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J.Nyiri, Ya.A.Smorodinsky

A NEW METHOD OF OBTAINING THE TRANSFORMATION COEFFICIENTS FOR THE THREE-BODY BASIS FUNCTIONS

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Нири Ю., Смородинский Я.А. E2 - 12191

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Исследуется новый способ вывода коэффициентов преобразования трехчастичных функции в шестимерном пространстве. Этот способ является обобщением предложенных прежде методов. Применяется теоретико-групповое представление. Полученный результат включает в себя как частный случай выражения для коэффициентов преобразования, известных раньше.

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A New Method of Obtaining the Transformation Coefficients for the Three-Body Basis Functions

A new method is proposed for obtaining the coefficient $\langle j_1' j_2' | j_1 j_2 \rangle^{\Phi}_{KJM}$ corresponding to transformations of the three-body basis function in the six-dimensional space.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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In the quantum mechanical three-body problem there are several ways for constructing basis functions. One of the easiest is to choose the set of the so-called tree-functions $\Phi^{j_1 j_2}_{KJM}(\vec{\eta},\vec{\xi})$ [1]. The transformations of the Jacobi coordinates generate transformations of the basis functions:

$$\Phi^{j_1 j_2}_{KJM}(\vec{\eta}',\vec{\xi}') = \sum_{j_1' j_2'} \langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM} \; \Phi^{j_1' j_2'}_{KJM}(\vec{\eta},\vec{\xi}) \; . \qquad (1)$$

The coefficients $\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM}$, i.e., matrix elements of these transformations, were first obtained by Raynal and Revai [2], and Efros and Smorodinsky [3]. The results, however, looked different to such an extent, that it made necessary to prove the identity of the two forms of $\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM}$. This was done in paper [4]. Looking for a common form of the two versions, we found a much simpler method of obtaining the transformation coefficient, which, by the way, allows quite surprising group theoretical generalizations. In the following we present this new way of constructing $\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM}$.

By definition, we can write

$$\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM} = \int (\Phi^{j_1' j_2'}_{KJM}(\vec{\eta},\vec{\xi}))^* \, \Phi^{j_1 j_2}_{KJM}(\vec{\eta}',\vec{\xi}') \, d\vec{\eta} \, d\vec{\xi} \, . \qquad (2)$$

Making use of the orthonormality of the basis functions, we can formally consider the matrix element of unity instead of the overlap integral:

$$\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM} = \langle j_1' j_2' | 1 | j_1 j_2 \rangle^{\phi}_{KJM} \qquad (3)$$

That means, that one can avoid a rather lengthy procedure of straightforward calculation of (2). Instead of that, one has to find such an expansion of unity into a series, in which $\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM}$ occurs as a coefficient. For that purpose it is necessary to write down 1 in a form which can be expanded in terms of the basis functions corresponding to J and M

The explicit form of the basis function $\Phi^{j_1 j_2}_{KJM}(\vec{\eta}, \vec{\xi})$ is the following:

$$\Phi^{j_1 j_2}_{KJM}(\vec{\eta}, \vec{\xi}) = Y^{j_1 j_2}_{JM}(\vec{m}, \vec{n}) \quad \Psi_{K j_1 j_2}(\Phi). \qquad (4)$$

The part of this function, corresponding to J M is

$$Y^{j_1 j_2}_{JM}(\vec{m}, \vec{n}) = \sum_{m_1 m_2} C^{JM}_{j_1 m_1 j_2 m_2} Y_{j_1 m_1}(\vec{m}) \; Y_{j_2 m_2}(\vec{n}), \qquad (5)$$

where

$$\vec{m} = \frac{\vec{\eta}}{\eta}, \qquad \vec{n} = \frac{\vec{\xi}}{\xi},$$

and j_1, j_2 are the angular momenta, corresponding to $\vec{\eta}$ and $\vec{\xi}$.

$$\Psi_{K j_1 j_2}(\Phi) = N_{K j_1 j_2} (\sin\Phi)^{j_1} (\cos\Phi)^{j_2} \, P^{(j_1 + \frac{1}{2}, \, j_2 + \frac{1}{2})}_{\frac{K - j_1 - j_2}{2}} (\cos 2\Phi). \qquad (6)$$

where we used $\xi^2 + \eta^2 = 1$ and therefore could introduce

$$\cos\Phi = \xi, \qquad \sin\Phi = \eta.$$

The normalization factor $N_{Kj_1j_2}$ is well-known from the theory of Jacobi polynomials:

$$N_{Kj_1j_2} = \left[\frac{2(K+2)\Gamma(\frac{K-j_1-j_2}{2}+1)\Gamma(\frac{K+j_1+j_2}{2}+2)}{\Gamma(\frac{K-j_1+j_2}{2}+\frac{3}{2})\Gamma(\frac{K+j_1-j_2}{2}+\frac{3}{2})} \right]^{1/2} \quad (7)$$

Let us introduce now two three-dimensional vectors $\vec{P}_i$ and $\vec{Q}_i$ having the properties

$$\vec{P}_i = \vec{Q}_i = 1$$

and (8)

$$\vec{P}_i \vec{Q}_i = 0 .$$

We can write then

$$1 = e^{2(P_i^2 - Q_i^2)} . \quad (9)$$

where the factor 2 is introduced for the sake of convenience. For reasons, which will become clear later, we consider in the following $i\vec{Q}_i$ instead of $\vec{Q}_i$. Further, we substitute one of the P_i-s and Q_i-s by P_k and Q_k, which are connected with the original vectors by the orthogonal transformation

$$i\vec{Q}_k = \cos\phi \, i\vec{Q}_i + \sin\phi \, \vec{P}_i \; ,$$

$$\vec{P}_k = \sin\phi \, i\vec{Q}_i - \cos \vec{P}_i \; . \quad (10)$$

(Because of the orthogonality of this transformation the properties (8) are invariant). Thus, instead of (9), we have

$$e^{-2P_iP_k\cos\phi - 2Q_iQ_k\cos\phi + 2iP_iQ_k\sin\phi + 2iQ_iP_k\sin\phi} . \quad (11)$$

Using the plane-wave expansion

$$e^{i\vec{p}\vec{x}} = \sum_{\lambda\mu} i^{\lambda} j_{\lambda}(px)\, Y^{*}_{\lambda\mu}(p)\, Y_{\lambda\mu}(x)$$

we can re-write (11) in the form

$$\sum_{prqs} (-1)^{\frac{p+r+q+s}{2}} j_p(2iP_iP_k\cos\delta)\, j_r(2Q_iP_k\sin\delta)\, j_q(2P_iQ_k\sin\delta)\, j_s(2iQ_iQ_k\cos\delta) \tag{12}$$

$$\cdot Y^{*}_{p\pi}(\hat{P}_i)\, Y^{*}_{q\chi}(\hat{P}_i)\, Y^{*}_{r\rho}(\hat{Q}_i)\, Y^{*}_{s\sigma}(\hat{Q}_i)\, Y_{p\pi}(\hat{P}_k)\, Y_{r\rho}(\hat{P}_k)\, Y_{q\chi}(\hat{Q}_k)\, Y_{s\sigma}(\hat{Q}_k),$$

where $\hat{P}_i$, $\hat{Q}_i$, $\hat{P}_k$ and $\hat{Q}_k$ denote the three-dimensional polar angles.

Remember now the so-called Bateman expansion, which is usually given in the form (/5/ p. 974, f. 8.442 (2)).

$$J_{\nu}(az)\, J_{\mu}(bz) = \frac{\left(\frac{az}{2}\right)^{\nu}\left(\frac{bz}{2}\right)^{\mu}}{\Gamma(\mu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{az}{2}\right)^{2k} F\left(-k, -\nu-k; \mu+1; \frac{b^2}{a^2}\right)}{k!\,\Gamma(\nu+k+1)}$$

and note, that the hypergeometrical function can be substituted by the Jacobi polynomial (/5/ p. 1050, f. 8.962 (4)):

$$P_k^{(\nu,\mu)}(x) = \frac{\Gamma(k+1+\mu)}{k!\,\Gamma(1+\mu)} \left(\frac{x-1}{2}\right)^k F\left(-k, -k-\nu; \mu+1; \frac{x+1}{x-1}\right)$$

or by Wigner's d-function. Substituting

$$j_{\lambda}(x) = \sqrt{\frac{\pi}{2x}}\, J_{\lambda+\frac{1}{2}}(x),$$

we can apply these formulae to the spherical Bessel functions which occur in (12). It is easy to see, that the step from the hypergeometrical function to the Jacobi polynomial is possible only, if we consider the imaginary vector $i\vec{Q}_i$. Replacing the lengths of $\vec{P}_i, \vec{Q}_i, \vec{P}_k, \vec{Q}_k$ by unity, we can write then

$$j_p(2i\cos\phi)\, j_q(2\sin\phi) = \frac{\pi}{4}\sum_{K_1}(-1)^{K_1-\frac{q}{2}+\frac{1}{2}}(\sin\phi\cos\phi)^{-1/2}\, d^{\frac{K_1+1}{2}}_{\frac{p+q+1}{2},\,\frac{q-p}{2}}(\cos 2\phi)\cdot$$

$$\left[\left(\frac{K_1-p+q}{2}+\frac{1}{2}\right)!\left(\frac{K_1+p-q}{2}+\frac{1}{2}\right)!\left(\frac{K_1+p+q}{2}+1\right)!\left(\frac{K_1-p-q}{2}\right)!\right]^{-1/2},$$

and

$$j_s(2i\cos\phi)\, j_r(2\sin\phi) = \frac{\pi}{4}\sum_{K_2}(-1)^{K_2-\frac{s}{2}+\frac{1}{2}}(\sin\phi\cos\phi)^{-1/2}\, d^{\frac{K_2+1}{2}}_{\frac{s+r+1}{2},\,\frac{r-s}{2}}(\cos 2\phi)\cdot$$

$$\left[\left(\frac{K_2-r+s}{2}+\frac{1}{2}\right)!\left(\frac{K_2+r-s}{2}+\frac{1}{2}\right)!\left(\frac{K_2+r+s}{2}+1\right)!\left(\frac{K_2-r-s}{2}\right)!\right]^{-1/2}. \tag{13}$$

Returning now to the matrix element between two states (5), we obtain a long but simple expression for (3):

$$\frac{\pi}{8}\sum_{\substack{m_1 m_2\\ m_1' m_2'}} C^{JM}_{j_1 m_1 j_2 m_2}\, C^{JM}_{j_1' m_1' j_2' m_2'} \int \sum_{\substack{prqs\\ \pi\rho\chi\sigma}} \sum_{K_1 K_2} (-1)^{K+\frac{q}{2}-\frac{r}{2}+1} \times$$

$$\times (\sin 2\phi)^{-1}\, d^{\frac{K_1+1}{2}}_{\frac{p+q+1}{2},\,\frac{q-p}{2}}(\cos 2\phi)\, d^{\frac{K_2+1}{2}}_{\frac{s+r+1}{2},\,\frac{r-s}{2}}(\cos 2\phi),$$

$$\times [(\frac{K_1-p+q}{2}+\frac{1}{2})!(\frac{K_1+p-q}{2}+\frac{1}{2})!(\frac{K_1+p+q}{2}+1)!(\frac{K_1-p-q}{2})!]^{-1/2}$$

$$\times [(\frac{K_2-r+s}{2}+\frac{1}{2})!(\frac{K_2+r-s}{2}+\frac{1}{2})!(\frac{K_2+r+s}{2}+1)!(\frac{K_2-r-s}{2})!]^{-1/2}.$$

(14)

$$\times Y^*_{p\pi}(\hat{P}_i)\, Y^*_{q\chi}(\hat{P}_i)\, Y_{j_1m_1}(\hat{P}_i) Y^*_{r\rho}(\hat{Q}_i) Y^*_{s\sigma}(\hat{Q}_i) Y_{j_2m_2}(\hat{Q}_i)$$

$$\times Y_{p\pi}(\hat{P}_k) Y_{r\rho}(\hat{P}_k) Y^*_{j_1'm_1'}(\hat{P}_k)\times$$
$$\times Y_{q\chi}(\hat{Q}_k) Y_{s\sigma}(\hat{Q}_k) Y^*_{j_2'm_2'}(\hat{Q}_k) d\hat{P}_i\, d\hat{Q}_i\, d\hat{P}_k\, d\hat{Q}_k .$$

Using the well-known properties of the spherical functions

$$Y^*_{\ell m}(\theta,\phi) = (-1)^m Y_{\ell -m}(\theta,\phi)$$

and

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, d\sin\theta\, Y_{\ell_1 m_1}(\theta,\phi)\, Y_{\ell_2 m_2}(\theta,\phi)\, Y^*_{\ell_3 m_3}(\theta,\phi) =$$

$$= [\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}]^{1/2}\; C^{\ell_3 0}_{\ell_1 0 \ell_2 0}\, C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}$$

(see [6] p. 124, f. (1) and p. 131, f. (5)), (14) can be rewritten in the form

$$\sum_{\substack{prqs \\ \pi\rho\kappa\sigma}} \sum_{\substack{m_1'm_2' \\ m_1m_2}} \sum_{K_1K_2} C^{JM}_{j_1m_1j_2m_2}\, C^{JM}_{j_1'm_1'j_2'm_2'}\, C^{j_1m_1}_{p\pi q\chi}\, C^{j_2m_2}_{r\rho s\sigma} \times$$

$$\times C^{j_1' m_1}_{p\pi r\rho} C^{j_2' m_2'}_{q\chi s\sigma} C^{j_1 0}_{p0 q0} C^{j_2 0}_{r0 s0} C^{j_1' 0}_{p0 r0} C^{j_2' 0}_{q0 s0} \times$$

$$\cdot \left[\left(\frac{K_1 - p + q + 1}{2}\right)! \left(\frac{K_1 + p - q + 1}{2}\right)! \left(\frac{K_1 - p - q}{2}\right)! \left(\frac{K_1 + p + q}{2} + 1\right)! \right]^{-1/2} \times$$

$$\cdot \left[\left(\frac{K_2 - s + r + 1}{2}\right)! \left(\frac{K_2 + s - r + 1}{2}\right)! \left(\frac{K_2 - s - r}{2}\right)! \left(\frac{K_2 + s + r}{2} + 1\right)! \right]^{-1/2} \cdot$$

$$\times (\sin 2\vartheta)^{-1} \, d^{\frac{K_1+1}{2}}_{\frac{p+q+1}{2}, \frac{p-q}{2}}(\cos 2\vartheta) \, d^{\frac{K_2+1}{2}}_{\frac{r+s+1}{2}, \frac{s-r}{2}}(\cos 2\vartheta) \quad . \qquad (15)$$

The parameters K_1 and K_2 are fixed by the condition $K = K_1 + K_2$. Instead of the sum over K_1, K_2 it is enough to consider a term with a definite K. The summation of the Clebsch- Gordan coefficients leads to the 9j coefficient

$$\sum_{m_i m_{ik}} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} C^{j_{34} m_{34}}_{j_3 m_3 j_4 m_4} C^{jm}_{j_{12} m_{12} j_{34} m_{34}}$$

$$\times C^{j_{13} m_{13}}_{j_1 m_1 j_3 m_3} C^{j_{24} m_{24}}_{j_2 m_2 j_4 m_4} C^{j' m'}_{j_{13} m_{13} j_{24} m_{24}} =$$

$$= \delta_{jj'} \delta_{mm'} [(2j_{12}+1)(2j_{13}+1)(2j_{24}+1)(2j_{34}+1)]^{1/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} .$$

Substituting this expression into (15), we obtain the following form for the transformation coefficient:

$$\sum_{prqs} \left[\left(\frac{K_1-p+q+1}{2}\right)! \left(\frac{K_1+p-q+1}{2}\right)! \left(\frac{K_1+p+q}{2}+1\right)! \left(\frac{K_1-p-q}{2}\right)! \right]^{-1/2}$$

$$\times \left[\left(\frac{K_2-r+s+1}{2}\right)! \left(\frac{K_2+r-s+1}{2}\right)! \left(\frac{K_2+r+s}{2}+1\right)! \left(\frac{K_2-r-s}{2}\right)! \right]^{-1/2}$$

$$\times \left\{\!\!\left\{\begin{matrix} p & r & j_1' \\ q & s & j_2' \\ j_1 & j_2 & J \end{matrix}\right\}\!\!\right\} (\sin 2\phi)^{-1}\, d^{\frac{K_1+1}{2}}_{\frac{q+p+1}{2},\frac{q-p}{2}}(\cos 2\phi)\, d^{\frac{K_2+1}{2}}_{\frac{r+s+1}{2},\frac{s-r}{2}}(\cos 2\phi). \quad (16)$$

where we introduced the notation

$$\left\{\!\!\left\{\begin{matrix} p & r & j_1' \\ q & s & j_2' \\ j_1 & j_2 & J \end{matrix}\right\}\!\!\right\} = \left[(2j_1+1)(2j_2+1)(2j_1'+1)(2j_2'+1)\right]^{1/2} \times$$

$$\times (2p+1)(2q+1)(2s+1)(2r+1) \times$$

$$\times \begin{pmatrix} p & q & j_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s & r & j_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p & r & j_1' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q & s & j_2' \\ 0 & 0 & 0 \end{pmatrix} \left\{\begin{matrix} p & r & j_1' \\ q & s & j' \\ j_1 & j_2 & J \end{matrix}\right\}. \quad (17)$$

Considering the whole basis function (4) instead of (5), we can write down the final form of the orthonormal transformation coefficient:

$$\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM} = \frac{\pi}{2} \left(\frac{K_1 - j_1}{2}\right)! \left(\frac{K_1 + j_1 + 1}{2}\right)! \left(\frac{K_2 - j_2}{2}\right)! \left(\frac{K_2 + j_2 + 1}{2}\right)!\,.$$

$$\times \frac{a_{Kj_1'j_2'}}{a_{Kj_1j_2}} \sum_{prqs} \left[\left(\frac{K_1 - p + q + 1}{2}\right)! \left(\frac{K_1 + p - q + 1}{2}\right)! \left(\frac{K_1 - p - q}{2}\right)! \left(\frac{K_1 + p + q}{2} + 1\right)! \right]^{-1/2}.$$

$$\times \left[\left(\frac{K_2 - s + r + 1}{2}\right)! \left(\frac{K_2 + s - r + 1}{2}\right)! \left(\frac{K_2 + r - s}{2}\right)! \left(\frac{K_2 + r + s}{2} + 1\right)! \right]^{-1/2}.$$

$$\times \left\{\begin{matrix} p & r & j_1' \\ q & s & j_2' \\ j_1 & j_2 & J \end{matrix}\right\} (\sin 2\phi)^{-1}\, d^{\frac{K_1+1}{2}}_{\frac{q+p+1}{2},\,\frac{p-q}{2}}(\cos 2\phi)\; d^{\frac{K_2+1}{2}}_{\frac{r+s+1}{2},\,\frac{s-r}{2}}(\cos 2\phi)$$

(18)

with

$$a_{Kj_1j_2} = \left[\left(\frac{K - j_1 - j_2}{2}\right)! \left(\frac{K + j_1 + j_2}{2} + 1\right)! \left(\frac{K - j_1 + j_2 + 1}{2}\right)! \left(\frac{K + j_1 - j_2 + 1}{2}\right)! \right]^{1/2}.$$

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The coefficients $\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM}$, i.e., matrix elements of these transformations, were first obtained by Raynal and Revai [2], and Efros and Smorodinsky [3]. The results, however, looked different to such an extent, that it made necessary to prove the identity of the two forms of $\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM}$. This was done in paper [4]. Looking for a common form of the two versions, we found a much simpler method of obtaining the transformation coefficient, which, by the way, allows quite surprising group theoretical generalizations. In the following we present this new way of constructing $\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM}$.

By definition, we can write

$$\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM} = \int (\Phi^{j_1' j_2'}_{KJM}(\vec{\eta},\vec{\xi}))^* \, \Phi^{j_1 j_2}_{KJM}(\vec{\eta}',\vec{\xi}') d\vec{\eta}\, d\vec{\xi} . \qquad (2)$$

Making use of the orthonormality of the basis functions, we can formally consider the matrix element of unity instead of the overlap integral:

$$\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM} = \langle j_1' j_2' | 1 | j_1 j_2 \rangle^{\phi}_{KJM} \tag{3}$$

That means, that one can avoid a rather lengthy procedure of straightforward calculation of (2). Instead of that, one has to find such an expansion of unity into a series, in which $\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM}$ occurs as a coefficient. For that purpose it is necessary to write down 1 in a form which can be expanded in terms of the basis functions corresponding to J and M.

The explicit form of the basis function $\Phi^{j_1 j_2}_{KJM}(\vec{\eta}, \vec{\xi})$ is the following:

$$\Phi^{j_1 j_2}_{KJM}(\vec{\eta}, \vec{\xi}) = Y^{j_1 j_2}_{JM}(\vec{m}, \vec{n}) \quad \Psi_{K j_1 j_2}(\Phi). \tag{4}$$

The part of this function, corresponding to J M is

$$Y^{j_1 j_2}_{JM}(\vec{m}, \vec{n}) = \sum_{m_1 m_2} C^{JM}_{j_1 m_1 j_2 m_2} Y_{j_1 m_1}(\vec{m}) \; Y_{j_2 m_2}(\vec{n}), \tag{5}$$

where

$$\vec{m} = \frac{\vec{\eta}}{\eta}, \qquad \vec{n} = \frac{\vec{\xi}}{\xi},$$

and j_1, j_2 are the angular momenta, corresponding to $\vec{\eta}$ and $\vec{\xi}$.

$$\Psi_{K j_1 j_2}(\Phi) = N_{K j_1 j_2} (\sin\Phi)^{j_1} (\cos\Phi)^{j_2} \, P^{(j_1 + \frac{1}{2}, j_2 + \frac{1}{2})}_{\frac{K - j_1 - j_2}{2}} (\cos 2\Phi). \tag{6}$$

where we used $\xi^2 + \eta^2 = 1$ and therefore could introduce

$$\cos\Phi = \xi, \qquad \sin\Phi = \eta.$$

The normalization factor $N_{Kj_1j_2}$ is well-known from the theory of Jacobi polynomials:

$$N_{Kj_1j_2} = \left[\frac{2(K+2)\Gamma(\frac{K-j_1-j_2}{2}+1)\Gamma(\frac{K+j_1+j_2}{2}+2)}{\Gamma(\frac{K-j_1+j_2}{2}+\frac{3}{2})\Gamma(\frac{K+j_1-j_2}{2}+\frac{3}{2})} \right]^{1/2} \qquad (7)$$

Let us introduce now two three-dimensional vectors $\vec{P}_i$ and $\vec{Q}_i$ having the properties

$$\vec{P}_i = \vec{Q}_i = 1$$

and (8)

$$\vec{P}_i \vec{Q}_i = 0 .$$

We can write then

$$1 = e^{2(P_i^2 - Q_i^2)} . \qquad (9)$$

where the factor 2 is introduced for the sake of convenience. For reasons, which will become clear later, we consider in the following $i\vec{Q}_i$ instead of $\vec{Q}_i$. Further, we substitute one of the P_i-s and Q_i-s by P_k and Q_k, which are connected with the original vectors by the orthogonal transformation

$$i\vec{Q}_k = \cos\phi \, i\vec{Q}_i + \sin\phi \, \vec{P}_i \; ,$$

$$\vec{P}_k = \sin\phi \, i\vec{Q}_i - \cos \vec{P}_i \; . \qquad (10)$$

(Because of the orthogonality of this transformation the properties (8) are invariant). Thus, instead of (9), we have

$$e^{-2P_iP_k\cos\phi - 2Q_iQ_k\cos\phi + 2iP_iQ_k\sin\phi + 2iQ_iP_k\sin\phi} . \qquad (11)$$

Using the plane-wave expansion

$$e^{i\vec{p}\vec{x}} = \sum_{\lambda\mu} i^{\lambda} j_{\lambda}(px)\, Y^{*}_{\lambda\mu}(p)\, Y_{\lambda\mu}(x)$$

we can re-write (11) in the form

$$\sum_{prqs} (-1)^{\frac{p+r+q+s}{2}} j_p(2iP_iP_k\cos\phi)\, j_r(2Q_iP_k\sin\phi)\, j_q(2P_iQ_k\sin\phi)\, j_s(2iQ_iQ_k\cos\phi) \tag{12}$$

$$\cdot Y^{*}_{p\pi}(\hat{P}_i)\, Y^{*}_{q\chi}(\hat{P}_i)\, Y^{*}_{r\rho}(\hat{Q}_i)\, Y^{*}_{s\sigma}(\hat{Q}_i)\, Y_{p\pi}(\hat{P}_k)\, Y_{r\rho}(\hat{P}_k)\, Y_{q\chi}(\hat{Q}_k)\, Y_{s\sigma}(\hat{Q}_k),$$

where $\hat{P}_i$, $\hat{Q}_i$, $\hat{P}_k$ and $\hat{Q}_k$ denote the three-dimensional polar angles.

Remember now the so-called Bateman expansion, which is usually given in the form (/5/ p. 974, f. 8.442 (2)).

$$J_{\nu}(az)\, J_{\mu}(bz) = \frac{\left(\frac{az}{2}\right)^{\nu}\left(\frac{bz}{2}\right)^{\mu}}{\Gamma(\mu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{az}{2}\right)^{2k} F\left(-k, -\nu-k; \mu+1; \frac{b^2}{a^2}\right)}{k!\,\Gamma(\nu+k+1)}$$

and note, that the hypergeometrical function can be substituted by the Jacobi polynomial (/5/ p. 1050, f. 8.962 (4)):

$$P_k^{(\nu,\mu)}(x) = \frac{\Gamma(k+1+\mu)}{k!\,\Gamma(1+\mu)} \left(\frac{x-1}{2}\right)^k F\left(-k, -k-\nu; \mu+1; \frac{x+1}{x-1}\right)$$

or by Wigner's d -function. Substituting

$$j_{\lambda}(x) = \sqrt{\frac{\pi}{2x}}\, J_{\lambda+\frac{1}{2}}(x),$$

we can apply these formulae to the spherical Bessel functions which occur in (12). It is easy to see, that the step from the hypergeometrical function to the Jacobi polynomial is possible only, if we consider the imaginary vector $i\vec{Q}_i$. Replacing the lengths of $\vec{P}_i, \vec{Q}_i, \vec{P}_k, \vec{Q}_k$ by unity, we can write then

$$j_p(2i\cos\phi)\, j_q(2\sin\phi) = \frac{\pi}{4}\sum_{K_1}(-1)^{K_1-\frac{q}{2}+\frac{1}{2}}(\sin\phi\cos\phi)^{-1/2}\, d^{\frac{K_1+1}{2}}_{\frac{p+q+1}{2},\,\frac{q-p}{2}}(\cos 2\phi)\cdot$$

$$\left[\left(\frac{K_1-p+q}{2}+\frac{1}{2}\right)!\left(\frac{K_1+p-q}{2}+\frac{1}{2}\right)!\left(\frac{K_1+p+q}{2}+1\right)!\left(\frac{K_1-p-q}{2}\right)!\right]^{-1/2},$$

and

$$j_s(2i\cos\phi)\, j_r(2\sin\phi) = \frac{\pi}{4}\sum_{K_2}(-1)^{K_2-\frac{s}{2}+\frac{1}{2}}(\sin\phi\cos\phi)^{-1/2}\, d^{\frac{K_2+1}{2}}_{\frac{s+r+1}{2},\,\frac{r-s}{2}}(\cos 2\phi)\cdot$$

$$\left[\left(\frac{K_2-r+s}{2}+\frac{1}{2}\right)!\left(\frac{K_2+r-s}{2}+\frac{1}{2}\right)!\left(\frac{K_2+r+s}{2}+1\right)!\left(\frac{K_2-r-s}{2}\right)!\right]^{-1/2}. \tag{13}$$

Returning now to the matrix element between two states (5), we obtain a long but simple expression for (3):

$$\frac{\pi}{8}\sum_{\substack{m_1m_2\\ m_1'm_2'}} C^{JM}_{j_1m_1j_2m_2}\, C^{JM}_{j_1'm_1'j_2'm_2'} \int \sum_{\substack{prqs\\ \pi\rho\chi\sigma}}\ \sum_{K_1K_2}(-1)^{K+\frac{q}{2}-\frac{r}{2}+1}\times$$

$$\times(\sin 2\phi)^{-1}\, d^{\frac{K_1+1}{2}}_{\frac{p+q+1}{2},\,\frac{q-p}{2}}(\cos 2\phi)\, d^{\frac{K_2+1}{2}}_{\frac{s+r+1}{2},\,\frac{r-s}{2}}(\cos 2\phi),$$

$$\times\left[\left(\frac{K_1-p+q}{2}+\frac{1}{2}\right)!\left(\frac{K_1+p-q}{2}+\frac{1}{2}\right)!\left(\frac{K_1+p+q}{2}+1\right)!\left(\frac{K_1-p-q}{2}\right)!\right]^{-1/2}$$

$$\times\left[\left(\frac{K_2-r+s}{2}+\frac{1}{2}\right)!\left(\frac{K_2+r-s}{2}+\frac{1}{2}\right)!\left(\frac{K_2+r+s}{2}+1\right)!\left(\frac{K_2-r-s}{2}\right)!\right]^{-1/2}.$$

(14)

$$\times Y^*_{p\pi}(\hat{P}_i)\,Y^*_{q\chi}(\hat{P}_i)\,Y_{j_1m_1}(\hat{P}_i)\,Y^*_{r\rho}(\hat{Q}_i)\,Y^*_{s\sigma}(\hat{Q}_i)\,Y_{j_2m_2}(\hat{Q}_i)$$

$$\times Y_{p\pi}(\hat{P}_k)\,Y_{r\rho}(\hat{P}_k)\,Y^*_{j_1'm_1'}(\hat{P}_k)\,\times$$
$$\times Y_{q\chi}(\hat{Q}_k)\,Y_{s\sigma}(\hat{Q}_k)\,Y^*_{j_2'm_2'}(\hat{Q}_k)\,d\hat{P}_i\,d\hat{Q}_i\,d\hat{P}_k\,d\hat{Q}_k\ .$$

Using the well-known properties of the spherical functions

$$Y^*_{\ell m}(\theta,\phi) = (-1)^m\,Y_{\ell -m}(\theta,\phi)$$

and

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\ d\sin\theta\ Y_{\ell_1 m_1}(\theta,\phi)\ Y_{\ell_2 m_2}(\theta,\phi)\ Y^*_{\ell_3 m_3}(\theta,\phi) =$$

$$= \left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}\right]^{1/2} C^{\ell_3 0}_{\ell_1 0 \ell_2 0}\, C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}$$

(see [6] p. 124, f. (1) and p. 131, f. (5)), (14) can be rewritten in the form

$$\sum_{\substack{prqs\\ \pi\rho\kappa\sigma}}\ \sum_{\substack{m_1'm_2'\\ m_1m_2}}\ \sum_{K_1K_2} C^{JM}_{j_1m_1j_2m_2}\, C^{JM}_{j_1'm_1'j_2'm_2'}\, C^{j_1m_1}_{p\pi q\chi}\, C^{j_2m_2}_{r\rho s\sigma}\ \times$$

$$\times C^{j_1' m_1}_{p\pi r\rho} C^{j_2' m_2'}_{q\chi s\sigma} C^{j_1 0}_{p0 q0} C^{j_2 0}_{r0 s0} C^{j_1' 0}_{p0 r0} C^{j_2' 0}_{q0 s0} \times$$

$$\cdot \left[\left(\frac{K_1 - p + q + 1}{2}\right)! \left(\frac{K_1 + p - q + 1}{2}\right)! \left(\frac{K_1 - p - q}{2}\right)! \left(\frac{K_1 + p + q}{2} + 1\right)! \right]^{-1/2} \times$$

$$\cdot \left[\left(\frac{K_2 - s + r + 1}{2}\right)! \left(\frac{K_2 + s - r + 1}{2}\right)! \left(\frac{K_2 - s - r}{2}\right)! \left(\frac{K_2 + s + r}{2} + 1\right)! \right]^{-1/2} .$$

$$\times (\sin 2\delta)^{-1}\, d^{\frac{K_1+1}{2}}_{\frac{p+q+1}{2},\, \frac{p-q}{2}} (\cos 2\delta)\, d^{\frac{K_2+1}{2}}_{\frac{r+s+1}{2},\, \frac{s-r}{2}} (\cos 2\delta) \quad . \qquad (15)$$

The parameters K_1 and K_2 are fixed by the condition $K = K_1 + K_2$. Instead of the sum over K_1, K_2 it is enough to consider a term with a definite K. The summation of the Clebsch- Gordan coefficients leads to the 9j coefficient

$$\sum_{m_i m_{ik}} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} C^{j_{34} m_{34}}_{j_3 m_3 j_4 m_4} C^{jm}_{j_{12} m_{12} j_{34} m_{34}}$$

$$\times C^{j_{13} m_{13}}_{j_1 m_1 j_3 m_3} C^{j_{24} m_{24}}_{j_2 m_2 j_4 m_4} C^{j' m'}_{j_{13} m_{13} j_{24} m_{24}} =$$

$$= \delta_{jj'} \delta_{mm'} [(2j_{12}+1)(2j_{13}+1)(2j_{24}+1)(2j_{34}+1)]^{1/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} .$$

Substituting this expression into (15), we obtain the following form for the transformation coefficient:

$$\sum_{prqs} \left[\left(\frac{K_1-p+q+1}{2}\right)! \left(\frac{K_1+p-q+1}{2}\right)! \left(\frac{K_1+p+q}{2}+1\right)! \left(\frac{K_1-p-q}{2}\right)! \right]^{-1/2}$$

$$\times \left[\left(\frac{K_2-r+s+1}{2}\right)! \left(\frac{K_2+r-s+1}{2}\right)! \left(\frac{K_2+r+s}{2}+1\right)! \left(\frac{K_2-r-s}{2}\right)! \right]^{-1/2} .$$

$$\times \left\{\left\{ \begin{matrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{matrix} \right\}\right\} (\sin 2\phi)^{-1}\, d^{\frac{K_1+1}{2}}_{\frac{q+p+1}{2},\frac{q-p}{2}}(\cos 2\phi)\, d^{\frac{K_2+1}{2}}_{\frac{r+s+1}{2},\frac{s-r}{2}}(\cos 2\phi) . \tag{16}$$

where we introduced the notation

$$\left\{\left\{ \begin{matrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{matrix} \right\}\right\} = \left[(2j_1+1)(2j_2+1)(2j'_1+1)(2j'_2+1) \right]^{1/2} \times$$

$$\times (2p+1)(2q+1)(2s+1)(2r+1) \times$$

$$\times \begin{pmatrix} p & q & j_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s & r & j_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p & r & j'_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q & s & j'_2 \\ 0 & 0 & 0 \end{pmatrix} \left\{ \begin{matrix} p & r & j'_1 \\ q & s & j' \\ j_1 & j_2 & J \end{matrix} \right\}. \tag{17}$$

Considering the whole basis function (4) instead of (5), we can write down the final form of the orthonormal transformation coefficient:

$$\langle j_1' j_2' | j_1 j_2 \rangle^{\phi}_{KJM} = \frac{\pi}{2} \left(\frac{K_1 - j_1}{2}\right)! \left(\frac{K_1 + j_1 + 1}{2}\right)! \left(\frac{K_2 - j_2}{2}\right)! \left(\frac{K_2 + j_2 + 1}{2}\right)! \cdot$$

$$\times \frac{a_{Kj_1'j_2'}}{a_{Kj_1j_2}} \sum_{prqs} \left[\left(\frac{K_1 - p + q + 1}{2}\right)! \left(\frac{K_1 + p - q + 1}{2}\right)! \left(\frac{K_1 - p - q}{2}\right)! \left(\frac{K_1 + p + q}{2} + 1\right)! \right]^{-1/2} \cdot$$

$$\times \left[\left(\frac{K_2 - s + r + 1}{2}\right)! \left(\frac{K_2 + s - r + 1}{2}\right)! \left(\frac{K_2 + r - s}{2}\right)! \left(\frac{K_2 + r + s}{2} + 1\right)! \right]^{-1/2} \cdot$$

$$\times \left\{\begin{matrix} p & r & j_1' \\ q & s & j_2' \\ j_1 & j_2 & J \end{matrix}\right\} (\sin 2\phi)^{-1} \, d^{\frac{K_1+1}{2}}_{\frac{q+p+1}{2}, \frac{p-q}{2}}(\cos 2\phi) \, d^{\frac{K_2+1}{2}}_{\frac{r+s+1}{2}, \frac{s-r}{2}}(\cos 2\phi) \tag{18}$$

with

$$a_{Kj_1j_2} = \left[\left(\frac{K - j_1 - j_2}{2}\right)! \left(\frac{K + j_1 + j_2}{2} + 1\right)! \left(\frac{K - j_1 + j_2 + 1}{2}\right)! \left(\frac{K + j_1 - j_2 + 1}{2}\right)! \right]^{1/2} .$$

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