FR7803458

LYCEN/7838
Juin 1978

A new approach of the vector analysis

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The Graphical Spin Algebra has been shown to be applicable in a cartesian coordinate system without major modification. Then the (G. S. A.) allows a new and very easy approach of the usual vector analysis. Some examples of application are given.

1. INTRODUCTION

The Graphical Spin Algebra (G. S. A.) (Elbaz and Castel 1972) is now well-known as a powerful tool to handle the Racah algebra of the SU_2 group. Different extension have been given for the SU_3 group or even for all compact groups (Agrawala and Belinfante 1968, Guichon 1975, Stedman 1975, 1976). The (G. S. A.) lies on a one-to-one diagrammatic representation of the elements of the group and on some fundamental rules of transformation based on the well-known rotational invariance orthogonality and completeness relations. For the purpose of this paper only some basic aspects of the (G. S. A.) have to be known (Elbaz and Castel 1971).

Recently we have shown (Elbaz and Nahabetian 1977) that one could define a graphical representation of the vectors or vector operators in a spherical coordinate system and use the (G. S. A.) to get very interesting results with the Wigner-Eckart theorem for instance. We have then been interested in writing directly a tensor in a spherical or in a cartesian basis (Elbaz and Meyer 1978) and shown that the vector and tensor polarization observables could be obtained without any difficulty and in a systematic way. But appeared then an interesting feature. When a graphical representation of the metric tensor (Coope and Snider 1970) $E^{j}(r| s)$ was given one could use the (G. S. A.) without important alteration in a spherical coordinate system as usual or even in a cartesian system.

The choice of a proper convention to link cartesian and spherical systems was then important and it appeared that the use of the Biedenharn-Rose convention allowed and identical graphical representation of the scalar and dot products of two vector operators. Moreover since the " 3nj " coefficients are scalars, independent of the coordinate system one could use the (G. S. A.) without specifying the reference frame and defining it only at convenience.

Such a result was sufficiently important to reconsider the graphical representation of the vector operators. It appeared effectively that the usual graphical rules of the (G. S. A.) and the knowledge of two special cartesian Clebsch-Gordan coefficients gave immediately all the usual results known as the vector analysis and allowed the obtention of a lot of new relations in that field. In a didactic point of view the method allowed to get easily some well-known results tedious to establish and difficult to remember.

In the first part of this paper we shall recall the relation between cartesian and spherical coordinates and the graphical representation of the corresponding vectors (or vector operators). Then we shall introduce the cartesian Clebsch-Gordan coefficients and evaluate two simple cases. It will define the proper convention as to obtain identical representation of the scalar and dot products in cartesian or spherical coordinates. We shall then show how to use these results in the vector analysis.

2. STANDARDIZATION OF VECTOR OPERATORS

An infinitesimal rotation α around an Ou axis transforms the $\vec{a}$ unitary vector of a cartesian coordinate system into $\vec{a}'$ with

$$\vec{a}' = \vec{a} + \alpha\, \vec{u} \wedge \vec{a} \tag{2.1}$$

A vector operator $\vec{A}$ thus becomes

$$A_{a'} = \vec{A} \,.\, \vec{a}' = \vec{A} \,.\, (\vec{a} + \alpha\, \vec{u} \wedge \vec{a}) = \vec{A} + \alpha \vec{A}.(\vec{u} \wedge \vec{a}) \tag{2.2}$$

As an operator $\vec{A}$ transforms into

$$A_{a'} = D_\alpha^+ A_a D_\alpha = (1 + i\alpha J_u) A_a (1 - i\alpha J_u) \tag{2.3}$$

a comparison between (2.2) and (2.3) gives the commutator

$$[J_u, A_a] = i \vec{A} \,.\, (\vec{u} \wedge \vec{a}) \tag{2.4}$$

If the $\vec{u}$ unitary vector is chosen as $\vec{e}_3$ unitary vector along the Oz axis in a cartesian coordinate system and $\vec{a}$ along the Ox, Oy, Oz axis, one finds immediately that

$$\begin{aligned} [J_z, A_x] &= i A_y \\ [J_z, A_y] &= -i A_x \\ [J_z, A_z] &= 0 \end{aligned} \tag{2.5}$$

We note on the other hand that

$$\begin{aligned} \vec{u} &= \vec{e}_1 \vec{e}_2 \vec{e}_3 \\ \vec{a} &= \vec{e}_1 \vec{e}_2 \vec{e}_3 \\ \vec{A} &= \vec{J} \end{aligned} \tag{2.6}$$

leads to the usual commutation relations

$$\vec{J} \wedge \vec{J} = i \vec{J} \qquad (2.7)$$

Let us now consider the transformation by rotation of an irreducible tensor operator (I. T. O.) T_{kq}

$$T'_{kq} = R \, T_{kq} \, R^{+} = \sum_{p} T_{kp} \, D^{k}_{pq}(R) \qquad (2.8)$$

An infinitesimal rotation α around the Ou axis gives the Racah's definition

$$[J_u , T_{kq}] = \sum_{p} < kp \mid J_u \mid kq > T_{kp} \qquad (2.9)$$

Such an expression can then be evaluated by setting the Ou axis along the cartesian axis and we get the well-known relations

$$\begin{aligned} [J_z , T_{kq}] &= q \, T_{kq} \\ [J_{\pm} , T_{kq}] &= T_{kq \pm 1} \left[(k \pm q + 1)(k \mp q) \right]^{1/2} \end{aligned} \qquad (2.10)$$

with $J_{\pm} = J_x \pm i J_y$.

If we now compare the commutators

$$[J_z , A_z] = 0 \quad \text{and} \quad [J_z , T_{10}] = 0$$

it becomes natural to set a linear relation between them

$$T_{10} = c \, A_z \qquad (2.11)$$

Substituting this value into (2.10) gives

$$c \, [J_{\pm} , A_z] = \sqrt{2} \; T_{1 \pm 1}$$

or equivalently

$$T_{1 \pm 1} = \frac{c}{\sqrt{2}} \left\{ [J_x , A_z] \pm i [J_y , A_z] \right\} \qquad (2.12)$$

The above commutators are easily determined with (2.4)

$$\begin{aligned} [J_x , A_z] &= - i A_y \\ [J_y , A_z] &= i A_x \end{aligned} \qquad (2.13)$$

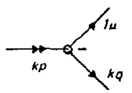
We now obtain a linear relation between the cartesian $(A_x\, A_y\, A_z)$ components of the $\vec{A}$ vector operator and the T_{1q} components of an I. T. O. . One defines the standard components $A_{1\mu}$ of the $\vec{A}$ vector operator

$$\begin{pmatrix} A_{11} \\ A_{10} \\ A_{1-1} \end{pmatrix} = \begin{pmatrix} -\frac{c}{\sqrt{2}} & -\frac{ic}{\sqrt{2}} & 0 \\ 0 & 0 & c \\ \frac{c}{\sqrt{2}} & -\frac{ic}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \qquad (2.14)$$

Note that two conventions are usually chosen to determine the c coefficients

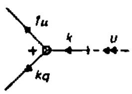
Edmonds (1967) $c = +1$
Biedenharn and Rose (1953) $c = +i$ (2.15)

We can finally note that the standardization of the kinetic momentum J allows an easy graphical expression of Racah's definition of the ITO since the Wigner-Eckart theorem applied to the $J_{1\mu}$ matrix element gives

$< kp \mid J_{1\mu} \mid kq > =$  (2.16)

where the marking circle brings the $< k \| J \| k >$ value.

We introduce (2. 16) into (2. 9) to get after summation over the magnetic quantum numbers

$[J_\mu, T_{kq}] =$  (2.17)

and since

$$< k \| J \| k > = c\sqrt{k(k+1)(2k+1)} \qquad (2.18)$$

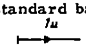
$$[J_\mu, T_{kq}] = T_{k\, q\pm\mu}\; c\sqrt{k(k+1)(2k+1)} \begin{pmatrix} q+\mu & 1 & k \\ k & \mu & q \end{pmatrix} \qquad (2.19)$$

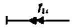
a result already mentioned by Brink and Satchler (1971).

3. DIAGRAMMATIC REPRESENTATION OF VECTOR OPERATORS

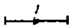
3.1 Spherical (standard) coordinates.

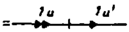
Using the (G. S. A.) diagrams (Elbaz and Nahabetian 1977), we define a spherical, or standard basis

$$|1\mu\rangle = $$ 

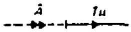
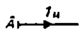
$$\langle 1\mu| = $$  (3.1)

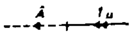
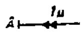
which verifies completness and orthonormalization relations

$$\sum_{\mu} |1\mu\rangle\langle 1\mu| = $$  $= 1$ (3.2)

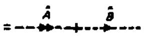
$$\langle 1\mu | 1\mu'\rangle = $$  $=$  $= \delta_{\mu\mu'}$ (3.3)

In such a basis a vector operator A will be defined by its components [18]

$$A_{1\mu} = \langle \hat{A} | 1\mu\rangle = $$  $=$  (3.4)

$$A^{+}_{1\mu} = \langle 1\mu | \hat{A}\rangle = $$  $=$  (3.5)

The scalar product is introduced through

$$\langle \hat{A} | \hat{B}\rangle = $$  $= \sum_{\mu} \langle \hat{A} | 1\mu\rangle\langle 1\mu | \hat{B}\rangle =$  (3.6)

$$= \sum_{\mu} A_{\mu} B^{+}_{\mu} = \sum_{\mu} (-)^{1-\mu} A_{\mu} B_{-\mu}$$

$$\langle \hat{A} | \hat{B}\rangle = (A_{1} B_{-1} + A_{-1} B_{1} - A_{0} B_{0}) \qquad (3.7)$$

The standardization (2.4) finally gives

$$\langle \hat{A} | \hat{B}\rangle = -c^{2}\, \vec{A}.\vec{B} \qquad (3.8)$$

As previously mentioned Edmond's convention gives

$$\langle \hat{A} | \hat{B}\rangle = -\vec{A}.\vec{B} \qquad (3.9)$$

while the Biedenharn's convention leads to

$$< \hat{A} \,|\, \hat{B} > \;=\; \vec{A} \cdot \vec{B} \tag{3.10}$$

3.2 Cartesian coordinates.

One introduces the cartesian vector basis $\vec{e}_r$ and for the sake of simplicity we set

$$|e_r> \;=\; |1\,r> \;=\; \overset{e_r}{\longmapsto} \;\equiv\; \overset{1r}{\longmapsto} \tag{3.11}$$

with $r = 1, 2, 3 = x, y, z$. They form a complete orthonormal basis

$$\begin{aligned} \sum_r |1\,r> <1\,r| &= \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} = 1 \\ <1\,r\,|\,1\,s> &= \overset{1r}{\twoheadrightarrow}\!\!\overset{1s}{\longmapsto} = \delta_{rs} \end{aligned} \tag{3.12}$$

At this stage one must point out that the above relations are particular cases of the description of tensors in cartesian coordinates.

If $A^{(j_1)}(m_1)$ defines a tensor of rank m_1 (3^{m_1} components) and of order j_1 (($2j_1 + 1$) independent components), a Cartesian Clebsch-Gordan coefficient defines the decomposition of two irreducible spaces H^{j_1} and H^{j_2} into a sum of subspace H^{j_3} (Coope and Snider 1970)

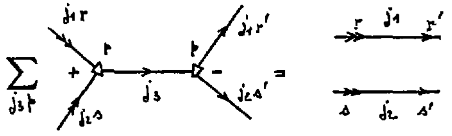
$$\left(A^{j_1}(m_1) \otimes B^{j_2}(m_2) \right)^{j_3 p}_{m_3} = \sum_{m_1 m_2} A^{j_1}(m_1)\, B^{j_2}(m_2) <j_1 m_1\, j_2 m_2 | j_3 m_3>_p \tag{3.13}$$

where $3^{m_1 + m_2} = 3^{m_3}$ is the dimension of the product space, and p is the multiplicity.

One then define the metric tensor

$$\overset{t \quad j_3 \quad t'}{\twoheadrightarrow\!\!\longrightarrow} \;=\; E^{j_3}(t\,|\,t') \tag{3.14}$$

and the more general completeness relation becomes



$$\sum_{j_3 p} \ldots = \ldots = E^{j_1}(r|r')\,E^{j_2}(s|s') \quad (3.15)$$

Since $E^{l}(r|s) = \delta_{rs}$ we find the above mentioned relation.

The cartesian components of vector operators are then

$$A_r = \langle \hat{A} \mid 1r \rangle = \langle \hat{A} \mid e_r \rangle = \hat{A} \twoheadrightarrow \vdash\!\xrightarrow{1r} = \hat{A} \vdash\!\xrightarrow{1r} \quad (3.16)$$

with $r = x, y, z = 1, 2, 3$.

The scalar product is now

$$\langle \hat{A} | \hat{B} \rangle = \sum_i \langle \hat{A} \mid 1i \rangle \langle 1i \mid \hat{B} \rangle = \hat{A} \vdash\!\overset{1}{\longrightarrow}\!\dashv \hat{B} = \sum_i A_i B_i^{+} \quad (3.17)$$

when dealing with hermitian operators $B_i^{+} = B_i$ and

$$\langle \hat{A} | \hat{B} \rangle = \vec{A} \cdot \vec{B} \quad (3.18)$$

4. CARTESIAN SPHERICAL TRANSFORMATION COEFFICIENTS

We can express a spherical component $A_{1\mu}$ of the $\vec{A}$ vector operator in a cartesian coordinate system

$$A_\mu = \langle \hat{A} | 1\mu \rangle = \sum_i \langle \hat{A} | e_i \rangle \langle e_i | 1\mu \rangle$$

$$A_\mu = \sum_i A_i \langle e_i | 1\mu \rangle \quad (4.1)$$

Graphically it follows that

$$\langle e_i | 1\mu \rangle = \overset{e_i}{\twoheadrightarrow}\!\!+\!\!\overset{1\mu}{\longrightarrow} \quad (4.2)$$

It defines the matrix element of the U-transformation matrix

$$A_\mu = U^i_\mu A_i = \sum_i \langle e_i | \mu \rangle A_i \quad (4.3)$$

$$U^i_\mu = \begin{pmatrix} \langle e_1 | 1 \rangle & \langle e_2 | 1 \rangle & \langle e_3 | 1 \rangle \\ \langle e_1 | 0 \rangle & \langle e_2 | 0 \rangle & \langle e_3 | 0 \rangle \\ \langle e_1 | -1 \rangle & \langle e_2 | -1 \rangle & \langle e_3 | -1 \rangle \end{pmatrix} = \begin{pmatrix} -\frac{c}{\sqrt{2}} & -\frac{ic}{\sqrt{2}} & 0 \\ 0 & 0 & c \\ \frac{c}{\sqrt{2}} & -\frac{ic}{\sqrt{2}} & 0 \end{pmatrix} \quad (4.4)$$

and

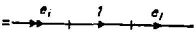
$$A_i = U^\mu_i A_\mu = \sum_\mu \langle \mu | e_i \rangle A_\mu \quad (4.5)$$

with

$$U^\mu_i = (U^i_\mu)^+ = \begin{pmatrix} \langle 1 | e_1 \rangle & \langle 0 | e_1 \rangle & \langle -1 | e_1 \rangle \\ \langle 1 | e_2 \rangle & \langle 0 | e_2 \rangle & \langle -1 | e_2 \rangle \\ \langle 1 | e_3 \rangle & \langle 0 | e_3 \rangle & \langle -1 | e_3 \rangle \end{pmatrix} = \begin{pmatrix} -\frac{c^*}{\sqrt{2}} & 0 & \frac{c^*}{\sqrt{2}} \\ \frac{ic^*}{\sqrt{2}} & 0 & \frac{ic^*}{\sqrt{2}} \\ 0 & c^* & 0 \end{pmatrix} \quad (4.6)$$

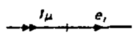
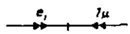
This transformation matrix is unitary and one can easily verify that

$$\sum_i \langle 1\mu' | e_i \rangle \langle e_i | 1\mu \rangle = U^{\mu'}_i U^i_\mu =$$  $$= \delta_{\mu\mu'} \quad (4.7)$$

$$\sum_\mu \langle e_i | \mu \rangle \langle \mu | e_j \rangle = U^i_\mu U^\mu_j =$$  $$= \delta_{ij} \quad (4.8)$$

An operator K^2 transforms the variance of the matrix element (Stone 1976)

$$K^2 \langle 1\mu | e_i \rangle = (-)^{1-\mu} \langle e_i | 1-\mu \rangle$$

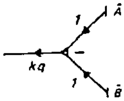
K^2  $=$  (4.9)

and one can verify that in any case $K^2 c^* = -c$ (4.10)

The Edmond's convention gives $K^2 = -1$ while the other gives $K^2 = +1$. Here again it seems more natural to give up the Edmond's convention and use $c = i$.

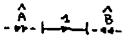
5. TENSOR PRODUCT IN SPHERICAL COORDINATES

The A_μ considered as ITO of rank 1 allows the determination of the tensorial product $(A_\mu \times B_\nu)_{kq}$ and graphically it immediately follows that

$$(A_\mu \times B_\nu)_{kq} =$$ 

(5. 1)

The tensorial product of zero rank is related to the scalar product since

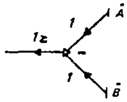
$$(A_\mu \times B_\nu)_{00} = \frac{1}{\sqrt{3}}$$ 

(5. 2)

$$= \frac{K^2}{\sqrt{3}}$$ 

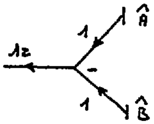
$$= \frac{K^2}{\sqrt{3}} (-c^2 \vec{A} \,.\, \vec{B})$$ (5. 3)

and the rank one to the cross-product

$$(A_\mu \times B_\nu)_{1z} = \frac{i\,c}{\sqrt{2}} (\vec{A} \wedge \vec{B})_{1z} =$$ 

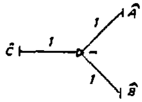
(5. 4)

or equivalently

$$(\vec{A} \wedge \vec{B})_{1z} = \frac{i\sqrt{6}}{c}$$ 

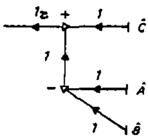
(5. 5)

One can then obtain the graphical representation of more complex products like

$$(\vec{A} \wedge \vec{B}) \,.\, \vec{C} = (\vec{B} \wedge \vec{C}) \,.\, \vec{A} = (\vec{C} \wedge \vec{A}) \,.\, \vec{B} = -i\sqrt{2}c$$ 

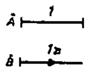
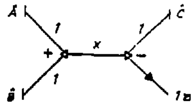
(5. 6)

and

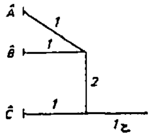
$$((\vec{A} \wedge \vec{B}) \wedge \vec{C})_{1z} = -\frac{2}{c^2}$$ 

(5. 7)

or even construct other tensors of rank one. We have for instance

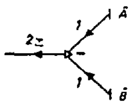
$(\vec{A}.\vec{C})\, B_{1z} = -c^2$  $= -c^2 \sum_x$  (5.8)

We then express the sum over $X = 0, 1, 2$ to get with (5.7)

$T_{1z}(\vec{A}, \vec{B}, \vec{C}) =$  $= \left[(\vec{A}.\vec{C})\, B_{1z} - \frac{1}{3}(\vec{A}.\vec{B})\, C_{1z} - \frac{1}{2}\left((\vec{A} \wedge \vec{B}) \wedge \vec{C} \right)_{1z} \right]$

(5.9)

One can also determine the $k = 2$ tensor components of the tensorial product of two vector operators

$(A_\mu \times B_\nu)_{2z} =$  (5.10)

which leads to

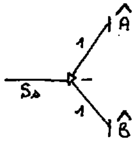
$$\begin{cases} (A_\mu \times B_\nu)_{22} = A_1 B_1 = \frac{c^2}{2}\left[(A_x B_x - A_y B_y) + i\,(A_y B_x + A_x B_y) \right] \\ (A_\mu \times B_\nu)_{21} = \frac{1}{\sqrt{2}}\,(A_0 B_1 + A_1 B_0) = -\frac{c^2}{2}\left[(A_z B_x + A_x B_z) + i\,(A_z B_y + A_y B_z) \right] \\ (A_\mu \times B_\nu)_{20} = \frac{1}{\sqrt{6}}\,(A_1 B_{-1} + A_{-1} B_1 + 2A_0 B_0) = -\frac{c^2}{\sqrt{6}}\,(\vec{A}.\vec{B} - 3A_z B_z) \\ (A_\mu \times B_\nu)_{2-1} = \frac{1}{\sqrt{2}}\,(A_0 B_{-1} + A_{-1} B_0) = \frac{c^2}{2}\left[(A_z B_x + A_x B_z) - i\,(A_z B_y + A_y B_z) \right] \\ (A_\mu \times B_\nu)_{2-2} = A_{-1} B_{-1} = \frac{c^2}{2}\left[(A_x B_x - A_y B_y) - i\,(A_y B_x + A_x B_y) \right] \end{cases}$$

(5.11)

6. SCALAR AND CROSS PRODUCTS IN CARTESIAN COORDINATES

Let us construct the tensor product of two vector operators in cartesian coordinates

$$(A_{r_1} \otimes B_{r_2})_{Ss} =$$



$$= \sum_{r_1 r_2} \langle 1 r_1 \; 1 r_2 | Ss \rangle A_{r_1} B_{r_2} \tag{6.1}$$

In order to obtain the scalar and the cross-product we have to evaluate two particular cartesian Clebsch-Gordan coefficients $\langle 1r_1 \; 1r_2 | 00 \rangle$ and $\langle 1r_1 \; 1r_2 | 1s \rangle$.

It can be easily found that

$$\langle 1r_1 \; 1r_2 | 00 \rangle = \sum_{\mu_1 \mu_2} \langle 1r_1 | 1\mu_1 \rangle \langle 1r_2 | 1\mu_2 \rangle \cdot \langle 1\mu_1 \; 1\mu_2 | 00 \rangle$$

$$= \sum_{\mu_1} \langle 1r_1 | 1\mu_1 \rangle \langle 1r_2 | 1{-\mu_1} \rangle \langle 1\mu_1 \; 1{-\mu_1} | 00 \rangle \tag{6.2}$$

$$= \sum_{\mu_1} \langle 1r_1 | 1\mu_1 \rangle \langle 1r_2 | 1{-\mu_1} \rangle \frac{1}{\sqrt{3}} (-)^{1-\mu_1}$$

We know that

$$\langle 1r_2 | 1{-\mu_1} \rangle (-)^{1-\mu_1} = K^2 \langle 1\mu_1 | 1r_2 \rangle \tag{6.3}$$

and then

$$\langle 1r_1 \; 1r_2 | 00 \rangle = \sum_{\mu_1} \langle 1r_1 | 1\mu_1 \rangle K^2 \langle 1\mu_1 | 1r_2 \rangle \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} K^2 \delta_{r_1 r_2} \tag{6.4}$$

One can obtain then

$$(A_{r_1} \otimes B_{r_2})_{00} = \frac{1}{\sqrt{3}} K^2 \vec{A} . \vec{B} \tag{6.5}$$

and a comparison with (3. 10) shows that the Biedenharn's convention leads to the same value of the tensor product of zero order

$$(A_{r_1} \otimes B_{r_2})_{00} = \frac{1}{\sqrt{3}} \vec{A} \, . \, \vec{B} \tag{6.6}$$

in any coordinate system

Let us now evaluate the cartesian Clebsch-Gordan coefficient

$$\langle 1r_1 \, 1r_2 \mid 1s \rangle = \sum_{\mu_1 \mu_2 \mu_3} \langle 1r_1 \mid 1\mu_1 \rangle \langle 1r_2 \mid 1\mu_2 \rangle \langle 1\mu_3 \mid \text{[illegible]} \rangle \, . \, \langle 1\mu_1 \, 1\mu_2 \mid 1\mu_3 \rangle \tag{6.7}$$

The use of the $U_{r_1}^{\mu_1}$ matrix elements gives without difficulty the value

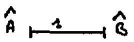
$$\langle 1r_1 \, 1r_2 \mid 1s \rangle = \frac{ic}{\sqrt{2}} \, \varepsilon_{r_1 r_2 s} \tag{6.8}$$

where $\varepsilon_{r_1 r_2 s} = 1$ if $r_1 r_2 s$ is an even permutation of 1, 2, 3 indices and $\varepsilon_{r_1 r_2 s} = -1$ for an odd permutation and zero elsewhere.

In cartesian coordinates one thus finds that

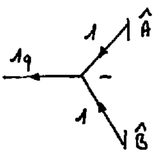
$$(A_{r_1} \times B_{r_2})_s = \frac{ic}{\sqrt{2}} \, (\vec{A} \wedge \vec{B})_s \tag{6.9}$$

It is exactly the result obtained in spherical coordinates. It thus appears that one can use the same graphical representation of the scalar and cross-product in spherical, cartesian coordinates if one choose the Biedenharn convention that is $c = i$ and in that case



$= \vec{A} \, . \, \vec{B}$

(6. 10)

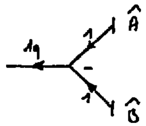


$$= \frac{1}{\sqrt{6}} \, (\vec{A} \wedge \vec{B})_q$$

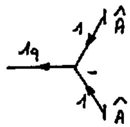
in any reference frame. When working in spherical coordinates $q = \mu = 1, 0, -1$ and in cartesian coordinates $q = s = 1, 2, 3 = x, y, z$.

7. VECTOR ANALYSIS

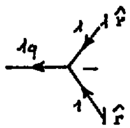
Let us first recall some obvious but useful results. The cross-product of two vector operators reads now

$$(\vec{A} \wedge \vec{B})_q = \sqrt{6}$$  (7.1)

If the components of $\vec{A}$ operator commute one can change the lecture order of the diagram without affecting the result ; one knows however that such a change multiplies the result by $(-)^{1+1+1}$. It then follows

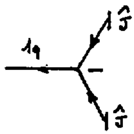
$$[\vec{A}, \vec{A}] = 0 \quad \Longleftrightarrow$$  $= 0$ (7.2)

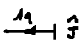
We obtain for instance



$= 0$ equivalent to $\vec{r} \wedge \vec{r} = 0$ (7.3)

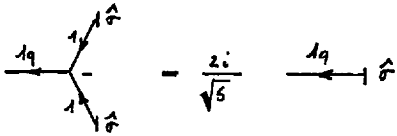
while



$= \frac{i}{\sqrt{6}}$  equivalent to $\vec{J} \wedge \vec{J} = i\,\vec{J}$

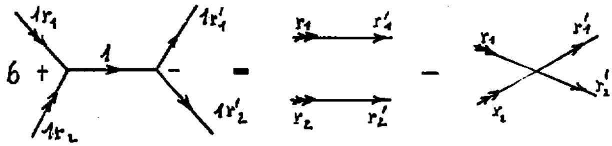
(7.4)

and with the Pauli matrices



(7.5)

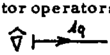
An other interesting result is obtain with the cartesian coordinates coefficients



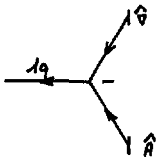
(7.6)

since

$$\sum_{s} \epsilon_{r_1 r_2 s}\, \epsilon_{r_1' r_2' s} = \delta_{r_1 r_1'}\, \delta_{r_2 r_2'} - \delta_{r_1 r_2'}\, \delta_{r_2 r_1'} \qquad (7.7)$$

Let us now examine different products obtained with vector operators. One notes that when one introduces the $\vec{\nabla}$ differential operator 

one obtains easily some well-known results

$$(\mathrm{c\vec{url}}\ \vec{A}\ ) = (\vec{\nabla} \wedge \vec{A}\ )_q = \sqrt{6}$$ 

(7.8)

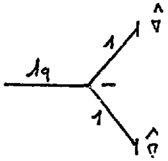
while

$\mathrm{div}\, A = \vec{\nabla} \cdot \vec{A} =$ 

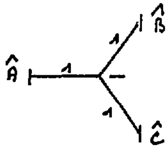
(7.9)

$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} =$ 

and with the use of (7.1)

$(\vec{\text{curl}}\ \vec{\text{grad}})_q = \sqrt{6}$  $= 0$ (7.10)

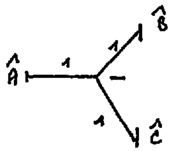
7.1 <u>The triple scalar product</u>.

$\vec{A} \cdot (\vec{B} \wedge \vec{C}) = \sqrt{6}$  (7.11)

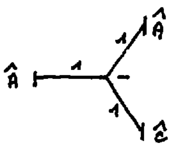
It represents the volume of the parallelepeped having $\vec{A}$, $\vec{B}$ and $\vec{C}$ as three of its edges. Due to the symmetry property of the " 3jn " coefficient one can start the lecture from any vector operator and thus obtain

$$\vec{A} \cdot (\vec{B} \wedge \vec{C}) = \vec{B} \cdot (\vec{C} \wedge \vec{A}) = \vec{C} \cdot (\vec{A} \wedge \vec{B}) \qquad (7.12)$$

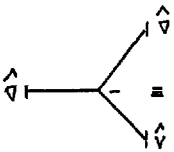
We can express the above diagram in term of the determinant

$$\det \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \sqrt{6}$$  (7.13)

If $[\vec{A}, \vec{A}] = 0$ one refinds that

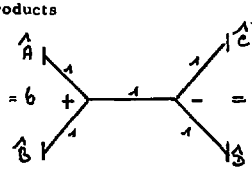
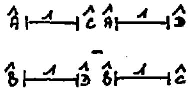
 $= 0$ (7.14)

When two of vector operators are differential operators

$$\text{div curl } \vec{V} = \vec{\nabla} \cdot \vec{\nabla} \wedge \vec{V} = \sqrt{6}$$  $$= 0 \qquad (7.15)$$

7.2 <u>Scalar products of two dot products.</u>

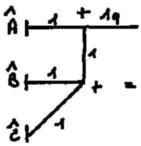
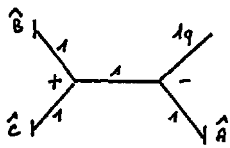
The use of (7.6) gives an interesting expression of the scalar products of two dot products

$$(\vec{A} \wedge \vec{B}) \cdot (\vec{C} \wedge \vec{D}) = 6$$  $$=$$ 

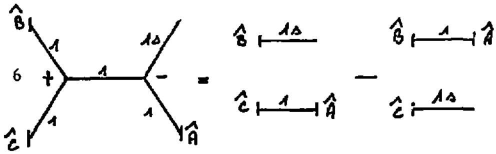
$$(\vec{A} \wedge \vec{B}) \cdot (\vec{C} \wedge \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \qquad (7.16)$$

When dealing with vector operators which do not necessarily commute, one must take care of the order of the operator in the left and right hand sides. When the above are only vectors the order is unimportant.

7.3 <u>The double cross-product</u>.

$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q = 6$$  $$= 6$$ 

One can consider that one works in cartesian coordinates and uses (7.6) to immediately obtain

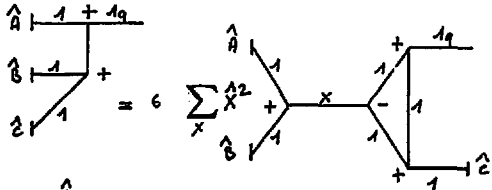
$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_s = 6$ 

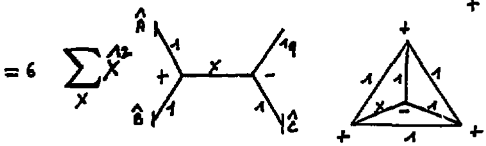
$$\vec{A} \wedge (\vec{B} \wedge \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B})\,\vec{C} \qquad (7.17)$$

If $\vec{A} = \vec{B} = \vec{\nabla}$ one immediately obtains the well-known result

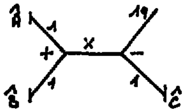
$$\text{curl curl } \vec{C} = \vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{C}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{C}) - \nabla^2 \vec{C} = \text{grad div } \vec{C} - \nabla^2 \vec{C} \qquad (7.18)$$

On can use now the graphical representation of the double cross-product and the usual rules of the (G.S.A.) to get the analytical expression of a particular tensor

$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q = 6$ 

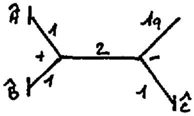


$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q = 6 \sum_X \hat{X}^2 (-)^X \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & X \end{Bmatrix} \qquad (7.19)$$

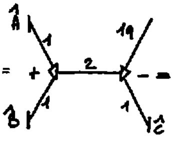
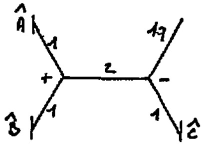


One can develop that expression since $X = 0, 1, 2$ and the corresponding " 6j " coefficient take the values $-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}$. One then obtains

$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q = -\frac{2}{3}(\vec{A}\,.\,\vec{B})\,C_q - \frac{1}{2}(\vec{C} \wedge (\vec{A} \wedge \vec{B}))_q +$$

5  (7.20)

One sets

$T_{1q}(\vec{A}, \vec{B}, \vec{C}) =$  $= 5$  (7.21)

and it follows that

$$T_{1q}(\vec{A}, \vec{B}, \vec{C}) = (\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q + \frac{2}{3}(\vec{A}\,.\,\vec{B})\,C_q + \frac{1}{2}(\vec{C} \wedge (\vec{A} \wedge \vec{B}))_q \tag{7.22}$$

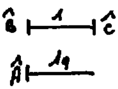
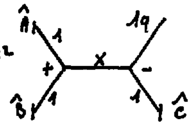
Or with (7. 17)

$$T_{1q}(\vec{A}, \vec{B}, \vec{C}) = (\vec{A}\,.\,\vec{C})\,B_q - (\vec{A}\,.\,\vec{B})\,C_q + \frac{2}{3}(A\,.\,B)\,C_q + \frac{1}{2}\left[(\vec{B}\,.\,\vec{C})\,A_q - (\vec{A}\,.\,\vec{C})\,B_q\right]$$

$$T_{1q} = \frac{1}{2}(\vec{A}\,.\,\vec{C})\,B_q - \frac{1}{3}(\vec{A}\,.\,\vec{B})\,C_q + \frac{1}{2}(\vec{B}\,.\,\vec{C})\,A_q \tag{7.23}$$

identical to (5. 9) .

One can rich the same result starting from the product $(\vec{B}\,.\,\vec{C})\,A_q$

$(\vec{B}\,.\,\vec{C})\,A_q =$  $= \sum \hat{X}^2$ 

and since X = 0, 1, 2

$$(\vec{B}\,.\,\vec{C})\,A_q = \frac{1}{3}\,(\vec{A}\,.\,\vec{B})\,C_q + \frac{1}{2}\,(\vec{C}\wedge(\vec{A}\wedge\vec{B}))_q + T_{1q}$$

or

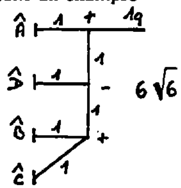
$$T_{1q} = (\vec{B}\,.\,\vec{C})\,A_q - \frac{1}{3}\,(\vec{A}\,.\,\vec{B})\,C_q - \frac{1}{2}\,(\vec{C}\wedge(\vec{A}\wedge\vec{B}))_q \qquad (7.24)$$

and if the double/cross-product is expressed with (7.17) one refinds (7.23).

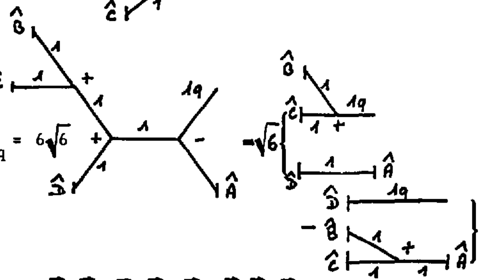
7.4 Multiple cross-product.

The use of the graphical representation of the scalar and cross-products and the rule (7.6) allow us to write in different ways a multiple cross-product. Let us take an example

$$(\vec{A}\wedge((\vec{B}\wedge\vec{C})\wedge\vec{D}))_q =$$



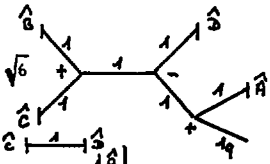
$$(\vec{A}\wedge((\vec{B}\wedge\vec{C})\wedge\vec{D}))_q = 6\sqrt{6}$$

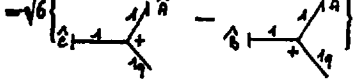


$$\vec{A}\wedge((\vec{B}\wedge\vec{C})\wedge\vec{D}) = (\vec{B}\wedge\vec{C})(\vec{A}\,.\,\vec{D}) - (\vec{A}\,.\,(\vec{B}\wedge\vec{C}))\vec{D} \qquad (7.25)$$

One can cut the diagram in an another way

$$(\vec{A}\wedge((\vec{B}\wedge\vec{C})\wedge\vec{D}))_q = 6\sqrt{6}$$

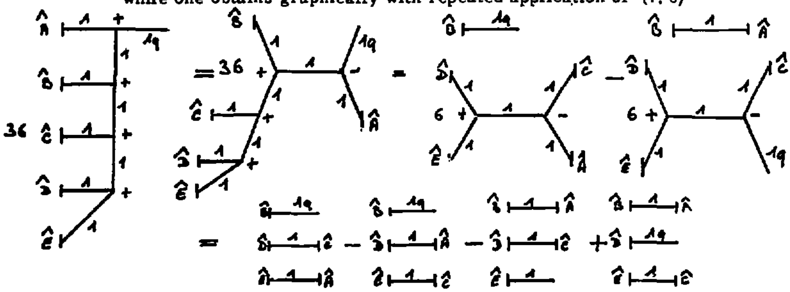




$$\vec{A}\wedge((\vec{B}\wedge\vec{C})\wedge\vec{D}) = (\vec{B}.\vec{D})(\vec{A}\wedge\vec{C}) - (\vec{C}.\vec{D})(\vec{A}\wedge\vec{B}) \qquad (7.26)$$

One sees on such a simple example the powerfull of our method. One can analytically verify the (7.25) or (7.26) equalities but it demands a high degree of intuition to discover them with other techniques. One can ask the reader to find directly the value of $\vec{A}\wedge(\vec{B}\wedge(\vec{C}\wedge(\vec{D}\wedge\vec{E})))$,

while one obtains graphically with repeated application of (7, 6)



$$\vec{A}\wedge(\vec{B}\wedge(\vec{C}\wedge(\vec{D}\wedge\vec{E}))) = (\vec{C}.\vec{D})(\vec{A}.\vec{E})\vec{B} - (\vec{A}.\vec{D})(\vec{E}.\vec{C})\vec{B}$$

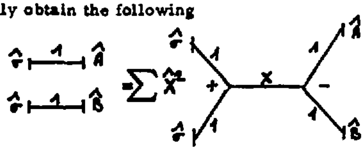
$$- (\vec{A}.\vec{B})(\vec{C}.\vec{D})\vec{E} + (\vec{A}.\vec{B})(\vec{E}.\vec{C})\vec{D} \qquad (7.27)$$

8. SOME EXAMPLES OF APPLICATION

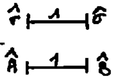
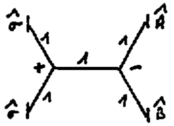
One can use (3, 17) to show that for the σ Pauli matrices considered as vector operators

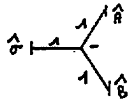
$$T_{2q}(\vec{\sigma},\vec{\sigma}) = 0 \qquad (8.1)$$

One can then easily obtain the following

$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) =$ 

We use the $X = 0, 1, 2$ and (7.5), (8.1) to get

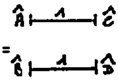
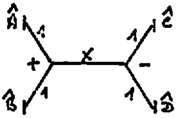
$$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \frac{1}{3}$$  $$+\ 3$$ 

$$= \frac{1}{3}\,\sigma^2\,\vec{A}.\vec{B} + 3i\sqrt{\frac{2}{3}}$$ 

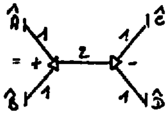
and since $\sigma^2 = 3$

$$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = (\vec{A}.\vec{B}) + i\,\vec{\sigma}.(\vec{A}\wedge\vec{B}) \tag{8.2}$$

One can obtain a more general expression when starting with the product of two scalar products of vector operators

$$(\vec{A}.\vec{C})(\vec{B}.\vec{D}) =$$  $$= \sum_X \hat{X}^2$$  (8.3)

We set

$$T_2(\vec{A},\vec{B}).\,T_2(\vec{C},\vec{D}) =$$  (8.4)

and one easily obtains

$$(\vec{A}.\vec{C})(\vec{B}.\vec{D}) = \frac{1}{3}(\vec{A}.\vec{B})(\vec{C}.\vec{D}) + \frac{1}{2}(\vec{A}\wedge\vec{B}).(\vec{C}\wedge\vec{D}) + T_2(\vec{A},\vec{B}).\,T_2(\vec{C},\vec{D})$$

or the well-known form

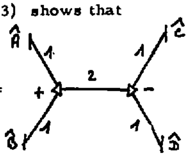
$$T_2(\vec{A},\vec{B}).\,T_2(\vec{C},\vec{D}) = (\vec{A}.\vec{C})(\vec{B}.\vec{D}) - \frac{1}{3}(\vec{A}.\vec{B})(\vec{C}.\vec{D}) - \frac{1}{2}(\vec{A}\wedge\vec{B}).(\vec{C}\wedge\vec{D}) \tag{8.5}$$

If $\vec{C} = \vec{D} = \vec{\sigma}$ one refinds (8.2). If all the vector operators are different one can express the scalar product of the two dot products with (7.6) getting

$$T_2(\vec{A},\vec{B}) \cdot T_2(\vec{C},\vec{D}) = \frac{1}{2}(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) + \frac{1}{2}(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) - \frac{1}{3}(\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{D}) \qquad (8.6)$$

A comparison with (7.21) and (7.23) shows that

$$T_2(\vec{A},\vec{B}) \cdot T_2(\vec{C},\vec{D}) = T_1(\vec{A},\vec{B},\vec{C})\cdot\vec{D} =$$



(8.7)

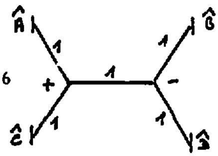
or equivalently

$$T_2(\vec{A},\vec{B}) \cdot T_2(\vec{C},\vec{D}) = T_1(\vec{C},\vec{D},\vec{B})\cdot\vec{A} = T_1(\vec{C},\vec{D},\vec{A})\cdot\vec{B} = T_1(\vec{A},\vec{B},\vec{D})\cdot\vec{C} \qquad (8.8)$$

since the above diagram can be cut by isolating any component.

One can evaluate the $T_2(\vec{A},\vec{B}) \cdot T_2(\vec{C},\vec{D})$ scalar product by starting from a mixed product

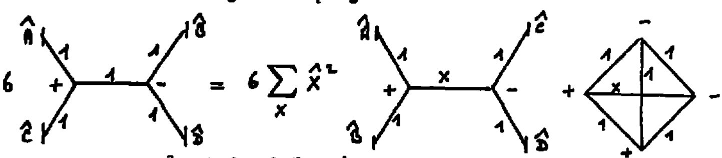
$$(\vec{A}\wedge\vec{C})\cdot(\vec{B}\wedge\vec{D}) = 6$$



$$= (\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{C}\cdot\vec{B})$$

(8.9)

One can now change the coupling scheme



$$= \frac{2}{3}(\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{D}) + \frac{1}{2}(\vec{A}\wedge\vec{B})\cdot(\vec{C}\wedge\vec{D}) - T_2(\vec{A},\vec{B})\cdot T_2(\vec{C},\vec{D})$$

(8.10)

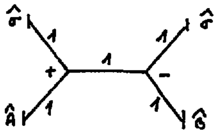
It comes finally that

$$T_2(\vec{A},\vec{B}).\,T_2(\vec{C},\vec{D}) = \frac{2}{3}(\vec{A}.\vec{B})(\vec{C}.\vec{D}) + \frac{1}{2}(\vec{A}\wedge\vec{B}).(\vec{C}\wedge\vec{D}) - (\vec{A}\wedge\vec{C}).(\vec{B}\wedge\vec{D}) \qquad (8.11)$$

This expression gives (8. 6) when one expresses the mixed products in terms of scalar products as in (8. 9) .

We note that when $\vec{C} = \vec{D} = \vec{\sigma}$ one can reach the dot product $(\vec{\sigma}\wedge\vec{A}).(\vec{\sigma}\wedge\vec{B})$ but it is easier to get it directly

$(\vec{\sigma}\wedge\vec{A}).(\vec{\sigma}\wedge\vec{B}) = 6$



$= \sigma^2\,\vec{A}.\vec{B} - (\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B})$

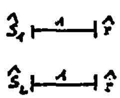
$$(\vec{\sigma}\wedge\vec{A}).(\vec{\sigma}\wedge\vec{B}) = 3\vec{A}.\vec{B} - (\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) \qquad (8.12)$$

or with (8. 2)

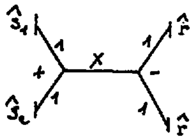
$$(\vec{\sigma}\wedge\vec{A}).(\vec{\sigma}\wedge\vec{B}) = 2\vec{A}.\vec{B} - i\,\vec{\sigma}.(\vec{A}\wedge\vec{B}) \qquad (8.13)$$

Let us finish by an example in which both cartesian and spherical aspects of the (G. S. A.) have to be used

$(\vec{S}_1.\vec{r})(\vec{S}_2.\vec{r}) =$



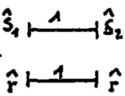
$= \sum_X \hat{X}^2$



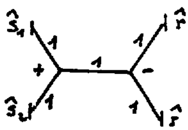
(8. 14)

and since X = 0, 1, 2

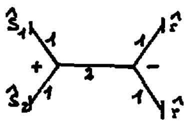
$(\vec{S}_1.\vec{r})(\vec{S}_2.\vec{r}) = \frac{1}{3}$



$+ 3$



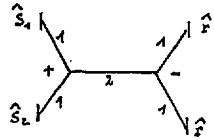
$+ 5$



(8. 15)

Since $\vec{r} \wedge \vec{r} = 0$ the second diagram vanishes and we are left with

$$(\vec{S}_1 . \vec{r})(\vec{S}_2 . \vec{r}) - \frac{1}{3}(\vec{S}_1 . \vec{S}_2)\, r^2 = 5$$

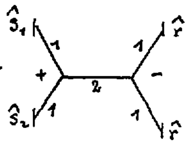


$$= T_2(\vec{S}_1 . \vec{S}_2) . T_2(\vec{r}, \vec{r})$$

(8. 16)

We divide the two sides by the length r^2 of the $\vec{r}$ vector and set

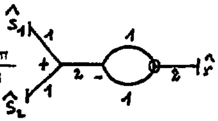
$$S_{12} = \frac{(\vec{S}_1 . \vec{r})(\vec{S}_2 . \vec{r})}{r^2} - \frac{1}{3}\vec{S}_1 . \vec{S}_2 = 5$$



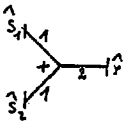
(8. 17)

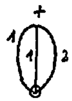
Since the only directionsof the $\vec{r}$ vector are now involved in the diagram, one can normalize it by $\sqrt{\frac{4\pi}{3}}$ in order to have $\overset{1m}{\longrightarrow}\!\hat{r} \equiv Y_{1m}(\hat{r})$ and use the usual technique of the (G. S. A.) on the two spherical harmonics thus left

$$S_{12} = 5 . \frac{4\pi}{3}$$



$$= \frac{4\sqrt{5}}{3}$$

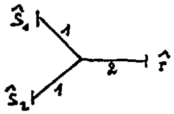




(8. 18)

where the marked triad takes the value $\frac{3\sqrt{5}}{\sqrt{4\pi}} \; \frac{\sqrt{2}}{\sqrt{3.5}}$ and we finally obtain

$$S_{12} = \frac{(\vec{S}_1 . \vec{r})(\vec{S}_2 . \vec{r})}{r^2} - \frac{1}{3}\vec{S}_1 . \vec{S}_2 = \sqrt{\frac{8\pi}{3}}$$

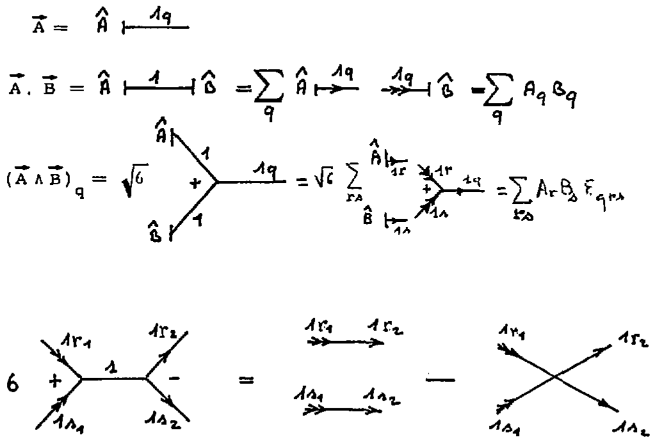


(8. 19)

where $\overset{\ell m}{\longrightarrow}\!\dashv \hat{r} \;=\; Y_{\ell m}(\hat{r})$ the usual spherical harmonic in the $\hat{r}$ direction, and $\hat{S} \;\vdash\!\overset{1\mu}{\longrightarrow} \;=\; S_{1\mu}$ is the standard form of the spin vector operator.

9. CONCLUSION

We have shown in this paper two important results. First if we use the Biedenharn-Rose convention $c = i$ for the transformation of the cartesian basis into a standard (spherical) basis, the (G.S.A.) is applicable without major modification in cartesian coordinates. Moreover, the graphical representations of the scalar and dot products and of the scalar " 3nj " coefficients in the two coordinates are identical. One can thus work without specifying a priori the coordinate system. The second important result is that the (G.S.A.) can give a new powerful approach of the vector analysis in its more usual aspect. In that case one can deal with the only few graphical representations and rules



One note that when dealing with these rules only, one can avoid the $\sqrt{6}$ numerical coefficient in the dot product, but the use of the other rules of the (G.S.A.) makes this coefficient indispensable. We have given here only few examples of the many possibilities of the (G. S. A.) and of this approach of the vector analysis.

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