FR 78035921

LYCEN/7832
Juin 1978

A Diagram Approach to Character Formulae for Finite and Compact Groups

M. KIBLER AND E. ELBAZ
Institut de Physique Nucléaire, Université Lyon-1 et IN2P3,
F-69621 Villeurbanne, France

Some basic relations for the representation theory and the Wigner-Racah algebra of a finite or compact continuous group are discussed and transcribed in terms of diagrams. Special emphasis is placed on the case of a simply reducible group and all the diagrams are applicable to SU_2 without any change.

Quelques relations essentielles pour la théorie de la représentation et l'algèbre de Wigner et Racah d'un groupe fini ou compact sont discutées et transcrites en termes de diagrammes. Une attention toute particulière est réservée au cas d'un groupe simplement réductible et tous les diagrammes donnés ici sont applicables à SU_2 sans aucun changement.

1. Introduction

The graphical techniques set up for angular momentum theory [1-7] and extended to the Wigner-Racah algebra of an arbitrary compact group [8-11] have been of invaluable help in nuclear and atomic physics. The interest for quantum chemistry of the graphical methods of spin algebras (hereafter abbreviated as GSA) developed by the senior author (E. E.) has been recently emphasized [12-14]. The graphical techniques provide simple representations for basis-dependent quantities (e. g., coupling coefficients) and non-basis-dependent quantities, i. e., invariants (e. g., recoupling coefficients).

The theory of invariants plays a fundamental role in numerous areas of mathematical physics, physics, and chemistry. Among the various branches of the theory of invariants, the character theory for a compact topological group (discrete or continuous) is probably the one that is the most familiar to the chemist.

The character theory is of paramount importance in many physical problems. From a qualitative point of view, the character theory is of invaluable help for determining selection rules in quantum chemistry as well as in molecular and solid state physics. Moreover, the splitting of a degenerate nuclear, atomic or molecular level under the action of a symmetry-breaking Hamiltonian (arising from internal or external fields) is a trivial problem once the characters for the invariance groups of the unperturbed and the symmetry-breaking Hamiltonians are known. From a quantitative point of view, the character theory is a very useful tool in chemistry and physics for calculating symmetry adapted state vectors (crystal and ligand field state vectors, molecular orbitals, normal coordinates, etc). In addition, the theory of level splitting recently discussed in an original way [15-18] might be approached on the basis of irreducible characters only.

The character theory is of essential importance in mathematical physics and more specifically in the representation theory and the Wigner-Racah algebra of groups. As a matter of fact, the irreducible characters appear as basic quantities in terms of which other invariants can be developed. In this direction, it is possible to connect recoupling coefficients and irreducible characters of a compact topological group. For instance, there exist character formulae for Wigner 6-j and 9-j symbols of finite or compact continuous groups [19-21]. These formulae, recently revived [22, 23], may be very convenient for computing recoupling coefficients [20, 24] or for obtaining asymptotic expressions of recoupling coefficients. In a similar vein, it is possible to derive formulae connecting isoscalar factors of a chain of compact topological groups and irreducible characters of the groups of the chain being considered [22, 23].

It is the aim of the present work to exhibit a diagrammatic representation of the main character formulae occuring in group theory. The material is organized in the following way. We devote Section 2 to some relations central to the representation theory and the Wigner-Racah algebra of a finite or compact continuous group. In this respect, it is not our purpose to give a summary of group theory. We limit ourselves to emphasizing the interdependence of various relations and some formal relations which are still in a state of incomplete development. In particular, the exact connection between orthogonality and convolution relations [25] is pointed out. The presentation adopted in Section 2 closely follows that of Ref. [26]. We develop in Section 3 a geometrical approach to the basic relations discussed in Section 2. The approach is achieved along the line of the GSA. Lastly, in an appendix the connection between the diagram technique for compact groups and the GSA is further investigated.

2. Basic Group Theoretical Formulae

A. Notational Preliminaries

Let G be a finite or compact continuous group with elements E (identity), R, S, T,.... The conjugation class of R is written as $\mathcal{C}_R$. We use j (or J) to denote an irreducible representations class of G, D^j the (unique) matrix representation associated with j, and [j] the dimension of D^j. In addition, we use the abbreviation $\hat{j}$ for $[j]^{1/2}$. (*Note* that [j] and $\hat{j}$ were denoted as $[j^2]$ and [j], respectively, in some previous works on SU_2 by the senior author.) The matrix $D^j(R)$, the elements of which are $D^j(R)_{mm'}$, is the representative of R in D^j. We write $\chi^j(R) \equiv \chi^j(\mathcal{C}_R)$ for the character of R in j. Furthermore, $\langle j_1 j_2 m_1 m_2 | b j m \rangle$ stands for a Clebsch-Gordan coefficient in the $\{jm\}$ scheme with b being necessary when j occurs more than once in the Kronecker product $j_1 \otimes j_2$.

In the compact case, $\int_G \dots dR$ *denotes a Haar integral over the* topological space of G, and the volume $\int_G dR$ of G is written as $|G|$. When G is finite, $\int_G \dots dR$ and $|G|$ are to be interpreted as being the sum $\sum_{R \in G} \dots$ and the order of G, respectively.

Finally, most of the symbols have their usual significance : z^* stands for the conjugate imaginary of z, $\delta(a, b)$ for the Kronecker delta of a and b, $\mathbb{1}_{[j]}$ for the $[j] \times [j]$ unit matrix, and tr X for the trace of X.

B. Orthogonality and Convolution Relations

Let us start from the so-called convolution relation

$$|G|^{-1} \int_G D^{j_2}(R^{-1}S)_{m_2' m_2} \, D^{j_1}(R)_{m_1 m_1'} \, dR$$

$$= \delta(j_2, j_1) \, \delta(m_2', m_1') \, [j_1]^{-1} \, D^{j_1}(S)_{m_1 m_2} \qquad (1)$$

Relation (1) was proved originally [25] for an arbitrary finite group. As emphasized by Löwdin [25], the extension from the finite case to the compact continuous one is straightforward. Such an extension yields (1).

Clearly the well-known orthogonality theorem

$$|G|^{-1} \int_G D^{j_2}(R^{-1})_{m_2' m_2} \, D^{j_1}(R)_{m_1 m_1'} \, dR$$

$$= \delta(j_2, j_1) \, \delta(m_2', m_1') \, \delta(m_2, m_1) \, [j_1]^{-1} \qquad (2)$$

appears as a specialization of the convolution relation (1). It is less evident (cf. Ref. [25]) that (1) may be derived from (2). This is actually the case, however, and the direct proof is left to the reader. The real reason for the equivalence of (1) and (2) is as follows [26]. Relations (2) and (1) restricted to $j_2 = j_1 \equiv j$ are obtained from the Schur equation

$$|G|^{-1} \int_G D^j(R^{-1}) \, X \, D^j(R) \, dR = 1\!\!1_{[j]} \, [j]^{-1} \, \mathrm{tr}\, X$$

by choosing $X = E_{m_2 m_1}$ and $X = D^j(S) \, E_{m_2 m_1}$, respectively, where $E_{m_2 m_1}$ stands for a canonical generator of the Lie algebra of the unitary group $U_{[j]}$ in $[j]$ dimensions. Note that another choice for X would lead to a relation apparently disctinct from (1) and (2). For instance, by taking $X = D^j(S) \, E_{m_2 m_1} \, D^j(T)$ we would obtain

$$|G|^{-1} \int_G D^{j_2}(R^{-1} S)_{m_2' m_2} \, D^{j_1}(T R)_{m_1 m_1'} \, dR$$

$$= \delta(j_2, j_1) \, \delta(m_2', m_1') \, [j_1]^{-1} \, D^{j_1}(T S)_{m_1 m_2} \qquad (3)$$

Indeed, $\{ E_{m_2 m_1} : m_2, m_1 \text{ ranging} \}$ constitutes a basis for the ring of the $[j] \times [j]$ matrices, so that the $E_{m_2 m_1}$ - choice is the more general one. Consequently, any (nontrivial) choice for X, as for example the $D^j(S) \, E_{m_2 m_1}$ - choice or the $D^j(S) \, E_{m_2 m_1} \, D^j(T)$ - choice, provides us with a relation that may be deduced from the great orthogonality theorem (2).

The reduction of $j_1 \otimes j_2$ as a direct sum of irreducible representations classes j is described in terms of representation matrix elements by

$$D^{j_1}(R)_{m_1 m_1'} \; D^{j_2}(R)_{m_2 m_2'} = \sum_{b j m m'} \langle j_1 j_2 m_1' m_2' | b j m' \rangle^* \, D^j(R)_{m m'} \langle j_1 j_2 m_1 m_2 | b j m \rangle \qquad (4)$$

As a simple consequence of (2), relation (4) may be worked out to lead to the so-called Gaunt's formula, namely,

$$|G|^{-1} \int_G D^j(R^{-1})_{m'm} \; D^{j_1}(R)_{m_1 m_1'} \; D^{j_2}(R)_{m_2 m_2'} \, dR = [j]^{-1} \sum_b \langle j_1 j_2 m_1' m_2' | b j m' \rangle^* \langle j_1 j_2 m_1 m_2 | b j m \rangle \qquad (5)$$

Relation (5) turns out to be of particular interest for the Wigner-Racah algebra of G. More precisely, Gaunt's formula lies at the root of character formulae for recoupling coefficients of the group G [19-21] and isoscalar factors of a chain starting from G [22, 23].

Passing on to the irreducible characters for G, we obtain from (3) the immediate corollary

$$|G|^{-1} \int_G \chi^{j_2}(R^{-1}S) \, \chi^{j_1}(R\,T) \, dR = \delta(j_2, j_1) \, [j_1]^{-1} \, \chi^{j_1}(S\,T) \qquad (6)$$

which specializes to

$$|G|^{-1} \int_G \chi^{j_2}(R^{-1}S) \, \chi^{j_1}(R) \, dR = \delta(j_2, j_1) \, [j_1]^{-1} \, \chi^{j_1}(S) \qquad (7)$$

The convolution (or *Dirichlet*) product $\chi^{j_1} * \chi^{j_2}$ defined through

$$(\chi^{j_1} * \chi^{j_2})(S) \equiv \int_G \chi^{j_1}(R^{-1}S) \, \chi^{j_2}(R) \, dR$$

is then easily seen to be [25]

$$\chi^{j_1} * \chi^{j_2} = \chi^{j_2} * \chi^{j_1} = \delta(j_2, j_1) \, [j_1]^{-1} \, |G| \, \chi^{j_1}$$

As a special case of (7), we have the classical character orthogonality relation

$$|G|^{-1}\int_G \chi^{j_2}(R^{-1})\,\chi^{j_1}(R)\,dR \equiv |G|^{-1}(\chi^{j_1} * \chi^{j_2})(E) = \delta(j_2, j_1) \qquad (8)$$

Alternatively, relation (8) directly follows from the great orthogonality theorem (2), a corollary usually emphasized in textbooks.

The little-known composition property of the irreducible characters

$$\chi^j(R)\ \chi^j(S) = [j]\ |G|^{-1}\int_G \chi^j(R\,T^{-1}\,S\,T)\,dT \qquad (9)$$

which parallels the fundamental property of the (irreducible) representation matrices

$$D^j(R)\ D^j(S) = D^j(R\,S)$$

may be obtained as a simple exercise in the algebra of the group G [27]. However, it should be noted that an alternative and more straightforward proof of (9) follows from the choice $X = D^j(S)$ in the Schur equation. We thus obtain

$$|G|^{-1}\int_G D^j(T^{-1}\,S\,T)\,dT = \mathbb{1}_{[j]}\ [j]^{-1}\,\chi^j(S)$$

which yields

$$|G|^{-1}\int_G D^j(R\,T^{-1}\,S\,T)\,dT = [j]^{-1}\,D^j(R)\ \chi^j(S)$$

to be compared with (9).

C. Completeness Relations

Let us now go to relations which are the duals of the above discussed orthogonality relations. In the finite case the dual of (2) appears in any textbook on group theory and may be written as

$$\sum_{j\,m\,m'} [j]\ D^j(R^{-1})_{m'm}\ D^j(S)_{m\,m'} = \delta(R, S)\ |G| \qquad (10)$$

The inverse orthogonality relation (10) *is a direct consequence of the* orthogonality relation (2) and thus does not contain any new information. The extension of (10) to the compact continuous case seems to have received very little attention. Following Sharp [20], we have

$$\sum_{j m m'} [j] \; D^j(R^{-1})_{m'm} \; D^j(S)_{m m'} = \not\delta(R^{-1}S) \; |G| \quad (11)$$

where the delta function of Dirac $\not\delta$ is defined by means of the group integration

$$\int_G F(R) \not\delta(R^{-1}S) \, dR = F(S)$$

for arbitrary F. By rewriting the completeness relations (10) and (11) in terms of irreducible characters, we (respectively) get

$$\sum_j [j] \; \chi^j(R^{-1}S) = \delta(R, S) \; |G| \quad (10')$$

and

$$\sum_j [j] \; \chi^j(R^{-1}S) = \not\delta(R^{-1}S) \; |G| \quad (11')$$

In the case of a finite group, (10') can be proved to be equivalent to

$$\sum_j \chi^j(\mathcal{C}_{R^{-1}}) \; \chi^j(\mathcal{C}_S) = \delta(\mathcal{C}_R, \mathcal{C}_S) \; |\mathcal{C}_R|^{-1} \; |G| \quad (10'')$$

where $|\mathcal{C}_R|$ stands for the order of $\mathcal{C}_R$; relation (10'') is the dual of

$$|G|^{-1} \sum_{\mathcal{C}_R} |\mathcal{C}_R| \; \chi^{j_2}(\mathcal{C}_{R^{-1}}) \; \chi^{j_1}(\mathcal{C}_R) = \delta(j_2, j_1)$$

which is an immediate consequence of (8). The transcription of (10'') to a compact topological group is far from easy. Formally, the right hand side of (10'') should be replaced by $\not\delta(\mathcal{C}_{R^{-1}} \mathcal{C}_S) \, |\mathcal{C}_R|^{-1} \, |G|$. In the case of a compact connected Lie group, the form $\not\delta(\mathcal{C}_{R^{-1}} \mathcal{C}_S) \, |\mathcal{C}_R|^{-1} \, |G|$ has received a precise mathematical meaning [20, 28]. Assuming G to be a compact connected Lie group, we have [20]

$$\sum_j \chi^j(x)^* \; \chi^j(x') = \delta(x - x') \; A_x^{-1} \; |G| \quad (11'')$$

where δ stands for a class Dirac delta function, each conjugation class being parametrized by an x-parameter with

$$|G|^{-1} \int A_x \, dx = 1$$

The relation (11'') should be equivalent to (11') and is the dual of

$$|G|^{-1} \int A_x \; \chi^{j_2}(x)^* \; \chi^{j_1}(x) \, dx = \delta(j_2, j_1)$$

D. Specific Group Integrals

Let us now turn our attention to the integrals

$$<s_n>_G = |G|^{-1} \int_G \chi^j(R^n) \, dR$$

and

$$<a_n>_G = |G|^{-1} \int_G [\chi^j(R)]^n \, dR$$

where n is a positive integer. (For G finite, $<s_n>_G$ and $<a_n>_G$ have been recently investigated [29].) The integrals $<s_n>_G$ and $<a_n>_G$ are (respectively) the averages over G of

$$s_n = \chi^j(R^n)$$

and

$$a_n = [\chi^j(R)]^n$$

The integral $<a_n>_G$ appears as a special case of

$$<a_n'>_G = |G|^{-1} \int_G \chi^{j_1}(R) \; \chi^{j_2}(R) \ldots \chi^{j_n}(R) \, dR$$

which is nothing but the frequency of the identity irreducible representation class j_o in the Kronecker product $j_1 \otimes j_2 \otimes \ldots \otimes j_n$.

The case $n=1$ is trivial since (8) gives

$$<s_1>_G \equiv <a_1>_G = \delta(j, j_o)$$

The cases $n=2$ and $n=3$ are of particular interest : they provide a criterion useful for the representation theory and the Wigner-Racah algebra of G, respectively. Indeed, in the case $n=2$ we have

$$<s_2>_G = \sum_{mm'} <jjmm'|j_o m_o>^* <jjm'm|j_o m_o>$$

$$= c^j$$

where c^j is 1, -1 or 0 according as D^j is orthogonal, sympletic or complex. Note that $<s_2>_G$ is generally different from

$$< a_2 >_G = \sum_{m m'} < j j m m' | j_o m_o >^* < j j m m' | j_o m_o >$$

$$= \delta(\bar{j}, j)$$

where $\bar{j}$ denotes the conjugate complex of j . In the case $n = 3$ we get

$$< s_3 >_G = \sum_{\substack{mm'm'' \\ bJMM'}} < j j m m' | bJM > < j j m'm'' | bJM' >^* < j J m'' M | j_o m_o > < j J m M' | j_o m_o >^*$$

to be compared with

$$< a_3 >_G = \sum_{\substack{m m'm'' \\ bJ MM'}} < j j m m' | bJM > < j j m m' | bJM' >^* < j J m'' M | j_o m_o > < j J m'' M' | j_o m_o >^*$$

The Clebsch-Gordan coefficients of the groups for which

$$\int_G \chi^j(R^3)\, dR = \int_G [\chi^j(R)]^3\, dR$$

holds for all j may be fully symmetrized in the sense discussed by Derome [30, 31] . In our modern terminology, we refer to such groups as simple phase groups. For G simply reducible, $< s_3 >_G$ and $< a_3 >_G$ reduce to

$$< s_3 >_G = < a_3 >_G = \{ j\, j\, j \}$$

where $\{ . . . \}$ stands for a generalized triad†. Consequently, a simply

† All the relations given in this paper for a simply reducible group apply to SU_2 with $[j] = 2j+1$, $\chi^j(R) \equiv \chi^j(\phi) = \sin [(2j+1)\phi/2] / \sin (\phi/2)$, $j_o = m_o = 0$, $c^j \equiv (-1)^{2j} = 1$ or -1 according as $2j+1$ is odd or even, and $\{ j_1 j_2 j_3 \} = 1$ or 0 according as j_1 , j_2, and j_3 satisfy the angular momenta addition rules or not.

reducible finite or compact continuous group is necessarily a simple phase group. The inverse statement is not true. As an illustration, all crystal point groups are simple phase groups (cf. Ref. [10]).

Going back to G finite or compact continuous, the case $n > 3$ may be handled with the following algorithm. The characters are developed as functions of representation matrix elements. The number of representation matrix elements is reduced step by step by repeated application of (4). At the final step, only one representation matrix element is left so that integration over G can be easily performed. This procedure leads to a (nonunique) expression for $< s_n >_G$ or $< a_n >_G$ in terms of Clebsch-Gordan coefficients of G. It is always possible to rewrite such an expression in terms of recoupling coefficients of G. For G simply reducible, the expression so-obtained assumes a reasonably simple form. By way of illustration, it can be shown that

$$< s_4 >_G = (-1)^{2j} \sum_{J_1} [J_1] \begin{Bmatrix} j & j & J_1 \\ j & j & J_1 \end{Bmatrix}$$

$$< s_5 >_G = (-1)^{2j} \sum_{J_1 J_2} (-1)^{J_1 + J_2} [J_1] [J_2] \begin{Bmatrix} j & j & J_2 \\ j & J_1 & J_2 \end{Bmatrix} \begin{Bmatrix} j & j & J_1 \\ j & J_2 & J_1 \end{Bmatrix}$$

where $\begin{Bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{Bmatrix}$ stands for a Wigner 6-j symbol of the simply reducible group G.

For G simply reducible, the above-mentioned examples for $< s_n >_G$ and $< a_n >_G$ satisfy : $< s_{2k+1} >_G = < a_{2k+1} >_G = 0$ when D^j is sympletic. This is actually a (trivial) general result.

For G finite, the algorithm just described could be appropriately modified, following van Zanten and de Vries [32], to obtain

$$< s_n >_{\mathcal{C}_R} = |\mathcal{C}_R|^{-1} \sum_{R \in \mathcal{C}_R} \chi^j(R^n)$$

(For G simply reducible, $< s_n >_{\mathcal{C}_R}$ has been recently investigated

[32].) At the final step, it would be sufficient to use†

$$\sum_{R\in\mathcal{C}_R} D^j(R) = \mathbb{1}_{[j]}\,[j]^{-1}\,\left|\mathcal{C}_R\right|\,\chi^j(\mathcal{C}_R) \qquad (12)$$

that comes from Schur's lemma. However, it should be realized that (12) is not really necessary at the final step ; in fact, s_n may be developed as

$$s_n = \sum_J \text{(recoupling coefficient)}\ \chi^J(\mathcal{C}_R)$$

which is compatible with

$$< s_n >_{\mathcal{C}_R} = s_n$$

As an example, we get

$$< s_2 >_{\mathcal{C}_R} = \sum_{\substack{m m'\\ bJ}} < j j m m' | b J M >^* \chi^J(\mathcal{C}_R) < j j m' m | b J M >$$

For G simply reducible, the latter relation particularizes to

$$< s_2 >_{\mathcal{C}_R} = (-1)^{2j} \sum_{J_1} (-1)^{J_1} \left\{ j\ j\ J_1 \right\} \chi^{J_1}(\mathcal{C}_R)$$

in agreement with Ref. [32]. For G simply reducible, we further obtain

$$< s_3 >_{\mathcal{C}_R} = \sum_{J_1 J_2} (-1)^{J_1} [J_1] \begin{Bmatrix} j & J_1 & j \\ j & J_1 & J_2 \end{Bmatrix} \chi^{J_2}(\mathcal{C}_R)$$

$$< s_4 >_{\mathcal{C}_R} = \sum_{J_1 J_2 J_3} (-1)^{J_1 + J_2} [J_1]\,[J_2] \begin{Bmatrix} j & j & J_1 \\ j & j & J_2 \\ J_1 & J_2 & J_3 \end{Bmatrix} \chi^{J_3}(\mathcal{C}_R)$$

---

† For G compact, the relation analogous to (12), if it exists, does not seem to be known. Formally, we should have

$$\int_{\mathcal{C}_R} D^j(R)\, d\mathcal{C}_R = \mathbb{1}_{[j]}\,[j]^{-1}\, A_x\, \chi^j(x)$$

from which we might obtain $A_x^{-1} \int_{\mathcal{C}_R} \chi^j(R^n)\, d\mathcal{C}_R$.

$$< s_5 >_{\mathcal{C}_R} = (-1)^{2j} \sum_{J_1 J_2 J_3 J_4} (-1)^{J_1 + J_2} [J_1][J_2][J_3] \left\{ \begin{matrix} J_1 & j & j & J_2 \\ j & J_2 & j & J_3 \\ j & J_1 & J_3 & J_4 \end{matrix} \middle| 2 \right\} \chi^{J_4}(\mathcal{C}_R)$$

where $\left\{ \begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \right\}$ and $\left\{ \begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \middle| 2 \right\}$ stand for a Wigner 9-j symbol and a 12-j symbol of the second kind, respectively, of the simply reducible group G. It should be noted that $< s_3 >_{\mathcal{C}_R}$ ($< s_4 >_{\mathcal{C}_R}$ and $< s_5 >_{\mathcal{C}_R}$) agree (disagree) with the results of Ref. [32].

E. Character Formulae for Recoupling Coefficients.
The Simply Reducible Case

Let us close Section 2 with some character formulae which have not attracted very much attention in the past. The recoupling coefficients are basis independent coefficients, i.e., they remain unchanged when going from the $\{jm\}$ scheme to another scheme, as for example the $\{ja\Gamma\gamma\}$ scheme (cf. Ref. [23]). It thus seems reasonable to connect recoupling coefficients and irreducible characters. Indeed, it is possible to express, modulo summation over all the internal multiplicity labels, any product Q of recoupling coefficients as a sum or integral of a product of irreducible characters, under the condition that Q involves an equal number of covariant and contravariant ordered generalized triads [19-21, 23]. The proof easily follows from repeated application of (5). Following this idea, Wigner [19] derived character formulae for 6-j symbols of a simply reducible finite or compact continuous group. These formulae, which are little-known, may be written in symmetric form as (cf. also Refs. [19, 20])

$$Q_1 = \left\{ \begin{matrix} j_1 & j_2 & j \\ j_1 & j_2 & j' \end{matrix} \right\}$$

$$= (-1)^{2j} |G|^{-2} \int_{G \otimes 2} \chi^{j_1}(RS)\, \chi^{j_2}(RS^{-1})\, \chi^{j}(R)\, \chi^{j'}(S)\, dR\, dS \qquad (13)$$

$$Q_2=\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}^2 = |G|^{-4} \int_{G\otimes 4} \chi^{j_1}(RS)\, \chi^{j_2}(RT)\, \chi^{j_3}(R^{-1}U)$$

$$\chi^{j_4}(TU)\, \chi^{j_5}(SU)\, \chi^{j_6}(S^{-1}T)\, dR\, dS\, dT\, dU \qquad (14)$$

$$Q_3=\begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_2 & j' \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j' \\ j_3 & j_2 & j'' \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j'' \\ j_3 & j_2 & j \end{Bmatrix}$$

$$= (-1)^{2j_1} |G|^{-6} \int_{G\otimes 6} \chi^{j_1}(RUT)\, \chi^{j_2}(RWTSUV)\, \chi^{j_3}(SWV)$$

$$\chi^{j}(TV^{-1})\, \chi^{j'}(RS^{-1})\, \chi^{j''}(UW^{-1})\, dR\, dS\, dT\, dU\, dV\, dW \qquad (15)$$

$$Q_4=\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \begin{Bmatrix} j_1 & j_5 & j_6 \\ j_1 & j_2 & j_3 \end{Bmatrix} \begin{Bmatrix} j_4 & j_2 & j_6 \\ j_1 & j_2 & j_3 \end{Bmatrix} \begin{Bmatrix} j_4 & j_5 & j_3 \\ j_1 & j_2 & j_3 \end{Bmatrix}$$

$$= (-1)^{2j_4} |G|^{-8} \int_{G\otimes 8} \chi^{j_1}(RS)\, \chi^{j_1}(TX^{-1}V)\, \chi^{j_2}(RVY^{-1}T^{-1}U)\, \chi^{j_3}(RXYTW)$$

$$\chi^{j_4}(UW^{-1}Y)\, \chi^{j_5}(S^{-1}XW)\, \chi^{j_6}(SU^{-1}V^{-1})\, dR\, dS\, dT\, dU\, dV\, dW\, dX\, dY \qquad (16)$$

where $\int_{G\otimes P}$ means $\int_G \int_G \cdots \int_G$ (P times) . Sharp [20] showed that the Wigner 6-j (or Racah W) symbols of a simply reducible finite or compact continuous group can be defined by the set of relations (13) , (14), (15), and (16) . The extension of (13) , (14), (15), and (16) to an arbitrary finite or compact continuous group were given by Derome and Sharp [21] . Such an extension requires taking the complex conjugate of some of the irreducible representations classes and introducing summation over internal multiplicity labels. For the simplicity of the diagrams in Section 3 we confine ourselves to the simply reducible case.

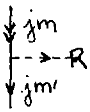
Let us go on to a geometric transcription of some of the main formulae reported in Section 2. The diagrammatic equivalent of relation (a), if any, will be quoted as (Da).

3. Diagrams for the Basic Group Theoretical Formulae

A. Diagrammatic Representation of the Character

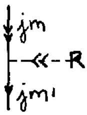
Let us start from the following diagrammatic representations :

$D^j(R)_{mm'}$ =



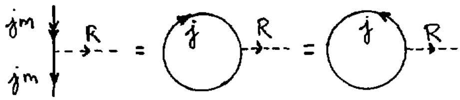
and

$D^j(R^{-1})_{mm'}$ =



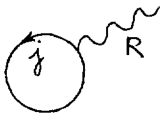
that are coherent with the GSA (cf. appendix). The usual summation rule over a (generalized) magnetic quantum number leads to

$\chi^j(R) = \sum_m D^j(R)_{mm} = \sum_m$



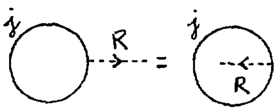
in agreement with Guichon [9]. The diagrams for $\chi^j(R)$ so-obtained compare with

$\chi^j(R)$ =



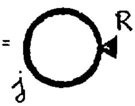
as given by Agrawala and Belinfante [8]. It should be noted that the direction of the j-arrow is here of no importance. Consequently, we can take

$\chi^j(R)$ =



which fully parallels the representation

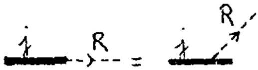
$\chi^j(R)$ =



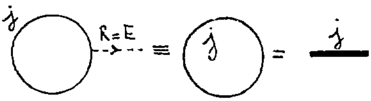
of Stedman [10]. Continuous deformation in our diagram for $\chi^j(R)$ yields

$\chi^j(R) =$ 

which, in turn, gives

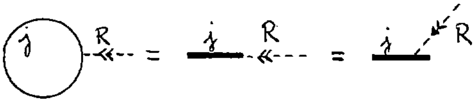
$\chi^j(R) =$ 

as a limiting case. The latter representation of $\chi^j(R)$ will sometimes prove to be useful. In the case where $R = E$ we simply omit the $R = E$ - line and thus get a closed loop, the value of which is well-known in the GSA (cf. also Ref. [10])

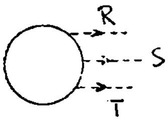
$\chi^j(E) =$ 

$= \hat{j}^2$

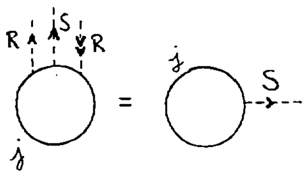
It is now straightforward to obtain

$\chi^j(R^{-1}) =$ 

and

$\chi^j(RST) =$ 

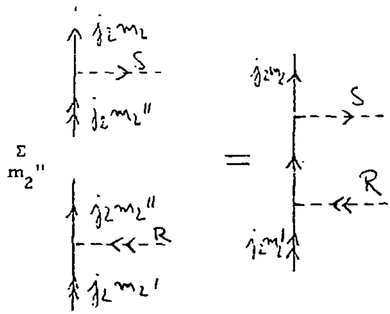
The character is a class function and this is diagrammatically depicted by

$\chi^j(RSR^{-1}) =$ 

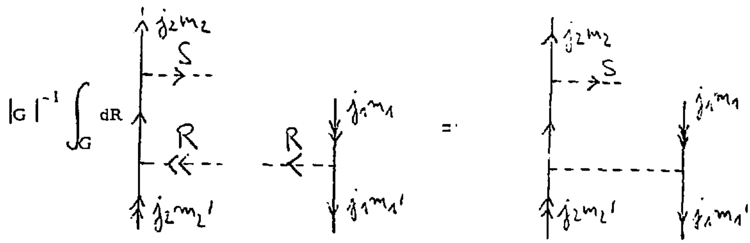
$= \chi^j(S)$

B. Diagrams for Orthogonality and Convolution Relations

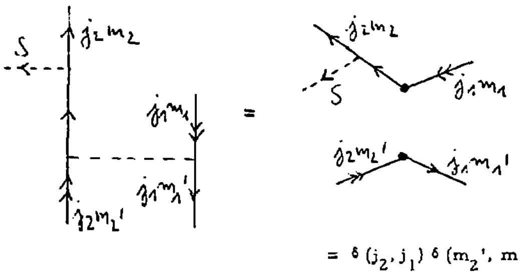
The diagrammatic equivalent of the convolution relation (1) may be obtained in the following way. First, the m-summation rule gives



Second, R-integration (described in the appendix for the $G \equiv SU_2$ case) yields



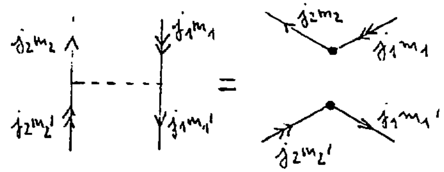
As a final result, we have



(D1)

$$= \delta(j_2, j_1)\,\delta(m_2', m_1')\,\hat{j}_1^{-2}\,D^{j_1}(S)_{m_1 m_2}$$

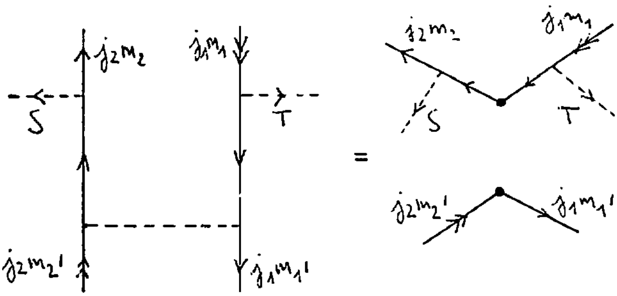
since a j-node on a j-line introduces a $\hat{j}^{-1}$ factor, a well-known fact in the GSA. In the same way, the great orthogonality theorem (2) is diagrammed as [9]



(D2)

$$= \delta(j_2, j_1)\ \delta(m_2', m_1')\ \delta(m_2, m_1)\ \hat{j}_1^{-2}$$

in agreement with Agrawala and Belinfante [8] and Stedman [10]. Clearly (D2) is a specialization of (D1). As a more general case we have

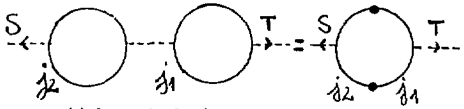


(D3)

$$= \delta(j_2, j_1)\ \delta(m_2', m_1')\ \hat{j}_1^{-2}\ D^{j_1}(TS)_{m_1 m_2}$$

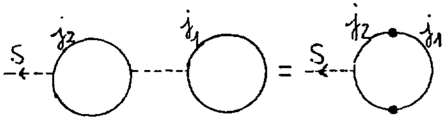
which covers both (D1) and (D2).

From (D3) it is straightforward to get



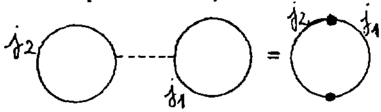
$$= \delta(j_2, j_1)\, \hat{j}_1^{\,-2}\, \chi^{j_1}(ST) \qquad \text{(D6)}$$

whichparticularizes to



$$= \delta(j_2, j_1)\, \hat{j}_1^{\,-2}\, \chi^{j_1}(S) \qquad \text{(D7)}$$

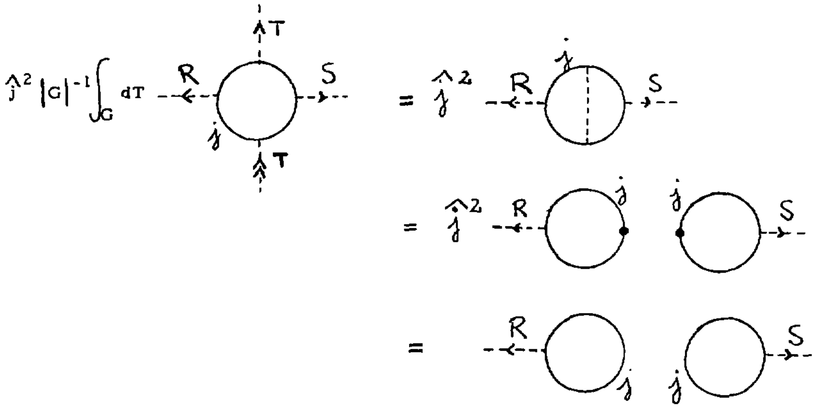
A further specialization yields the character orthogonality theorem



$$= \delta(j_2, j_1) \qquad \text{(D8)}$$

in agreement with Agrawala and Belinfante [8] and Stedman [10]. It is to be noted that (D7) and (D8) may also be deduced from (D1) and (D2), respectively.

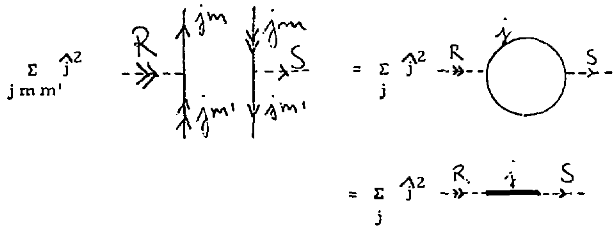
Finally, starting from the right hand side of (9) we obtain



$$= \quad \chi^j(R)\, \chi^j(S) \qquad \text{(D9)}$$

C. Diagrams for Completeness Relations

The distinction between finite and compact continuous groups does not need to be maintained any longer as far as diagrams are concerned. As a matter of fact, in the present state of development of the graphical techniques in group theory, it is not possible to distinguish between Kronecker and Dirac deltas. Consequently, both (10') and (11') have the same diagrammatic representation, viz :



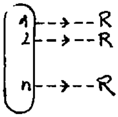
$$= \begin{cases} \delta(R, S)\ |G| & \text{for } G \text{ finite} \quad \text{(D10')} \\ \delta(R^{-1}S)\ |G| & \text{for } G \text{ compact continous} \quad \text{(D11')} \end{cases}$$

All the diagrams in Subsections A, B, and C apply to an arbitrary finite or compact continuous group. For simplicity, we shall restrict ourselves from now on to the case of a simply reducible group, as for example the group SU_2. The case of an arbitrary group requires the introduction of multiplicity lines that render the diagrams more complicated and do not offer anything really new from a graphical standpoint. However, such an introduction may be easily achieved, where necessary, following the works by Agrawala and Belinfante [8], Guichon [9], and Stedman [10].

D. Diagrams for Specific Group Integrals. The Simply Reducible Case

Let us now turn our attention to

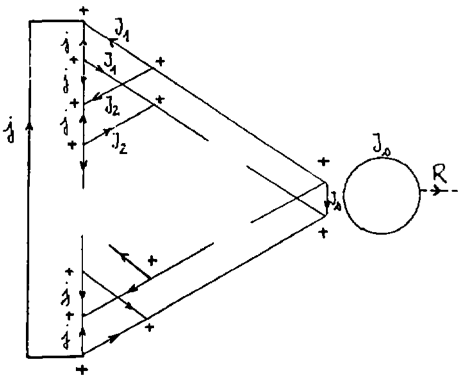
$$s_n = \sum_{\text{all } m} D^j(R)_{m m_1} D^j(R)_{m_1 m_2} \cdots D^j(R)_{m_{n-1} m}$$

$=$ 

The GSA allow a closed expression for the product of n representation matrix elements to be easily obtained . By repeatedly using the diagram of (4) transcribed in terms of 3-jm symbols for a simply reducible group, we get

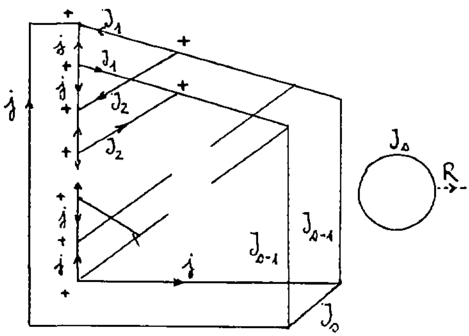
n even :

$$s_n = \sum_s \widehat{J_1}^2 \widehat{J_2}^2 \cdots \widehat{J_{s-1}}^2$$



n odd :

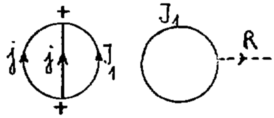
$$s_n = \sum_s \widehat{J_1}^2 \widehat{J_2}^2 \cdots \widehat{J_{s-1}}^2$$



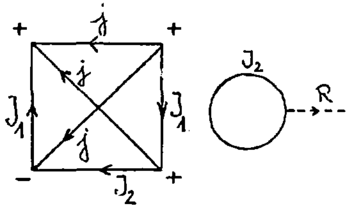
In both cases, we have

$$s_n = \sum_{J_s} \left\{ 3(n-1)-j \text{ symbol} \right\} \chi^{J_s}(R)$$

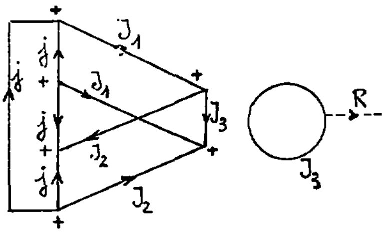
Let us give some examples

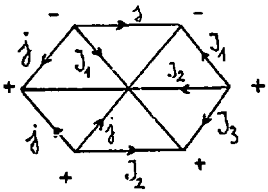
$s_2 = \sum_{J_1}$ 

$$= \sum_{J_1} (-1)^{2j+J_1} \{j\,j\,J_1\}\, \chi^{J_1}(R)$$

$s_3 = \sum_{J_1 J_2} \hat{J}_1^2$ 

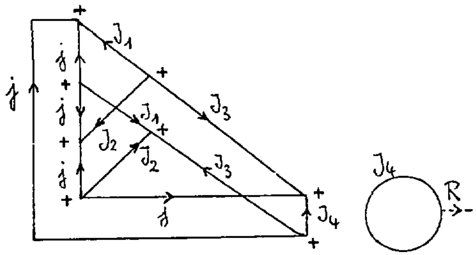
$$= \sum_{J_1 J_2} \hat{J}_1^2 (-1)^{J_1} \begin{Bmatrix} J_2 & J_1 & j \\ j & J_1 & j \end{Bmatrix} \chi^{J_2}(R)$$

$s_4 = \sum_{J_1 J_2 J_3} \hat{J}_1^2 \hat{J}_2^2$ 

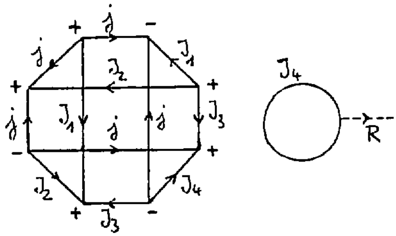
$= \sum_{J_1 J_2 J_3} \hat{J}_1^2 \hat{J}_2^2$ 

$$= \sum_{J_1 J_2 J_3} \hat{J}_1^2 \hat{J}_2^2 (-1)^{J_1+J_2+J_3} \left\{ \begin{array}{ccc} j & j & J_1 \\ j & j & J_2 \\ J_2 & J_1 & J_3 \end{array} \right\} \chi^{J_3}(R)$$

$$s_5 = \sum_{J_1 J_2 J_3 J_4} \hat{J}_1^2 \hat{J}_2^2 \hat{J}_3^3$$



$$= \sum_{J_1 J_2 J_3 J_4} \hat{J}_1^2 \hat{J}_2^2 \hat{J}_3^2$$

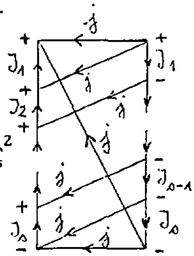


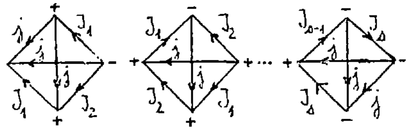
$$= \sum_{J_1 J_2 J_3 J_4} \hat{J}_1^2 \hat{J}_2^2 \hat{J}_3^2 (-1)^{2j+J_1+J_2} \left\{ \begin{array}{cccc|l} J_1 & j & j & J_2 & \\ j & J_2 & j & J_3 & 2 \\ j & J_1 & J_3 & J_4 & \end{array} \right\} \chi^{J_4}(R)$$

To go from s_n to $< s_n >_G$ we just have to average over G. This is achieved diagrammatically by removing the J_s-line. It thus arises $\delta(J_{s-1}, J_{s-2})$ or $\delta(J_{s-1}, j)$ according as n is even or odd. Therefore, we have

$$< s_n >_G = \Sigma \left\{ 3(n-3) - j \text{ symbol} \right\} \left\{ 6\text{-}j \text{ symbol} \right\}$$

Alternatively, we can express $< s_n >_G$ as a sum of product of 6-j coefficients. Indeed, by graphical integration we directly obtain

$$\langle s_n \rangle_G = \sum_{J_1 J_2 \ldots J_s} \hat{J}_1^2 \hat{J}_2^2 \ldots \hat{J}_s^2$$ 

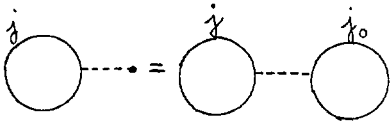
$$= \sum_{J_1 J_2 \ldots J_s} \hat{J}_1^2 \hat{J}_2^2 \ldots \hat{J}_s^2$$ 

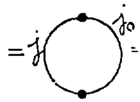
An analytical transcription of such a result is :

$$\langle s_n \rangle_G = \sum_{J_1 J_2 \ldots J_s} \hat{J}_1^2 \hat{J}_2^2 \ldots \hat{J}_s^2 \, (-1)^{2j + J_1 + J_s}$$

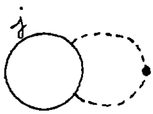
$$\begin{Bmatrix} j & J_1 & j \\ J_2 & J_1 & j \end{Bmatrix} \begin{Bmatrix} J_1 & J_2 & j \\ J_2 & J_1 & j \end{Bmatrix} \ldots \begin{Bmatrix} J_{s-2} & J_{s-1} & j \\ J_s & J_{s-1} & j \end{Bmatrix} \begin{Bmatrix} J_{s-1} & J_s & j \\ j & J_s & j \end{Bmatrix}$$

By way of illustration, let us particularize the diagrams for $\langle s_n \rangle_G$ (and $\langle a_n \rangle_G$) to some given values of n. As a trivial example, we have

$\langle s_1 \rangle_G \equiv \langle a_1 \rangle_G =$ 

 $= \delta(j, j_o)$

Further,

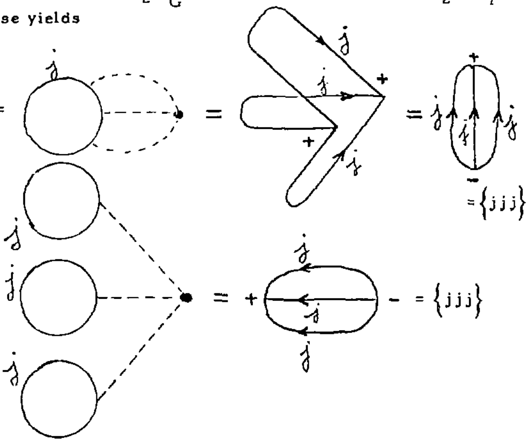
$\langle s_2 \rangle_G =$ 

provides a diagrammatic representation of the Frobenius-Schur coefficient c^j while the diagram for $\langle a_2 \rangle_G$ is nothing but (D8) with $j_2 = j_1 \equiv j$. The $n = 3$ case yields

$\langle s_3 \rangle_G =$

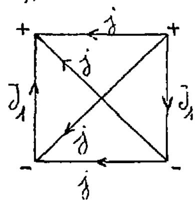
and

$\langle a_3 \rangle_G =$



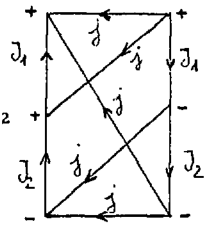
which compare with the diagrams immediately preceding relation (48) of Ref. [10]. Finally, we obtain

$\langle s_4 \rangle_G = \sum_{J_1} \hat{J}_1^2$



$= \sum_{J_1} \hat{J}_1^2 (-1)^{2j} \begin{Bmatrix} j & J_1 & j \\ j & J_1 & j \end{Bmatrix}$

$\langle s_5 \rangle_G = \sum_{J_1 J_2} \hat{J}_1^2 \hat{J}_2^2$



$$= \sum_{J_1 J_2} \hat{J}_1^2 \hat{J}_2^2 (-1)^{2j+J_1+J_2} \begin{Bmatrix} j & J_1 & j \\ J_2 & J_1 & j \end{Bmatrix} \begin{Bmatrix} J_1 & J_2 & j \\ j & J_2 & j \end{Bmatrix}$$

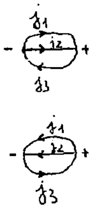
E. Diagrams for Character Formulae for Recoupling Coefficients The Simply Reducible Case

Let us begin with the trivial 3n-j symbol

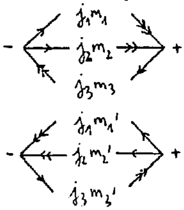
$$Q_0 = \{ j_1 j_2 j_3 \} = \{ j_1 j_2 j_3 \}^2$$

By applying GSA rules, we have the decomposition

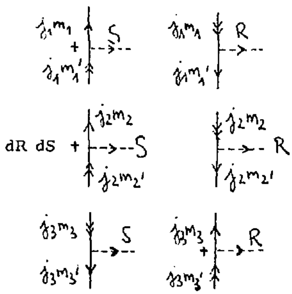
$Q_0 =$



$= \sum_{\substack{m_1 m_2 m_3 \\ m_1' m_2' m_3'}}$



$= \sum_{\substack{m_1 m_2 m_3 \\ m_1' m_2' m_3'}} |G|^{-2} \int_{G \otimes 2} dR\, dS$



The diagrams in which the variance is not natural have received a lecture order (cf. appendix) . We thus get

$$Q_0 = \sum_{\substack{m_1 m_2 m_3 \\ m_1' m_2' m_3'}} |G|^{-2} \int_{G \otimes 2} dR\, dS$$

$$(-1)^{2j_1} D^{j_1}(S^{-1})_{m_1' m_1} \; (-1)^{2j_2} D^{j_2}(S^{-1})_{m_2' m_2} \; D^{j_3}(S)_{m_3 m_3'}$$

$$D^{j_1}(R)_{m_1 m_1'}\ D^{j_2}(R)_{m_2 m_2'}\ (-1)^{2j_3}\ D^{j_3}(R^{-1})_{m_3' m_3}$$

$$= |G|^{-2} \int_{G\otimes 2} dR\, dS\ \chi^{j_1}(R\, S^{-1})\ \chi^{j_2}(R\, S^{-1})\ \chi^{j_3}(R^{-1}\, S)$$

Let us go now to the general case of a product Q of 3n-j symbols involving an equal number of covariant and contravariant generalized triads. It is not necessary to achieve the preceding decomposition in order to get a character formula for Q . Such a formula may be obtained from the following graphical rules.

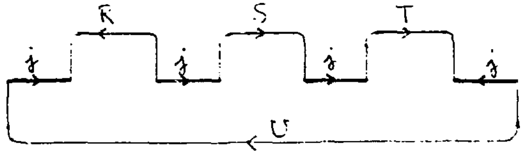
Rule 1 : We draw the diagram for Q and modify it, if necessary, according to GSA rules, to get pairs of corresponding nodes. (Two nodes are said to be correspondent if they have the same generalized triad, opposite variances, and identical lecture order.)

Rule 2 : We link the corresponding nodes two by two by means of a R-line the direction of which may be arbitrarily chosen. It is to be noted that the R-lines have to be distinguished from the usual lines of the GSA, i.e., the j-lines, the ξ-lines, and the integration R-lines.

Rule 3 : The character formula for Q is then

$$Q = (-1)^{\varphi}\ |G|^{-P} \int_{G\otimes P} \chi^{j}(\ldots)\ \chi^{j'}(\ldots) \ldots\ dR\ dS \ldots$$

where P is the number of distinct R-lines and where the phase φ and the argument in each χ are obtained as follows. Starting from a given line j, we move on a loop involving alternatively R-lines and other lines j . The argument of χ^j is then the product of the various encountered group elements, each element being taken to the power 1 or -1 according as the associated R-line has the direction of the lines j or not. Each negative power introduces the phase factor $(-1)^{2j}$. Note that in Rule 1, it is not crucial to have opposite variances for two corresponding nodes. It thus may arise on a loop lines j with opposite directions. In the case where we met a contra-flowing line j , we have to introduce the phase factor $(-1)^{2j}$. Note also that we may loop the loop in any direction since G is ambivalent. By way of illustration, the loop



gives the contribution

$$\chi^j(R^{-1}\,S\,T\,U)\,(-1)^{2j+2j} = \chi^j(R^{-1}\,S\,T\,U)$$

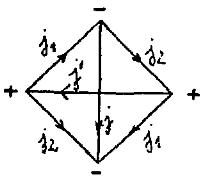
or identically

$$\chi^j(U^{-1}\,T^{-1}\,S^{-1}\,R)\,(-1)^{2j+2j+2j+2j+2j+2j}$$

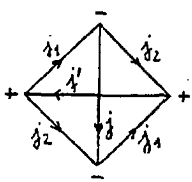
$$= \chi^j(U^{-1}\,T^{-1}\,S^{-1}\,R) \equiv \chi^j(R^{-1}\,S\,T\,U)$$

according as we move on the loop in a clockwise or a contra-clockwise way.

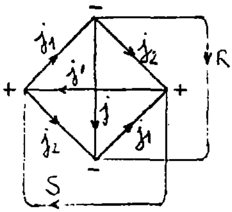
As a trivial example, let us consider

$Q_1 =$ 

In order to get two pairs of corresponding nodes we have to change the direction of one line j_1 thus introducing the phase factor $(-1)^{2j_1}$:

$Q_1 = (-1)^{2j_1}$ 

We link the corresponding nodes by *means of two* R-lines, viz, the lines R and S :

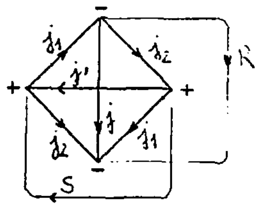
$Q_1 = (-1)^{2j_1}$ 

We then obtain four loops which lead to

$$Q_1 = (-1)^{2j_1} \, |G|^{-2} \int_{G \otimes 2} dR \, dS$$

$$\chi^{j_1}(R\,S)\, \chi^{j_2}(R^{-1} S)\, (-1)^{2j_2}\, \chi^{j}(R^{-1})\, (-1)^{2j}\, \chi^{j'}(S^{-1})\, (-1)^{2j'}$$

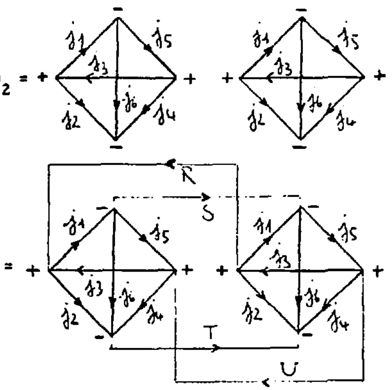
in agreement with (13). Remark that if we do not change the direction of one line j_1, we directly obtain

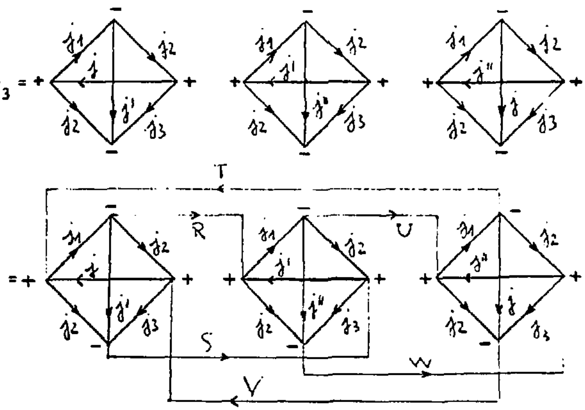
$Q_1 =$ 

$$= |G|^{-2} \int_{G \otimes 2} dR \, dS$$

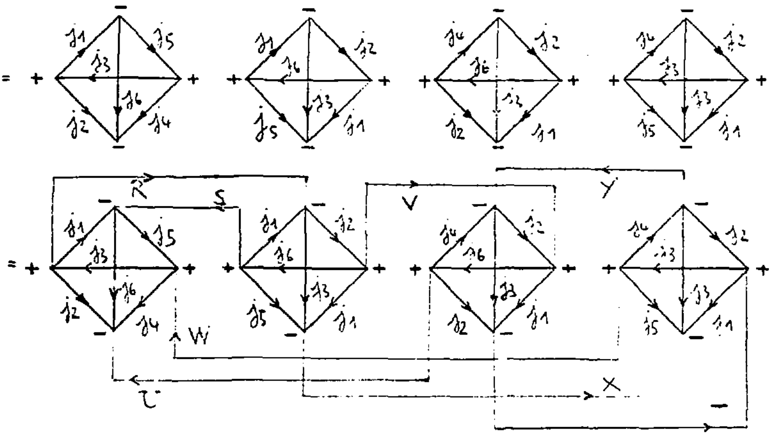
$$\chi^{j_1}(R\,S)\, (-1)^{2j_1}\, \chi^{j_2}(R^{-1} S)\, (-1)^{2j_2}\, \chi^{j}(R^{-1})\, (-1)^{2j}\, \chi^{j'}(S^{-1})\, (-1)^{2j'}$$

In a similar way, we have

$Q_2 =$ 

$Q_3 =$ 

and

$Q_4 =$ 

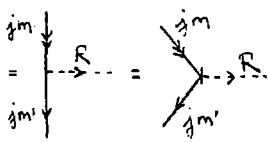
from which it is straightforward to deduce the character formulae (14), (15), and (16), respectively.

Appendix

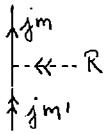
The Rotation Matrix in the GSA

In a previous series of works, one of us (E. E.) has developed a diagram technique, the so-called GSA, for the Wigner-Racah algebra of SU_2 (cf. Refs. [2, 3, 7]). The diagrammatic representation of $D^j(R)_{mm'}$ introduced in Ref. [2] is not entirely consistent with the representation of the state vectors $\langle jm|$ and $|jm'\rangle$. Indeed, the Elbaz et al. [2] representation of $D^j(R)_{mm'} \equiv \langle jm|P_R|jm'\rangle$ does not exhibit the variance of $\langle jm|$ and $|jm'\rangle$ so that a difficulty arises for the representation of $\chi^j(R)$. This difficulty may be overcome by using the diagrammatic representation of $D^j(R)_{mm'}$ as postulated by Guichon [9] for a compact group. We devote the remaining part of this paper to the latter representation, which presents a high degree of coherence with the basic axioms of the GSA.

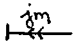
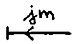
Following Guichon [9] we take

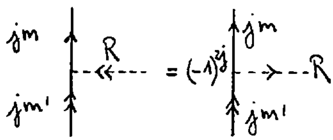
$D^j(R)_{mm'} =$ 

which is markedly different from the diagrammatic representations of $D^j(R)_{mm'}$ given by Agrawala and Belinfante [8] and Stedman [10] in the case of a finite or compact continuous group. We further assume

$$D^j(R^{-1})_{m'm} \equiv D^j(R)_{mm'}^{*} =$$ 

in accordance with the usual practice in the GSA that complex conjugation changes the variance of each arrow. Note that the variance for m , m' , and R are clearly specified both for $D^j(R)_{mm'}$ and $D^j(R)_{mm'}^{*}$. This probably constitutes the main advantage of the representation of Guichon [9] over the ones of Elbaz et al. [2], Agrawala and Belinfante [8], and Stedman [10].

At this point it is perhaps whorthwhile to mention that the GSA rule "  transforms contragrediently to $(-1)^{j-m}$  " may lead to some ambiguity when considered as an equality. In particular, starting from the well-known symmetry relation $D^j(R)_{mm'}^{*} = (-1)^{m-m'} D^j(R)_{-m-m'}$ and using such an equality, we would obtain

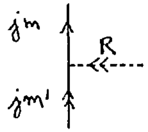


or

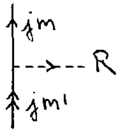
$$D^j(R^{-1})_{m'm} = (-1)^{2j} D^j(R)_{m'm}$$

a result which is evidently wrong. Such an ambiguity may be overcome by introducing a lecture order starting from the R-line for the second diagram:

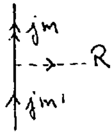
$D^j(R)_{mm'}^{\,*} \equiv$



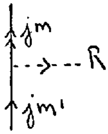
$= (-1)^{2j} +$



$= (-1)^{2j}\,(-1)^{j-m'}\,(-1)^{j+m}$



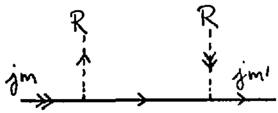
$= (-1)^{m-m'}$



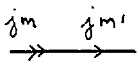
$\equiv (-1)^{m-m'}\, D^j(R)_{-m-m'}$

Application of the m-summation rule of the GSA yields

$$\sum_{m''} D^j(R)_{mm''}\, D^j(R)_{m'm''}^{\,*} =$$



$=$



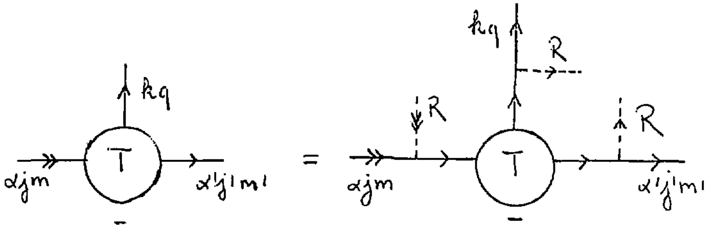
$= \delta(m', m)$

whence the unitarity property of D^j amounts in last analysis to omitting the R - and R^{-1} - lines exactly as in the Agrawala-Belinfante-Stedman techniques.

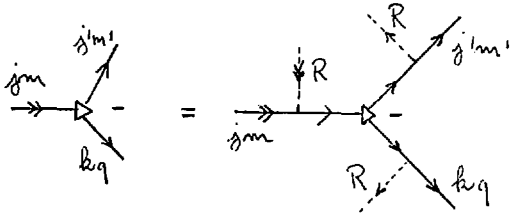
The invariance property

$$\langle \alpha j m \,|\, T^k_q \,|\, \alpha' j' m' \rangle = \sum_{\bar{m}\bar{q}\bar{m}'} D^j(R)_{\bar{m}m}^{\,*}\, \langle \alpha j \bar{m} \,|\, T^k_{\bar{q}} \,|\, \alpha' j' \bar{m}' \rangle\, D^k(R)_{\bar{q}q}\, D^{j'}(R)_{\bar{m}'m'}$$

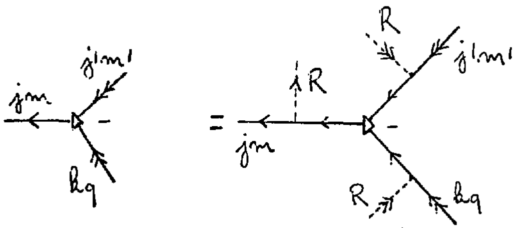
is diagrammed as [9]



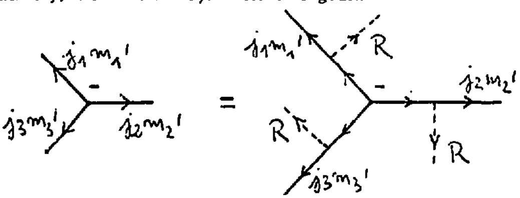
that particularizes to



or identically



for the Clebsch-Gordan coefficient $\langle j'km'q | jm \rangle^* \equiv \langle j'km'q | jm \rangle$.
Alternatively, we have the symmetric diagram

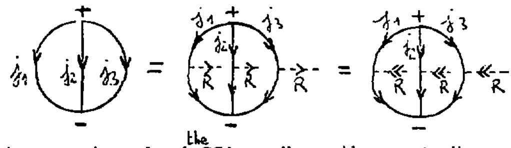


which corresponds to the 3-jm Wigner symbol

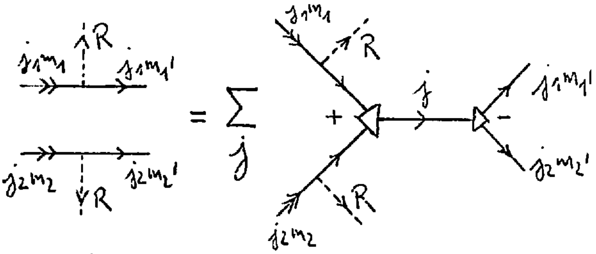
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix} = \sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} D^{j_1}(R)_{m_1 m_1'} \, D^{j_2}(R)_{m_2 m_2'} \, D^{j_3}(R)_{m_3 m_3'}$$

The rotational invariance of the 3-j symbol $\{j_1 j_2 j_3\}$ is expressed diagrammatically by

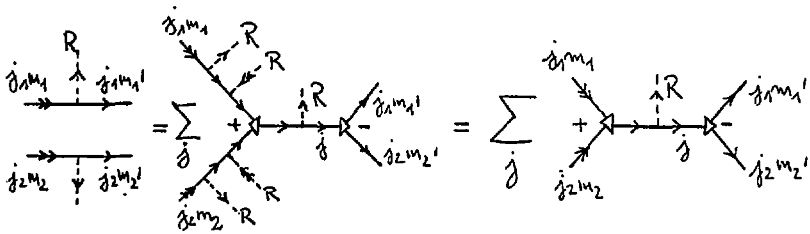
$\{j_1 j_2 j_3\}$ =



The j-summation rule of the GSA easily enables us to diagram (4) for $G \equiv SU_2$. We have

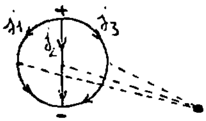


By using the above-mentioned invariance property for the Clebsch-Gordan coefficients, we get

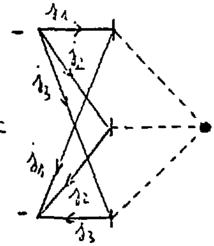


Following Sandars [5] and Stedman [10], a group integral of the type $|SU_2|^{-1}\int_{SU_2} f(R)\, dR$ will be transcribed diagrammatically merely by connecting the R-lines relative to the diagram for f to a solid circle. When only two R-lines with opposite variance are connected the solid circle will be omitted. Such a graphical manipulation defines the R-integration, which turns out to be a simple extension of the Ω- and ξ- integrations (cf. Refs. [2, 3, 7]). As a trivial example, we have

$$\{j_1 j_2 j_3\} = |SU_2|^{-1}\int_{SU_2} \{j_1 j_2 j_3\}\, dR =$$

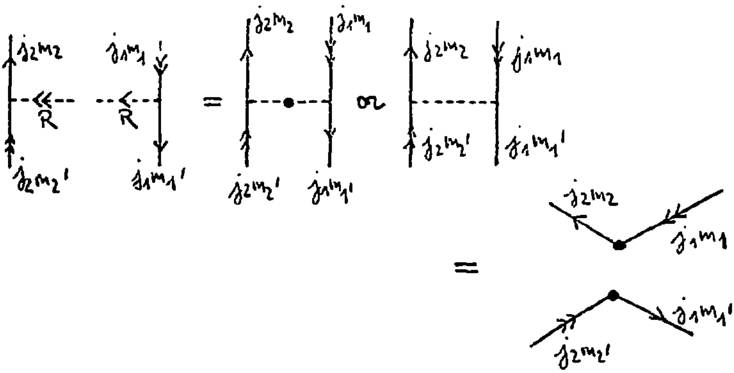


=



A more elaborate example is supplied by the great orthogonality theorem (2) for $G \equiv SU_2$: the diagram relation

$$|SU_2|^{-1}\int_{SU_2} dR$$



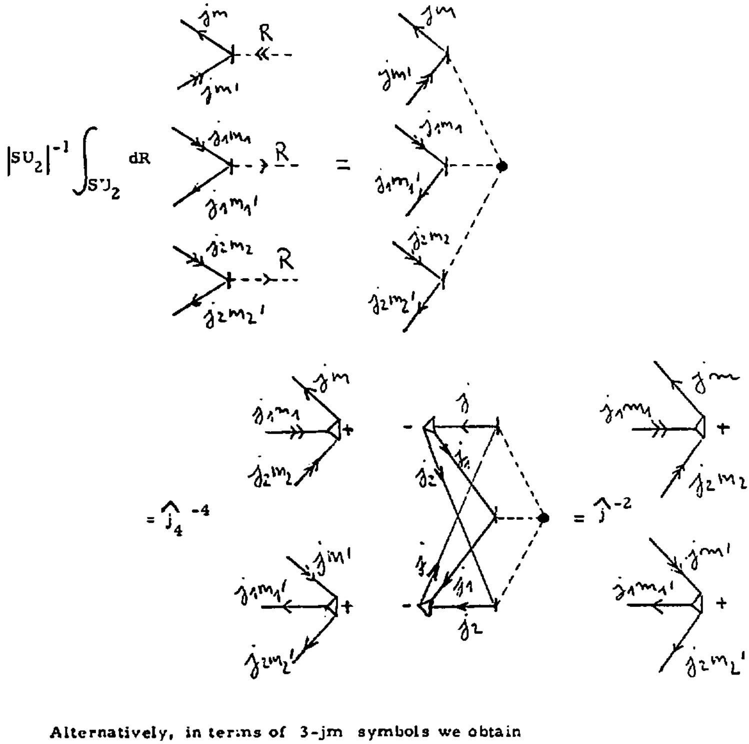
is the diagrammatic transcription of

$$|SU_2|^{-1}\int_{SU_2} D^{j_2}(R)_{m_2 m_2'}{}^{*}\, D^{j_1}(R)_{m_1 m_1'}\, dR$$

$$= \hat{j}_1^{-1}\,\delta(j_2, j_1)\,\delta(m_2, m_1)$$

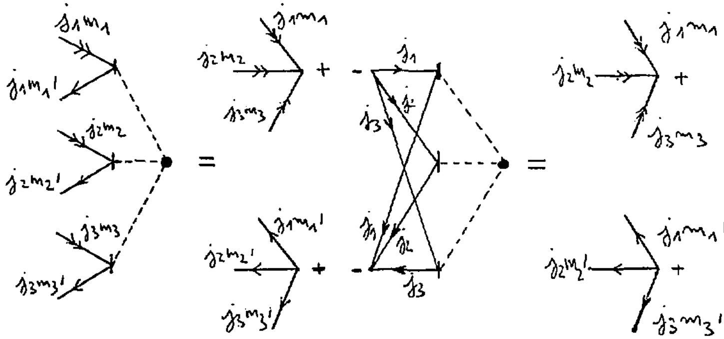
$$\hat{j}_1^{-1}\,\delta(j_2, j_1)\,\delta(m_2', m_1')$$

Along the same lines, the formula of Gaunt (5) for $G \equiv SU_2$ yields



Alternatively, in terms of 3-jm symbols we obtain

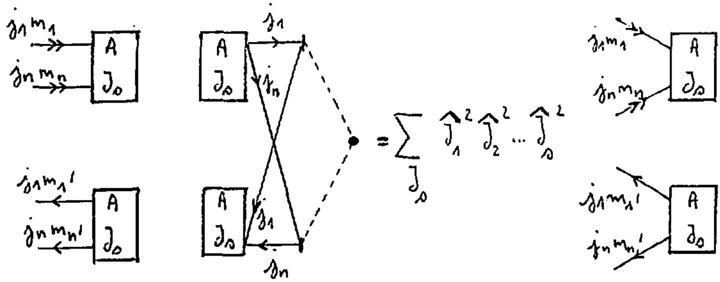
$$|SU_2|^{-1}\int_{SU_2} D^{j_1}(R)_{m_1 m_1'}\, D^{j_2}(R)_{m_2 m_2'}\, D^{j_3}(R)_{m_3 m_3'}\, dR \;=$$



$$= \begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix}$$

An immediate generalization of the latter relation leads to

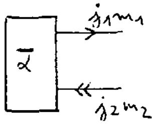
$$|SU_2|^{-1}\int_{SU_2} D^{j_1}(R)_{m_1 m_1'}\, D^{j_2}(R)_{m_2 m_2'} \cdots D^{j_n}(R)_{m_n m_n'}\, dR \;=\; \sum_{J_s} \hat{J}_1^2\, \hat{J}_2^2 \cdots \hat{J}_s^2$$



where the closed and integrated diagram is easily recognized to be equal to 1 or 0 according as $\vec{J}_1 + \vec{J}_2 + \ldots + \vec{J}_n = 0$ or not.

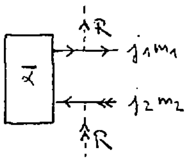
The R-integration is at the root of the YLV2 Yutsis-Levinson-Vanagas theorem [1], which is known as the pinching rule in the GSA. To be more precise, let us consider

$$F\begin{pmatrix} j_1 & m_2 \\ m_1 & j_2 \end{pmatrix}\!\!\left|\bar{\alpha}\right) =$$



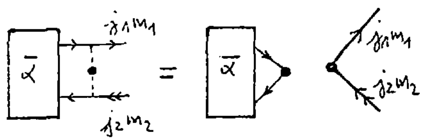
The use of the above-mentioned invariance property gives :

$$F\begin{pmatrix} j_1 & m_2 \\ m_1 & j_2 \end{pmatrix}\!\!\left|\bar{\alpha}\right) =$$



Therefore we have

$$F\begin{pmatrix} j_1 & m_2 \\ m_1 & j_2 \end{pmatrix}\!\!\left|\bar{\alpha}\right) \equiv |SU_2|^{-1}\int_{SU_2} F\begin{pmatrix} j_1 & m_2 \\ m_1 & j_2 \end{pmatrix}\!\!\left|\bar{\alpha}\right) dR =$$



as given by the YLV2 theorem.

Acknowlegments

Interesting discussions on convolution algebra which one of the authors (M. K.) had six years ago with Professors P. -O. Löwdin and Y. Öhrn are gratefully acknowledged. The authors wish to thank Dr. P. Guichon for permission to reproduce some diagrams of his 1975 unpublished report (obtainable from M. K.) and Drs. G. Grenet and P. Guichon for comments and criticism during the course of this work.

Bibliography

[1] A. P. Yutsis, I. B. Levinson, and V. V. Vanagas, Mathematical Apparatus of the Theory of Angular Momentum (Israel Program for Scientific Translations, Jerusalem, 1962).

[2] E. Elbaz, J.-N. Massot, and J. Lafoucrière, Nucl. Phys. 86, 625 (1966).

[3] J. N. Massot, E. El-Baz, and J. Lafoucrière, Rev. Mod. Phys. 39, 288 (1967).

[4] D. M. Brink and G. R. Satchler, Angular Momentum (Oxford University Press, London, 1968).

[5] P. G. H. Sandars, in Lectures in Theoretical Physics, Brandeis University Summer Institute, Vol. 1 (M. Chrétien and E. Lipworth, Eds., Gordon and Breach, New York, 1969).

[6] J. S. Briggs, Rev. Mod. Phys. 43, 189 (1971).

[7] E. Elbaz and B. Castel, Graphical Methods of Spin Algebras (Marcel Dekker, New York, 1972).

[8] V. K. Agrawala and J. G. Belinfante, Ann. Phys. (N Y) 49, 130 (1968).

[9] P. A. M. Guichon, Report LYCEN 7570, University of Lyon, 1975.

[10] G. E. Stedman, J. Phys. A : Math. Gen. 8, 1021 (1975).

[11] G. E. Stedman, J. Phys. A : Math. Gen. 9, 1999 (1976).

[12] J. Paldus, B. G. Adams, and J. Čížek, Int. J. Quant. Chem. 11, 813 (1977).

[13] B. G. Adams, J. Paldus, and J. Čížek, Int. J. Quant. Chem. 11, 849 (1977).

[14] J. Paldus and P. E. S. Wormer, Calculation of permutation matrices using graphical methods of spin algebras : explicit expressions for the Serber coupling case, Quantum Theory Group preprint, University of Waterloo, 1978.

[15] L. C. Biedenharn and A. Gamba, Revista Brasileira de Física 2, 319 (1972).

[16] J. D. Louck, Phys. Rev. A 9, 2273 (1974).

[17] E. de Vries and A. J. van Zanten, J. Phys. Math. A : Math., Nucl. Gen. 7, 807 (1974).

[18] K. Fox, H. W. Galbraith, B. J. Krohn, and J. D. Louck, Phys. Rev. A 15, 1363 (1977).

[19] E. P. Wigner, in Quantum Theory of Angular Momentum (L. C. Biedenharn and H. van Dam, Eds., Academic Press, New York, 1965).

[20] W. T. Sharp, Report AECL-1098, Atomic Energy of Canada Ltd, Ontario, 1960.

[21] J.-R. Derome and W. T. Sharp, J. Math. Phys. 6, 1584 (1965).

[22] G. Fieck, J. Phys. B : Atom. Molec. Phys. 10, L1 (1977).

[23] M. R. Kibler, J. Phys. A : Math. Gen. 10, 2041 (1977).

[24] E. de Vries and A. J. van Zanten, J. Math. Phys. 13, 862 (1972).

[25] P.-O. Löwdin, Rev. Mod. Phys. 39, 259 (1967).

[26] M. Kibler, Eléments de Théorie des Groupes, Lecture Notes, University of Lyon, 1970.

[27] H. Weyl, The Theory of Groups and Quantum Mechanics (Dover, New York, 1931).

[28] H. Weyl, Math. Zeitschrift 24, 377 (1926).

[29] A. J. van Zanten and E. de Vries, J. Algebra 25, 475 (1973).

[30] J.-R. Derome, J. Math. Phys. 7, 612 (1966).

[31] J.-R. Derome, J. Math. Phys. 8, 714 (1967).

[32] A. J. van Zanten and E. de Vries, Physica 49, 536 (1970).