportional to I_k with a frequency factor γ expressing the efficiency of acceleration. In various limiting cases the familiar acceleration processes of magnetic pumping and Fermi acceleration emerge naturally.

The relative importance of these processes is illustrated by two numerical examples.

The theory while systematic is really only valid for sufficiently small amplitudes. However, reasons are given for believing the same results are applicable even for reasonably finite amplitudes, at least for the case of Fermi acceleration.

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