MAY 1979 PPPL-1543

UC-20g

MASTER

RENORMALIZED COMPTON SCATTERING AND NONLINEAR DAMPING OF COLLISIONLESS DRIFT WAVES

BY

J. A. KROMMES

PLASMA PHYSICS LABORATORY

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

This work was supported by the U. S. Department of Energy Contract No. EY-76-C-02-3073. Reproduction, translation, publication, use and disposal, in whole or in part, by or for the United States Government is permitted.

Renormalized Compton Scattering and Nonlinear Damping of Collisionless Drift Waves

John A. Krommes

Plasma Physics Laboratory, Princeton University

Princeton, New Jersey 08544

A kinetic theory for the nonlinear damping of collisionless drift waves in a shear-free magnetic field is presented. The general formalism is a renormalized version of induced scattering on the ions and reduces correctly to weak turbulence theory. The approximation studied explicitly reduces to Compton scattering, systematizes the earlier calculations of Dupree and Tetreault (DT) [Phys. Fluids 21, 425 (1978)], and extends that theory to finite ion gyroradius. Certain conclusions differ significantly from those of DT. In particular, at long wavelengths the nonlinear ion "growth" rate is large and positive, proportional to $k_\perp^2 D$ both at zero and at finite gyroradius. (Here $k_\perp$ is the perpendicular wavenumber and D is the test particle diffusion coefficient.) Nevertheless, the rate of change of total mean kinetic energy because of the nonlinear interaction is small, proportional to $\bar{q}_{\parallel}^2$, where $\bar{q}_{\parallel}$ is a typical parallel wavenumber.

I. Introduction

The importance of a turbulence theory for plasma fluctuations has been recognized at least since the pioneering work of Dupree[1] and Galeev.[2] Quite naturally, the early theories were based on physical intuition and heuristic arguments. In particular, Dupree's lucid and persuasive physical arguments for the resonance broadening effect[1] (scattering of particles away from their unperturbed orbits) have influenced all later theories of weak plasma turbulence.

In the resonance broadening theories (RBT)[1,3] the turbulent fluctuations are considered to comprise a <u>fixed</u> (statistically specified) <u>bath</u> which imparts random accelerations or $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts to the particles. These particles are thus assumed to be <u>test</u> particles, in that they neither participate in the wave motions of the medium nor affect the bath in any way. Such a theory does not describe properly the nonresonant, weak turbulence limit in which the fluctuations can be systematically expanded in powers of the first-order wave fields; it can be expected that this trouble carries over to the resonant, renormalized limit as well. This fact was noted by several authors,[2,4,5] who generalized the simple RBT in a number of directions. However, although these generalizations were not unreasonable, they were asystematic and incomplete--that is, not all effects of a given order in the fluctuation intensity were included. In part this can be traced to the primitive and cumbersome nature of the mathematical techniques used.

In the usual RBT, the nonlinear dispersion relation is obtained from the linear one by the replacement $\omega \rightarrow \omega + i\omega_c$, where ω is the frequency and ω_c is the inverse of the turbulent decorrelation time. Recently, Catto[6] proposed another generalization of RBT. Catto argued heuristically that an improved theory would result if one would apply the resonance broadening approximation only to the nonadiabatic part of the fluctuating particle distribution.[7] In such a theory ω is not replaced everywhere by $\omega + i\omega_c$, reflecting the fact that only the nonadiabatic part can be resonant with the waves. Unfortunately, Catto did not provide a systematic justification for this prescription. In large measure the problem with such justification lies in the fact that the wave-particle resonance is not well-defined at finite turbulence levels; new concepts are required to effect a systematic renormalization. Furthermore, Catto considered only electrons. While Catto's arguments imply that a related correction should also exist for the ions, the form of this correction is not clear from Catto's work. For any mode such as the drift wave for which the physics of the electron and ion response is very different, one should not expect *a priori* that the form of the correction is structurally the same for both species.

Recently Dupree and Tetreault (DT)[8] considered the problem of turbulent saturation of long wavelength, collisionless drift waves in a shear-free magnetic field. They also emphasized that simple RBT was not correct for this problem. Because in their model the dominant nonlinearity arises from turbulent $\tilde{E} \times \tilde{B}$ convection, the resonance broadening prescription becomes $\omega \to \omega + ik_\perp^2 D$, where $k_\perp$ is the perpendicular wavenumber and D is the perpendicular test particle diffusion coefficient. DT pointed out that this prescription leads to an energy loss, from the waves to the ions, which persists even in the limit of purely perpendicular fluctuations ($k_{\parallel} \to 0$). Such energy transfer is unphysical; the $\underset{\sim}{E} \times \underset{\sim}{B}$ coupling is nondissipative and in itself merely redistributes energy in k space. Only for finite $k_{\parallel}$ can resonant wave-particle interaction and energy transfer occur. (For finite ion gyroradius, resonant energy transfer can in principle occur directly into the perpendicular modes; however, this requires an unrealistically large fluctuation amplitude which is not observed in experiments.)

While this argument is correct and describes an essential part of the drift wave physics, it is noteworthy that DT proved it only indirectly and only in the limit of zero gyroradius. What they showed was that at zero gyroradius any proper theory must satisfy $\langle \delta \underset{\sim}{j}_{\perp} \cdot \delta \underset{\sim}{E}_{\perp} \rangle \equiv 0$ (where $\delta \underset{\sim}{j}$ is the fluctuating current), and that the RBT violates this property. The RBT fails because it focuses on a given test wave k and treats the remaining waves as a fixed, turbulent bath, or reservoir of unlimited energy. In reality, however, the total ensemble of waves (including the test wave) is an isolated system; to conserve total energy the test wave must constantly exchange energy with its environment. RBT describes absorption of energy from the environment, but omits emission to the environment. Although this back reaction would be small for one test wave, each wave can be considered in turn to be a test wave and the effect is cumulative; the net size of the terms omitted is of the same order $[O(E^2)]$ as the term retained. DT corrected this defect by adding certain terms to the RBT for the drift-kinetic equation which preserved $\langle \delta \underset{\sim}{j}_{\perp} \cdot \delta \underset{\sim}{E}_{\perp} \rangle \equiv 0$.

That $\langle \delta \underset{\sim}{j}_\perp \cdot \delta \underset{\sim}{E}_\perp \rangle$ vanishes in the drift-kinetic description implies that the heating is proportional to $k_\parallel^2$ at zero gyroradius, but says little about the total heating at finite gyroradius. Furthermore, one must distinguish carefully between the energy transfer to or from a single mode, $\langle \delta \underset{\sim}{j} \cdot \delta \underset{\sim}{E} \rangle_k$, and the total energy transfer to the particles $\langle \delta \underset{\sim}{j}(\underset{\sim}{x},t) \cdot \delta \underset{\sim}{E}(\underset{\sim}{x},t) \rangle = \sum_k \langle \delta \underset{\sim}{j} \cdot \delta \underset{\sim}{E} \rangle_k$. (By $\langle ab \rangle_k$ for arbitrary a and b, we mean $\langle a(\underset{\sim}{x},t) b(\underset{\sim}{x}',t') \rangle$ Fourier-transformed in $\underset{\sim}{x}-\underset{\sim}{x}'$ and $t-t'$; specifically, the conjugate variables are $\{\underset{\sim}{x}-\underset{\sim}{x}', t-t'\} \leftrightarrow \{\underset{\sim}{k}, \omega_k\} \equiv k$.) Only the total transfer need vanish as $k_\parallel \to 0$; for any given wavenumber, $\langle \delta \underset{\sim}{j} \cdot \delta \underset{\sim}{E} \rangle_k$ can be large. There is no contradiction because particles can absorb energy from one mode and re-emit into another. A familiar example occurs in the weak turbulence theory of induced scattering, where total plasmon number and energy are conserved by the nonlinear interaction although the individual nonlinear growth rates are finite. (By "nonlinear growth rate" we mean the nonlinear part $\gamma_{\underset{\sim}{k}}^{(n)}$ of the coefficient in the rate equation for the plasmon number density $N_{\underset{\sim}{k}}$: $\partial_t N_{\underset{\sim}{k}} = 2\gamma_{\underset{\sim}{k}} N_{\underset{\sim}{k}} + \ldots$. Of course, $\gamma_{\underset{\sim}{k}}^{(n)}$ can be positive or negative.) Now what DT actually computed was the nonlinear growth rate $\gamma_{\underset{\sim}{k}}^{(n)}$ in the long wavelength limit ("Markovian approximation"). They found that, with the new terms added, $\gamma_{\underset{\sim}{k}}^{(n)} \sim -(k_\parallel v_{ti}/\omega)^2 k_\perp^2 D$. Unlike simple RBT, which predicts $\gamma_{\underset{\sim}{k}}^{(n)} \sim -k_\perp^2$, the new result vanishes as $k_\parallel^2$. DT used this fact to argue for the consistency of their theory and to support their physical picture of total parallel heating. However, we do not agree with their result. First of all, no <u>a priori</u> conclusions about the total heating can be drawn from the long-wavelength limit, because that limit does not hold uniformly over the spectrum and a sum over all k is required. More importantly we have recomputed $\gamma_{\underset{\sim}{k}}^{(n)}$ in the long-wavelength limit and find that

$$\gamma_{\underset{\sim}{k}}^{(n)} \sim + k_{\perp}^{2} D \quad ,$$

both at zero and at finite ion gyroradius. The difference in sign between our result and the simple resonance broadening theory should be noted. The positivity of our result reflects the fact that the induced scattering processes favor energy transfer to long wavelengths. Furthermore, we can exhibit the term in our theory whose asystematic omission leads to the DT result.

Although $\gamma_{\underset{\sim}{k}}^{(n)}$ is not explicitly proportional to $k_{\parallel}^{2}$, we can eschew the Markovian approximation and show that when the non-Markovian version of $\gamma_{\underset{\sim}{k}}^{(n)}$ is summed appropriately over all modes, the result for $\langle \delta \underset{\sim}{j} \cdot \delta \underset{\sim}{E} \rangle$ is indeed proportional to the square of a typical parallel wavenumber and thus correctly vanishes in the perpendicular limit. Futhermore, let us define the plasmon number density $N_{\underset{\sim}{k}}$ by

$$N_{\underset{\sim}{k}} \equiv \frac{1}{8\pi} \left(\frac{\partial \varepsilon_r}{\partial \omega(\underset{\sim}{k})} \right) |\underset{\sim}{k}|^2 \langle \delta\phi^2 \rangle_{\underset{\sim}{k}} \quad , \tag{1}$$

where ε_r is the real part of the dielectric and $\varepsilon_r[\underset{\sim}{k}, \omega(\underset{\sim}{k})] = 0$, and the total modal energy E by

$$E = \sum_{\underset{\sim}{k}} \omega(\underset{\sim}{k}) N_{\underset{\sim}{k}} \quad . \tag{2}$$

We can then show that the usual conservation law

$$\frac{\partial E}{\partial t} = - \langle \delta \underset{\sim}{j} \cdot \delta \underset{\sim}{E} \rangle \tag{3}$$

holds, which reduces to the statement of energy conservation

$$\frac{\partial}{\partial t} (\langle K \rangle + E) = \text{(divergence of heat flow)} \quad , \tag{4}$$

where $\langle K \rangle$ is the mean kinetic energy:

$$\langle K \rangle = \int d\underset{\sim}{v} \, \tfrac{1}{2} mn \, v^2 \langle f \rangle \quad , \tag{5}$$

n being the mean density and f being the distribution function.

That $\gamma_{\underset{\sim}{k}}^{(n)}$ is large while the total heating is small means that spectral energy transfer between stable and unstable modes is likely to be important. However, a number of details concerned with the effects of shear, electron nonlinearities, and an effective energy sink at long wavelengths remain to be resolved before this process can be studied in depth. The present calculation should therefore be viewed as a preliminary model calculation which illustrates several important aspects of the nonlinear drift wave physics.

One can understand qualitatively why most approaches to the plasma turbulence questions have fallen short of a systematic derivation. Almost always, the approaches begin with the usual coupled equations for the statistical mean $\langle f \rangle$ and the fluctuation δf of the particle distribution. Generally, truncations or other approximations are made on the equation for δf . Straightforward truncations, of course, lead to secularities. One can attempt more sophisticated procedures such as summation of infinite subseries of perturbation theory,[9] or reorganization of perturbation theory, via operator techniques,[1,5] followed by some truncation. In such methods it is very difficult to understand what has been omitted. Mathematically, this is related to the fact that the expansions are probably asymptotic at best. There is also a more philosophical point. The problem is that, in the statistical sense, δf is not an <u>observable</u>. Observables are quantities, like the fluctuation spectrum $\langle \delta \underset{\sim}{E} \delta \underset{\sim}{E} \rangle_{\underset{\sim}{k},\omega}$ which result from averages over a statistical ensemble; they are thus smooth functions of their arguments. On the other hand, functions like δf describe the "microscopic" dynamics of individual realizations and can thus be quite jagged and, at certain phase space points, very large. If one wishes to employ any sort of ordering, it would be much better to have a formally exact set of coupled equations for relevant <u>observables</u> (which have smooth shapes and definite sizes), rather than for fluctuations (whose shapes and sizes are indefinite).

Such a theory was provided by the fundamental and elegant theory of Martin, Siggia, and Rose (MSR).[10] MSR defined the observables by means of functional derivatives of a certain generating functional, a procedure quite analogous to the derivation of equilibrium thermodynamic variables by derivatives of the free energy. In the MSR theory, and also physically, the most important observables are the mean distribution $\langle f \rangle$, the correlation function $C(1,2) \equiv \langle \delta f(1) \delta f(2) \rangle$, and the mean response function $R(1;2) \equiv \langle \delta f(1)/\delta \hat{n}(2) \rangle$. Here "1" denotes the complete set of phase space variables including the time, and $\hat{\eta}$ is a nonrandom source inserted on the right-hand side of the relevant dynamical equation, quadratically nonlinear in the dependent variable Ψ: schematically, $\partial_t \Psi = a\Psi + b\Psi\Psi + \hat{\eta}$, where a and b may be operators. Examples of such equations are, of course, the Vlasov equation and the drift-kinetic equation. Additional observables of importance are certain vertex functions $\Gamma_i(1,2,3)$, which describe the three-point correlations in the system. By means of straightforward functional manipulations, MSR find coupled equations, relating $\langle f \rangle$, C, R, and Γ_i, which are exact for Gaussian initial conditions. (The assumption of Gaussian initial conditions is usually adequate; it can be relaxed if necessary.[11]) Furthermore, MSR demonstrate that a certain reasonable expansion around Gaussian statistics of the fully interacting nonlinear system leads to the well-known Direct Interaction Approximation (DIA).[12,13]

The DIA is an extremely interesting and useful approximation. Kraichnan has shown[12] that a model dynamical system exists for which the DIA represents an <u>exact</u> statistical description. This implies that the DIA satisfies certain realizability constraints (e.g., the energy spectrum is positive definite; time correlations decay to zero). The DIA is thus a well-motivated starting point from which to derive further approximate theories. Orszag and Kraichnan[14] have written down the DIA specifically for Vlasov plasma by using the Kraichnan method of stochastic models. This important paper is often overlooked.[15]

Krommes[16] was the first to discuss the possibility of employing the MSR formalism for plasma physics applications. In particular, he pointed out that the DIA reduced naturally to renormalized Vlasov weak turbulence theory. Krommes also attempted to give a precise definition of the terms "coherent" and "incoherent" introduced by Dupree[17] in his discussions of renormalizations based on the fluctuating distribution δf. However, while the formal factorization of the DIA given by Krommes was exact, the propagator which appeared did not correspond to the test particle propagator $g \equiv \langle \delta f/\delta \hat{\eta} \rangle |_{\langle \underset{\sim}{E} \rangle}$. This was discussed by Dubois and Espedal,[18] who provided an alternative, more physical factorization in which both g and the dielectric function ϵ, relating the mean response of the nonlinear medium to a linear external disturbance, appear naturally. We have recently discussed[19] aspects of the relation of the Dubois factorization to renormalized weak turbulence theory. We shall utilize results of that discussion in the present paper.

In the present work, we discuss and extend the DT model of collisionless drift wave turbulence by beginning with the DIA or, more specifically, with the renormalized weak turbulence theory approximation to the DIA discussed by Krommes and Kleva.[19] In Dupree's language, the approximation corresponds to the neglect of all incoherent noise; we call this the Coherent Approximation to the DIA (DIAC). Although the DT remark in physical terms on the neglect of incoherent noise, there are some nontrivial differences between their procedure and ours. Dupree attempts to define incoherent noise in terms of certain ("mode-coupling") contributions to the fluctuating distribution δf. Any such definition is of necessity imprecise because of the nonobservable nature of δf. We, on the other hand, can give a precise mathematical definition of the incoherent source term in terms of expressions involving only observables. The Dupree procedure is analogous to a Langevin

$$g^{(o)} \rightarrow g \quad , \tag{10a}$$

$$f^{(o)} \rightarrow \bar{f} \equiv \langle f \rangle + \delta \bar{f} \quad . \tag{10b}$$

Essentially all renormalized theories enforce some form of Eq. (10a). However, many early theories[1,3] replace Eq. (10b) by $f^{(o)} \rightarrow \langle f \rangle$, thus omitting $\delta\bar{f}$. It was originally Galeev[2] who emphasized that $\delta\bar{f}$ was necessary to ensure energy conservation. The "new" terms which DT add and which we shall also discuss represent a certain approximate form of $\delta\bar{f}$ appropriate to the drift-wave problem. Catto's procedure of resonance broadening only the nonadiabatic part of the distribution also follows from a certain approximate treatment of $\delta\bar{f}$ (see Sec. VI).

In constructing a renormalized theory, it is very useful to keep in mind the relation of the renormalization to the limit of weak turbulence. For example, away from the linear wave-particle resonance $\omega_k - k_{\parallel} v_{\parallel} = 0$ one can expand the solution of Eq (6):

$$g_k \simeq g_k^{(o)} - g_k^{(o)} \Sigma_k g_k^{(o)} \quad . \tag{11}$$

When this is inserted into Eq. (8), the first term of Eq. (11) produces the linear dielectric while the second term generates part of the induced scattering processes (with unsymmetrized matrix elements) when the order E^2 version of Σ_k is used. The remaining part of the induced scattering (necessary for symmetrization) arises from the $O(E^2)$ approximation to $\delta\bar{f}_k$. Furthermore, we can write both in weak and renormalized turbulence theory

$$\Sigma = \Sigma^{(d)} + \Sigma^{(p)} ,$$
$$\delta\bar{f} = \delta\bar{f}^{(d)} + \delta\bar{f}^{(p)} , \tag{12}$$

where "d" stands for diffusion and "p" stands for polarization. In weak turbulence theory, the diffusion terms generate Compton scattering while the polarization terms generate nonlinear scattering from the shielding clouds. In the renormalized theory $\Sigma^{(d)}$ represents the familiar orbit diffusion effect of the turbulent fluctuations on the test particles while $\Sigma^{(p)}$ represents polarization drag; $\delta\bar{f}^{(d)}$ and $\delta\bar{f}^{(p)}$ describe the associated inverse effects of the test particles on the medium. In the DT theory and also the one we present here, the polarization terms are neglected. Although it is easy to write down formal expressions for these terms, it is difficult to evaluate their practical effects. Preliminary estimates and analogies with weak turbulence theory suggest that omission of the polarization terms may not alter the qualitative conclusions of the present paper significantly, particularly at finite gyroradius. However, substantial quantitative errors are likely to arise from this omission. We will address the polarization terms in detail in a future paper.

The remainder of the paper is organized as follows. In Sec. II we briefly discuss the Coherent Approximation to the DIA and set down the relevant formulas for $\Sigma^{(d)}$ and $\delta\bar{f}^{(d)}$. In Sec. III we apply the DIAC to the drift-kinetic equation and review the DT calculation. In Sec. IV we discuss certain conservation laws and energetics of the DIAC and find an expression for the total heating rate $\langle\delta\underset{\sim}{j}\cdot\delta\underset{\sim}{E}\rangle$. In Sec. V we find the nonlinear growth rate in the limit of long wavelength, both at zero and at finite gyroradius. We summarize the work in Sec. VI. Finally, we devote the Appendix to a review of some aspects of Compton scattering of dri waves which emphasizes the similarities between the renormalized and unrenormalized theories.

II. Statistical Theory of the Vlasov Equation: Coherent Approximation to the DIA

We tabulate here the important ideas and formulas which we will need in the subsequent sections. The discussion is brief and incomplete; further details can be found in Ref. 19 and in Refs. 10, 16, and 18.

Consider the Vlasov equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f + \left(\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v}\times\underset{\sim}{B}_o\right)\cdot\underset{\sim}{\partial} f = 0 \quad . \tag{13}$$

Here $\underset{\sim}{\partial} \equiv (q/m)\partial/\partial\underset{\sim}{v}$, $\underset{\sim}{B}_o$ is constant, and the electrostatic electric field $\underset{\sim}{E} \equiv -\underset{\sim}{\nabla}\phi$ is determined from Poisson's equation:

$$\nabla^2\phi(\underset{\sim}{x},t) = -4\pi\sum_s (ne)_s\int d\underset{\sim}{v}\, f^{(s)}(\underset{\sim}{x},\underset{\sim}{v},t) \quad . \tag{14}$$

Here "s" is a species label equal to either "e" or "i", n_s is the mean density and e_s is the signed charge of species s . It is convenient to adopt a compact argument notation wherein the set $\{\underset{\sim}{x}_1,\underset{\sim}{v}_1,s_1,t_1\}$ is represented by the single number "1" and the integration-summation convention for repeated indices is adopted.

Thus, we may represent the solution of Eq. (14) by the operator expression

$$\phi = \Phi f \quad , \tag{15a}$$

which appears in x space as

$$\phi(1) = \Phi(1,\bar{1})f(\bar{1}) \quad , \tag{15b}$$

or in k space as

$$\phi_k = \Phi_k(\bar{1})f_k(\bar{1}) \quad . \tag{15c}$$

(When a k subscript appears explicitly, the explicit coordinate dependence and summation convention ref... to just species and velocity.) Explicitly,

$$\Phi_k(\bar{1}) = \frac{4\pi}{|k|^2}(ne)_{\bar{s}} \quad . \tag{16}$$

We also define the zeroth-order particle propagator $g^{(o)}$ by

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} + \frac{1}{c}\underset{\sim}{v}\times\underset{\sim}{B}\cdot\underset{\sim}{\partial}\right)_1 g^{(o)}(1;1') = \delta(1-1') \quad . \tag{17}$$

An explicit representation of $g^{(o)}$ is given in Eq. (62). Furthermore, define the mean response function R by

$$R(1;1') \equiv \langle \delta f(1)/\delta\hat{\eta}(1')\rangle_{|\hat{\eta}=0} \quad , \tag{18}$$

where $\hat{\eta}$ is an arbitrary source function inserted on the right-hand side of Eq. (13). One can show[19] that

$$R_k(1;1') = g_k(1;\bar{1})[\delta(1-1') + (i\underset{\sim}{k}\cdot\underset{\sim}{\partial}\bar{f}_k)_{\bar{1}}\varepsilon_k^{-1}\phi_k(2)g_k(2;1')] \quad , \tag{19}$$

where the renormalized propagator g and the renormalized distribution $\bar{f}$ obey Eqs. (6), (10b), and (12). It is possible[10] to write explicit expressions for Σ and $\delta\bar{f}$ in terms of $C_k(1,1') \equiv \langle\delta f(1)\delta f(1')\rangle_k$, $\langle\delta f\delta\phi\rangle_k$, and $I_k \equiv \langle\delta\phi\delta\phi\rangle_k$. The correlation function C obeys an equation, sometimes called the Wyld equation, of the form[16]

$$C = R\tilde{\Sigma}R^T \quad , \tag{20}$$

where "T" denotes transpose $[R(1;2)^T = R(2;1)]$ and $\tilde{\Sigma}$ is another functional of the observables for which an explicit expression can also be written. Equation (20) is the formal representation of the physical statement that fluctuations are driven by a certain "incoherent noise" source $\tilde{\Sigma}$. This can be seen more clearly when the balance equation for the spectrum I is formed. Using the identity

$$\Phi R = \varepsilon^{-1}\phi g \quad , \tag{21}$$

one finds

$$I_k = \frac{\phi_k(g_k\tilde{\Sigma}_k g_k^T)\phi_k^T}{|\varepsilon_k|^2} \quad , \tag{22}$$

which we may write suggestively as

$$I_k = \frac{\langle\delta\tilde{\phi}\delta\tilde{\phi}\rangle_k}{|\varepsilon_k|^2} \; . \tag{23}$$

[It is to be understood that $\langle\delta\tilde{\phi}\delta\tilde{\phi}\rangle$ is just a shorthand notation for the numerator in Eq. (22).] In particular examples, Dupree has written an expression of the form (23). The source term $\langle \delta\tilde{\phi}\delta\tilde{\phi} \rangle$ describes n-wave coupling, clumps, etc.

In this paper we shall be concerned only with the dielectric properties of the turbulent medium and shall ignore the incoherent noise altogether:

$$\tilde{\Sigma} = 0 \quad . \tag{24}$$

This is called the Coherent Approximation[19] and has several consequences. First, in order that the spectrum not vanish identically, one must satisfy the operator expression $\varepsilon I = 0$. As is well known, when this is written out explicitly for a medium slightly inhomogeneous in space time, one finds that

$$I_k \simeq 2\pi I_{\underset{\sim}{k}} \delta[\omega_k - \omega(\underset{\sim}{k})] \quad , \tag{25}$$

where

$$\mathrm{Re}\, \varepsilon[\underset{\sim}{k}, \omega(\underset{\sim}{k})] = 0 \quad , \tag{26}$$

and that

$$\frac{\partial I_{\underset{\sim}{k}}(t)}{\partial t} = 2\gamma_{\underset{\sim}{k}} I_{\underset{\sim}{k}} + \text{(additional wave-kinetic terms)} \quad , \tag{27}$$

where

$$\gamma_{\underset{\sim}{k}} \equiv - \frac{\mathrm{Im}\, \varepsilon[\underset{\sim}{k}, \omega(\underset{\sim}{k})]}{\mathrm{Re}\, \partial\varepsilon[\underset{\sim}{k}, \omega(\underset{\sim}{k})]/\partial\omega(\underset{\sim}{k})} \quad . \tag{28}$$

Another consequence of the Coherent Approximation is that the relation $R^{-1}C = 0$ allows us to write

$$C_k = g_k(i\underset{\sim}{k} \cdot \underset{\sim}{\partial} \bar{f}_k) \langle \delta\phi\delta f \rangle_k \tag{29}$$

and, by applying the Φ operator from the right to Eq. (29),

$$\langle \delta f \delta \phi \rangle_k = g_k (i\underset{\sim}{k} \cdot \underset{\sim}{\partial} \bar{f}_k) I_k \quad . \tag{30}$$

Thus, all correlations involving δf can be expressed in terms of I_k. The procedure which Dupree uses to renormalize the Vlasov equation is closely related to the Coherent Approximation.

The results (29) and (30) can be used to simplify the forms of Σ and $\delta \bar{f}$. In the DIA (on which we do not expound here) the expressions for the diffusive parts of these operators become[19]

$$\Sigma_k^{(d)}(1,\bar{1}) = -\sum_q \underset{\sim}{q} \cdot \underset{\sim}{\partial}_1 g_{k-q}(1;2) I_q \underset{\sim}{q} \cdot \underset{\sim}{\partial}_2 \delta(2-\bar{1}) \quad , \tag{31}$$

$$\underset{\sim}{\partial} \delta \bar{f}_k^{(d)}(1) = \sum_q \underset{\sim}{q} \cdot \underset{\sim}{\partial}_1 g_{k-q}(1;2) I_q \underset{\sim}{\partial}_2 g_{-q}(2;3) \underset{\sim}{q} \cdot \underset{\sim}{\partial}_3 \bar{f}_{-q} \quad . \tag{32}$$

Similar expressions exist for the polarization terms, but we shall not study those here.

III. Review of the Dupree-Tetreault Calculation

In this section we introduce the basic model for electrostatic collisionless drift waves studied by DT, and review their calculations and approximations. We give this review in some detail because our description is kinetic while theirs was based on the fluid equations. Of course, each step in the kinetic theory has an analogue in the fluid treatment.

DT assume that the electrons obey linear dynamics and are destabilizing. That is, if we write the dielectric in terms of the susceptibilities $\chi^{(s)}$,

$$\varepsilon = 1 + \chi^{(e)} + \chi^{(i)} \quad , \tag{33}$$

the electron contribution $\gamma_k^{(e)}$ to the total growth rate will be positive,

$$\gamma_k^{(e)} = - \frac{\mathrm{Im}\, \chi^{(e)} [\underset{\sim}{k}, \omega(\underset{\sim}{k})]}{\mathrm{Re}\, \partial\varepsilon[\underset{\sim}{k}, \omega(\underset{\sim}{k})]/\partial\omega(\underset{\sim}{k})} > 0 \quad . \tag{34}$$

We shall write

$$\chi^{(e)} = \frac{1}{(k\lambda_{De})^2} (1 - i\alpha_k) \quad , \tag{35}$$

where $\lambda_{Ds} \equiv (T/4\pi n e^2)_s^{1/2}$ is the Debye length of species s. Basically, $\alpha_k = \gamma_k^{(e)}/\omega(k)$.

We assume a constant magnetic field $\underset{\sim}{B}_o$ in the z direction and a constant density gradient in the x direction. Shear is ignored. The drift frequency of species s is defined by

$$\begin{aligned} \omega_*^{(s)}(\underset{\sim}{k}) &\equiv \left(\frac{cT}{eB}\right)_s \underset{\sim}{k} \cdot \hat{z} \times \underset{\sim}{\nabla} \ln n_s \\ &= - k_y \left(\frac{cT}{eB}\right)_s L_n^{-1} \quad , \end{aligned} \tag{36}$$

where $L_n^{-1} \equiv - \nabla_x \ln n$. In the DT model the ions are assumed to obey the drift-kinetic equation (limit of zero gyroradius):

$$\frac{\partial f}{\partial t} + \left(v_\parallel \frac{\partial}{\partial z} + \underset{\sim}{V}_E \cdot \underset{\sim}{\nabla} \right) f + E_\parallel \partial_\parallel f = 0 \quad , \tag{37}$$

where the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift velocity $\underset{\sim}{V}_E$ is defined by

$$\underset{\sim}{V}_E \equiv (c/B)\underset{\sim}{E} \times \hat{z} = -(c/B)\hat{z} \times (i\underset{\sim}{k})\delta\phi_k \quad . \tag{38}$$

(In Secs. IV and V we shall consider the more relevant case of finite ion gyroradius, but here we wish to follow the DT calculation as closely as possible to exhibit clearly the parallels and differences in the formalisms.) We take the mean distribution to be Maxwellian:

$$\langle f(v_{||})\rangle = [n(x)/n] f_M(v_{||}) \quad ,$$

$$f_M(v_{||}) \equiv (2\pi v_t^2)^{-1/2} \exp(-v_{||}^2/2v_t^2) \quad , \tag{39}$$

where $v_t^2 \equiv T/m$. Spatial variations of fluctuations in the direction perpendicular to $\underset{\sim}{B}$ are taken to be short compare to the equilibrium scale length: $k_\perp L_n >> 1$.

The renormalized theory for Eq. (37) follows immediately from that for Eq. (13) with the transcription

$$g_k^{(o)} \rightarrow \bar{g}_k^{(o)} \equiv [-i(\omega_k - k_{||}v_{||})]^{-1} \quad , \tag{40}$$

$$\underset{\sim}{\partial} \rightarrow \hat{z}\partial_{||} + (c/B)\hat{n} \times \underset{\sim}{\nabla} \quad . \tag{41}$$

Following DT we shall ignore the effects of parallel nonlinearities. We then have the more explicit rule

$$\underset{\sim}{k}\cdot\underset{\sim}{\partial}\langle f\rangle \rightarrow \left(\frac{e}{T}\right)(\omega_* - k_{||}v_{||}) f_M \quad , \tag{42a}$$

$$\underset{\sim}{k}\cdot\underset{\sim}{\partial} g_q \rightarrow (c/B)\underset{\sim}{k}\cdot\hat{z} \times (i\underset{\sim}{q})\bar{g}_q \quad , \tag{42b}$$

and Eqs. (8), (31), and (32) become

$$\chi_k^{(i)} = -i\Phi_k^{(i)}\bar{g}_k\left[\left(\omega_*^{(i)}(\underset{\sim}{k}) - k_\parallel v_\parallel\right)\left(\frac{e}{T}\right)_i f_M + \underset{\sim}{k}\cdot\underset{\sim}{\partial}\delta f_k^{(d)}\right] \quad , \tag{43}$$

$$\Sigma_k^{(d)} = \sum_q M_{\underset{\sim}{k},\underset{\sim}{q}}\,\bar{g}_{k-q}\, I_q \quad , \tag{44}$$

$$\underset{\sim}{k}\cdot\underset{\sim}{\partial}\delta\bar{f}_k^{(d)} = -\sum_q M_{\underset{\sim}{k},\underset{\sim}{q}}\,\bar{g}_{k-q}\, I_q\, \bar{g}_{-q}\left[\left(\frac{e}{T}\right)[\omega_*(\underset{\sim}{q}) - q_\parallel v_\parallel] f_M + \underset{\sim}{q}\cdot\underset{\sim}{\partial}\delta\bar{f}_{-q}\right] \quad , \tag{45}$$

where we have introduced the coupling coefficient

$$M_{\underset{\sim}{k},\underset{\sim}{q}} \equiv (c/B)^2 |\hat{n}\cdot(\underset{\sim}{k}\times\underset{\sim}{q})|^2 \quad . \tag{46}$$

Consider long-wavelength fluctuations, $|\underset{\sim}{k}| \ll |\bar{\underset{\sim}{q}}|$, where $\bar{\underset{\sim}{q}}$ is a typical wavenumber of the spectrum. One can then write

$$\Sigma_k^{(d)} \simeq k_\perp^2 D \quad , \tag{47}$$

where

$$D = \sum_q D_q \quad , \tag{48a}$$

$$D_q \equiv \tfrac{1}{2}(c/B)^2 \mathrm{Re}(\bar{g}_q) q_\perp^2 I_q \quad . \tag{48b}$$

In writing expressions (48) we assumed that the spectrum is isotropic and averaged over the angle between $\underset{\sim}{k}_\perp$ and $\underset{\sim}{q}_\perp$. We also used $g_{-q} = g_q^*$ to show that D is real. Furthermore, we can use the identity

$$\bar{g}_k(\omega_* - k_\parallel v_\parallel) = i + \bar{g}_k\left(\omega_*(\underset{\sim}{k}) - \omega_k - ik_\perp^2 D\right) \tag{49}$$

to write Eq. (43) as

$$\chi_k^{(i)} = \frac{1}{(k\lambda_{Di})^2} - \frac{i}{(k\lambda_{Di})^2} \int dv_{\parallel} \bar{g}_k \left[(\omega_*^{(i)}(\underset{\sim}{k}) - \omega_k - ik_\perp^2 D) f_M + \left(\frac{T}{e}\right) \underset{\sim}{k} \cdot \underset{\sim}{\partial} \delta \bar{f}_k^{(d)} \right] . \qquad (50)$$

To simplify expression (45), we again make the Markovian approximation and also neglect $\underset{\sim}{q} \cdot \underset{\sim}{\partial} \delta f_{-q}^{(d)}$:

$$\left(\frac{T}{e}\right) \underset{\sim}{k} \cdot \underset{\sim}{\partial} \delta \bar{f}_k^{(d)} \simeq - k_\perp^2 \sum_q D_q \bar{g}_{-q} [\omega_*^{(i)}(\underset{\sim}{q}) - q_{\parallel} v_{\parallel}] f_M \quad . \qquad (51)$$

Noting that $\omega_*^{(i)} = -\tau^{-1} \omega_*^{(e)}$, where $\tau \equiv T_e/T_i$, using the fact that $\omega_q \simeq \omega_*^{(e)}(\underset{\sim}{q})$ (true only in the zero gyroradius limit), and recalling that $\omega_q \gg q_{\parallel} v_{\parallel}$, we can approximate

$$\bar{g}_{-q} [\omega_*^{(i)}(\underset{\sim}{q}) - q_{\parallel} v_{\parallel}] \simeq i\tau^{-1} \quad . \qquad (52)$$

To arrive at Eq. (52) we have also neglected $q_\perp^2 D$ in $\bar{g}_{-q}$ following an implied argument of DT that this term is small near the stochasticity threshold (for direct interaction of the waves with the bulk ions). This approximation is crucial and we do not believe it is consistent. We shall return to this point in Sec. V; however, we continue here with the review of the DT calculation. With Eq. (52), Eq. (51) simplifies to

$$\underset{\sim}{k} \cdot \underset{\sim}{\partial} \delta \bar{f}_k^{(d)} \simeq - i (k_\perp^2 D) \tau^{-1} \left(\frac{e}{T}\right)_i f_M \quad , \qquad (53)$$

and the susceptibility becomes

$$\chi_k^{(i)} = \frac{1}{(k\lambda_{Di})^2} + \frac{1}{(k\lambda_{Di})^2}\left(\frac{\omega_k - \omega_*^{(i)} + ik_\perp^2 D(1+\tau^{-1})}{\sqrt{2}\, k_\parallel v_{ti}}\right) Z\left(\frac{\omega + ik_\perp^2 D}{\sqrt{2}\, k_\parallel v_{ti}}\right) \quad , \tag{54}$$

where

$$Z(w) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} du \, \frac{\exp(-u^2)}{u-w} \quad (\mathrm{Im}\, w > 0) \tag{55a}$$

$$\sim -\frac{1}{w}\left(1 + \frac{1}{2w^2}\right) \qquad (|w| >> 1) \quad . \tag{55b}$$

The dispersion relation then follows from quasineutrality:

$$\chi_k^{(e)} + \chi_k^{(i)} \simeq 0 \quad . \tag{56}$$

The solution of Eq. (56) is straightforward if one treats both α and $(k_\parallel v_{ti}/\omega)^2$ as small and also neglects terms of order αD or D^2. One finds

$$\frac{\omega(\underset{\sim}{k})}{\omega_*^{(e)}} \simeq 1 + (1+\tau)\left(\frac{k_\parallel v_{ti}}{\omega_*^{(e)}}\right)^2 \quad , \tag{57a}$$

$$\frac{\gamma_{\underset{\sim}{k}}}{\omega_*^{(e)}} \simeq \alpha_k - (1+\tau)\left(\frac{k_\parallel v_{ti}}{\omega_*^{(e)}}\right)^2 \left(\frac{k_\perp^2 D}{\omega_*^{(e)}}\right) \quad . \tag{57b}$$

Noting that DT took $\tau = 1$ and defined their thermal velocity to be larger than ours by $\sqrt{2}$, we verify that Eqs. (57) agree with their result.

In the calculations leading to the form (57b), sensitive cancellations occur between dominant terms of order $k_\perp^2 D$. This can be seen by examining the form (54). If we retain only the first term of the asymptotic expansion (55b) and set $\omega_* \simeq \omega_*^{(e)}$, a term $\omega_*^{(e)} + ik_\perp^2 D$ cancels between numerator and denominator in Eq. (54) and $\chi_k^{(i)}$ becomes purely real. Such cancellations are suspicious. They imply that the calculation may be sensitive to the precise way in which the imaginary parts are handled. Furthermore, the result (57b) is also sensitive to the fact that $\omega_k \simeq \omega_*^{(e)}$, which does not hold in the finite gyroradius case. These problems motivate us to examine the calculation more critically and in the absence of the zero gyroradius approximation. We begin by examining some consequences of the DIAC which do not depend on the Markovian approximation.

IV. Symmetries and Fiergetics

As DT emphasized in somewhat different language, it must be true in any sensible theory that the equal time, equal space point correlation $<\delta V_E \cdot \nabla(\delta f)\delta\phi>$ vanishes identically, because by statistical stationarity

$$<\delta \underset{\sim}{V}_E(\underset{\sim}{x},t) \cdot \underset{\sim}{\nabla}[\delta f(\underset{\sim}{x},v_\parallel,t)]\delta\phi(\underset{\sim}{x},t)> = -<\delta f \delta \underset{\sim}{V}_E \cdot \underset{\sim}{\nabla}\delta\phi> = <\delta f \delta \underset{\sim}{V}_E \cdot \delta \underset{\sim}{E}> \equiv 0 \quad , \tag{58}$$

by the orthogonality of the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift with the field. It is easy to check that this identity remains true in the DIAC. We note that the function on the left-hand side of Eq. (58) is the diagonal version of the displaced correlation $<\delta \underset{\sim}{V}_E(\underset{\sim}{x},t) \cdot \underset{\sim}{\nabla}[\delta f(\underset{\sim}{x},v_\parallel,t)]\delta\phi(\underset{\sim}{x}',t')>$, so that

$$<\delta \underset{\sim}{V}_E(\underset{\sim}{x},t) \cdot \underset{\sim}{\nabla}[\delta f(\underset{\sim}{x},v_\parallel,t)]\delta\phi(\underset{\sim}{x},t)> = \sum_k <[\delta \underset{\sim}{V}_E \cdot \underset{\sim}{\nabla}(\delta f)]\delta\phi>_k \quad . \tag{59}$$

The latter triplet correlation is the sum of all nonlinear terms in the evolution equation for $\langle \delta f \delta \phi \rangle_k$, which we obtain by multiplying Eq. (30) by g^{-1}. We find

$$\langle \delta \underset{\sim}{V}_E \cdot \underset{\sim}{\nabla} (\delta f) \delta \phi \rangle = \sum_k \left[\Sigma_k^{(d)} \langle \delta f \delta \phi \rangle_k - i \underset{\sim}{k} \cdot \underset{\sim}{\partial} \delta \bar{f}_k^{(d)} \langle \delta \phi \delta \phi \rangle_k \right]$$

$$= \sum_k \left[\Sigma_k^{(d)} g_k i \underset{\sim}{k} \cdot \underset{\sim}{\partial} \bar{f}_k^{(d)} - i \underset{\sim}{k} \cdot \underset{\sim}{\partial} \delta \bar{f}_k^{(d)} \right] I_k$$

$$= i \sum_{k,q} M_{\underset{\sim}{k},\underset{\sim}{q}} \, g_{k-q} I_k I_q \left(g_k \underset{\sim}{k} \cdot \underset{\sim}{\partial} \bar{f}_k^{(d)} + g_{-q} \underset{\sim}{q} \cdot \underset{\sim}{\partial} \bar{f}_{-q}^{(d)} \right)$$

$$\equiv 0 \quad , \tag{60}$$

where the last step followed upon interchanging $\{k,q\} \to \{-q,-k\}$. This proof casts into our language of statistical observables a similar calculation of DT. The proof remains true in the Markovian limit, which consists merely of neglecting the k in g_{k-q}, or when the polarization terms are included.

Of more direct interest is the net ion heating rate $\langle \delta \underset{\sim}{j}^{(i)} \cdot \delta \underset{\sim}{E} \rangle = \sum_k \langle \delta \underset{\sim}{j}^{(i)} \cdot \delta \underset{\sim}{E} \rangle_k$. This quantity was not computed by DT and indeed is not accesible from a Markovian theory, as we remarked in the Introduction. Although DT state that when a certain "stochastic threshold" is exceeded wave energy goes "directly" into the ions, implying that one could compute the heating wavenumber by wavenumber, this assertion must be viewed cautiously. The assumption of stochasticity merely justifies the usual expansions about Gaussian statistics (the random phase approximation in WTT). In weakly turbulent induced scattering, for example, the beat resonances are stochastic and the $\gamma_k^{(n)}$'s are individually finite, yet the nonlinear interaction conserves total plasmon energy, $\sum_k \omega(k)\gamma_k^{(n)} N_k = 0$. Here, there is no "direct" energy channel from the waves to the particles although the particles do participate in the turbulent motions. In the renormalized theory a net loss of wave energy to the particles is to be expected; however, the size of the effect is unclear. We now attempt to compute it.

It is convenient to consider immediately the finite gyroradius case; the drift-kinetic results then follow as a special case. We shall proceed directly from the magnetized Vlasov equation (13); one might equally well begin with a so-called gyro-kinetic equation (drift-kinetic equation with finite gyroradius corrections). If we introduce a cylindrical coordinate system in velocity space (Fig. 1), the particle propagator is formally

$$g_k(v;v') = \left[-i\left(\omega_k - k\cdot v - \Omega\frac{\partial}{\partial\phi} + i\Sigma_k\right)\right]^{-1}\delta(v - v') \quad , \qquad (61)$$

where $\Omega \equiv eB/mc$. When the ϕ dependence of Σ_k can be ignored, one has the explicit representation

$$g_k(v;v') = g_k(\phi;\phi')\frac{1}{v_\perp}\delta(v_\perp - v_\perp')\delta(v_\parallel - v_\parallel') \quad , \qquad (62a)$$

$$g_k(\phi;\phi') = \int_0^\infty d\tau \exp\Big(i\tilde{\omega}_k\tau + ik_\perp\rho[\sin(\phi-\alpha_k) - \sin(\phi'-\alpha_k)]$$

$$- \Sigma_k\tau\Big)\Delta(\phi + \Omega\tau - \phi') \quad , \qquad (62b)$$

where $\tilde{\omega}_k \equiv \omega_k - k_\parallel v_\parallel$, $\rho \equiv v_\perp/\Omega$, and $\Delta(\phi)$ is a periodic Dirac function of period 2π. In general, Σ_k has ϕ dependence, which complicates the problem considerably.

We are now prepared to compute the net wave damping on the ions:

$$<\delta \underset{\sim}{j}^{(i)}(\underset{\sim}{x},t)\cdot\delta\underset{\sim}{E}(\underset{\sim}{x},t)> = ine\int d\underset{\sim}{v}\sum_k \underset{\sim}{k}\cdot\underset{\sim}{v}<\delta f\delta\phi>_k$$

$$= ine\int d\underset{\sim}{v}\sum_k \underset{\sim}{k}\cdot\underset{\sim}{v}\, g_k^{(o)}\Big(-\Sigma^{(d)}<\delta f\delta\phi>_k$$

$$+ i\underset{\sim}{k}\cdot\underset{\sim}{\partial}\left(<f>+\delta\bar{f}_k^{(d)}\right)<\delta\phi\delta\phi>_k\Big) \quad . \tag{63}$$

The part of Eq. (63) explicitly proportional to $<f>$ is exponentially small because $g_k^{(o)}$ is nonresonant: $k_\parallel v_{ti} << \omega(\underset{\sim}{k}) << \Omega_i$. The remainder can be written in the symmetrized form

$$<\delta\underset{\sim}{j}^{(i)}\cdot\delta\underset{\sim}{E}> = -\tfrac{1}{2}ne\int d\underset{\sim}{v}\sum_{k,q}[\underset{\sim}{k}\cdot\underset{\sim}{v}\, g_k^{(o)}\underset{\sim}{q}\cdot\underset{\sim}{\partial} + (k\leftrightarrow q)]$$

$$\times\, g_{k+q} I_k I_q [\underset{\sim}{q}\cdot\underset{\sim}{\partial}\, g_k \underset{\sim}{k}\cdot\underset{\sim}{\partial}\bar{f}_k^{(d)} + (k\leftrightarrow q)] \quad . \tag{64}$$

Formula (42b) remains valid to lowest order in ω/Ω. We can also write

$$\oint d\phi\, \underset{\sim}{k}\cdot\underset{\sim}{v} g_k^{(o)}\ldots = \oint d\phi\left(-i+\omega_k g_k^{(o)}\right)\ldots \quad , \tag{65}$$

so that Eq. (64) reduces to

$$<\delta\underset{\sim}{j}^{(i)}\cdot\delta\underset{\sim}{E}> = \tfrac{1}{2}ne\int d\underset{\sim}{v}\sum_{k,q} M_{\underset{\sim}{k},\underset{\sim}{q}} I_k I_q\left(\omega_k g_k^{(o)} - \omega_q g_q^{(o)}\right) g_{k+q}$$

$$\times\, (g_k \underset{\sim}{k}\cdot\underset{\sim}{\partial}\bar{f}_k - g_q \underset{\sim}{q}\cdot\underset{\sim}{\partial}\bar{f}_q) \quad . \tag{66}$$

Note that one can add any symmetric function of k and q to either of the groups in parentheses in Eq. (66) without changing the value of the expression.

To understand Eq. (66) in more detail, we shall approximate $\bar{f}_k$ by $\langle f \rangle$. Aspects of this approximation are assessed in a related calculation in Sec. V. In order to perform the velocity integrations, we shall approximate $\Sigma^{(d)}$ in all g's by its angle-averaged value:

$$\Sigma_k^{(d)} \to \left\langle \Sigma_k^{(d)} \right\rangle \equiv \oint d\phi d\phi' \, \Sigma_k^{(d)}(\phi,\phi') = \sum_q M_{\underset{\sim}{k},\underset{\sim}{q}} \bar{g}_{k-q} J_0^2(|\underset{\sim}{k}-\underset{\sim}{q}|) I_q \quad . \tag{67}$$

[We use the notation $J_0(k) \equiv J_0(|\underset{\sim}{k}|) \equiv J_0(k_\perp \rho)$.] We believe that this procedure makes a quantitative but not qualitative error; it is exact at zero gyroradius. With the representation (62) the remaining ϕ integrals can be performed. Details of a related calculation are given in the Appendix. If we take

$$\langle f \rangle = [1 + (v_y/\Omega + x)/L_n] f_M(\underset{\sim}{v})$$

we find

$$\langle \delta \underset{\sim}{j}^{(i)} \cdot \delta \underset{\sim}{E} \rangle = \frac{i}{8\pi\lambda_{Di}^2} \int d\underset{\sim}{v} \, f_M \sum_{k,q} M_{\underset{\sim}{k},\underset{\sim}{q}} J_0^2(k) J_0^2(q) I_k I_q \left(\frac{\omega_k}{\bar{\omega}_k} - \frac{\omega_q}{\bar{\omega}_q}\right) \bar{g}_{k+q}$$
$$\times \left[\bar{g}_k \left(\omega_*^{(i)}(\underset{\sim}{k}) - \omega_k\right) - \bar{g}_q \left(\omega_*^{(i)}(\underset{\sim}{q}) - \omega_q\right)\right] \quad . \tag{68}$$

Because ω_k and ω_q are nonresonant, the form of the group in parentheses makes this expression vanish in the limit of zero parallel wavenumber. Dimensionally, Eq. (68) is of order

$$\langle \delta \underset{\sim}{j}^{(i)} \cdot \delta \underset{\sim}{E} \rangle \sim \left(\frac{\bar{q}_\parallel v_{ti}}{\omega}\right)^2 \sum_k (k_\perp^2 D) N_k \quad . \tag{69}$$

This is reminiscent of the DT damping appropriately summed over the spectrum, but arises here from a quite different point of view.

V. Nonlinear Dispersion Relation at Finite Gyroradius

We now study in detail the nonlinear growth rate in the Markovian approximation. We allow for finite ion gyroradius. The ion susceptibility is in this case

$$\chi_k^{(i)} = -\frac{i}{(k\lambda_{Di})^2}\int d\underset{\sim}{v}\, g_k\left[\left(\omega_*^{(i)}(\underset{\sim}{k}) - \underset{\sim}{k}\cdot\underset{\sim}{v}\right)f_M + \left(\frac{T}{e}\right)\underset{\sim}{k}\cdot\underset{\sim}{\partial}\delta\bar{f}_k^{(d)}\right] \quad . \tag{70}$$

For clarity, we shall initially neglect the $\delta\bar{f}_{-q}$ term of Eq. (32); we return later to assess this approximation. We have then

$$\left(\frac{T}{e}\right)\underset{\sim}{k}\cdot\underset{\sim}{\partial}\delta\bar{f}_k^{(d)} \simeq -\sum_q M_{\underset{\sim}{k},\underset{\sim}{q}}\, g_{k-q} I_q g_{-q}\left(\omega_*^{(i)}(\underset{\sim}{q}) - \underset{\sim}{q}\cdot\underset{\sim}{v}\right)f_M \quad . \tag{71}$$

We may use the identity

$$g_k[\omega_*(\underset{\sim}{k}) - \underset{\sim}{k}\cdot\underset{\sim}{v}]f_M = \left[i + g_k\left(\omega_*(\underset{\sim}{k}) - \omega_k - i\,\Sigma_k\right)\right]f_M \tag{72}$$

[which is similar but <u>not</u> identical to Eq. (49)] and a similar expression for $k \to -q$ to rewrite Eq. (70) as

$$\chi_k^{(i)} = \frac{1}{(k\lambda_{Di})^2} - \frac{i}{(k\lambda_{Di})^2}\int d\underset{\sim}{v}\, g_k\left[\left(\omega_*^{(i)}(\underset{\sim}{k}) - \omega_k - i\,\Sigma_k^{(d)}\right)f_M + i\sum_q M_{\underset{\sim}{k},\underset{\sim}{q}}\, g_{k-q} I_q f_M - \sum_q M_{\underset{\sim}{k},\underset{\sim}{q}}\, g_{k-q} I_q g_{-q}\left(\omega_*^{(i)}(\underset{\sim}{q}) - \omega_q + i\,\Sigma_{-q}^{(d)}\right)f_M\right] \quad . \tag{73}$$

We recognize the first term under the q sum as $\Sigma_k^{(d)}$, so a cancellation occurs. Near the stochasticity threshold we can neglect Σ_{-q}

in the last term. If we again replace Σ by $\langle\Sigma\rangle$ in the renormalized propagators, we can carry out the ϕ integrals to find

$$\chi_k^{(i)} = \frac{1}{(k\lambda_{Di})^2} - \frac{i}{(k\lambda_{Di})^2} \int d\underset{\sim}{v} f_M J_o^2(k) \bar{g}_k \Big[\omega_*^{(i)}(\underset{\sim}{k}) - \omega_k - \sum_q M_{\underset{\sim}{k},\underset{\sim}{q}} \bar{g}_{k-q} J_o^2(q) I_q \bar{g}_{-q}\left(\omega_*^{(i)}(\underset{\sim}{q}) - \omega_q\right)\Big] . \tag{74}$$

To take the imaginary part of $\chi_k^{(i)}$, it is convenient to write

$$\bar{g}_k = \left[- i\left(\tilde{\omega}_k + i \left\langle \Sigma_k^{(d)}\right\rangle\right)\right]^{-1}$$

$$= \frac{1 - i\,\bar{\Sigma}_k^*}{- i\hat{\omega}_k} , \tag{75}$$

where

$$\bar{\Sigma}_k \equiv \left\langle\Sigma_k^{(d)}\right\rangle/\tilde{\omega}_k , \tag{76}$$

$$\hat{\omega}_k \equiv \tilde{\omega}_k\left(1 + |\bar{\Sigma}_k|^2\right) . \tag{77}$$

We have

$$\left\langle\Sigma_k\right\rangle = \sum_q M_{\underset{\sim}{k},\underset{\sim}{q}}\left(\frac{i + \bar{\Sigma}_{k-q}^*}{\hat{\omega}_{k-q}}\right) J_o^2(|\underset{\sim}{k}-\underset{\sim}{q}|) I_q , \tag{78}$$

which becomes in the Markovian approximation

$$\left\langle\Sigma_k\right\rangle = \sum_q M_{\underset{\sim}{k},\underset{\sim}{q}}\left(\frac{\bar{\Sigma}_q^*}{\hat{\omega}_q}\right) J_o^2(q) I_q . \tag{79}$$

If we also make the Markovian approximation on Σ_q, the solution of Eq. (79) is purely real. This approximation is somewhat ill-motivated; it cannot be correct for sufficiently large q. Nevertheless, we shall treat $\langle\Sigma_k\rangle$ as real for all k. This ignores nonlinear frequency shifts which may be substantial; however, comparable frequency shifts have already been ignored by neglecting the polarization terms. For real $\bar{\Sigma}_k$, then, one finds

$$\mathrm{Im}\,\chi_k^{(i)} = -\frac{1}{(k\lambda_{Di})^2}\int d\underset{\sim}{v} f_M J_o^2(k)\Big[\frac{\bar{\Sigma}_k}{\hat{\omega}_k}\Big(\omega_*^{(i)}(\underset{\sim}{k}) - \omega_k\Big) + \sum_q M_{\underset{\sim}{k},\underset{\sim}{q}} J_o^2(q)\, I_q \frac{(\bar{\Sigma}_k + \bar{\Sigma}_{k-q} + \bar{\Sigma}_{-q})}{\hat{\omega}_k \hat{\omega}_{k-q} \hat{\omega}_{-q}}\Big(\omega_*^{(i)}(\underset{\sim}{q}) - \omega_q\Big)\Big] \quad , \tag{80}$$

where because of the Markovian approximation we have

$$\bar{\Sigma}_k + \bar{\Sigma}_{k-q} + \bar{\Sigma}_{-q} \simeq -2\bar{\Sigma}_q \quad . \tag{81}$$

It is at this point that we find a substantial difference with DT. In their calculation, the term $\bar{\Sigma}_{-q}$, which stems from the real part of g_{-q} in Eqs. (51) and (74), was neglected, thus changing the factor of 2 in Eq. (81) to 1. This approximation would be correct in the weak turbulence limit (see the Appendix). In the present renormalized theory the error is of order unity, and the additional factor $\bar{\Sigma}_{-q}$ will prevent the cancellation of dominant terms about which we remarked in Sec. III.

If we continue to work near the stochasticity threshold and ignore thermal corrections, we can write $\hat{\omega}_k \simeq \omega_k$. Using the finite gyroradius linear theory result

$$\mathrm{Re}\,\varepsilon \simeq \frac{1}{(k\lambda_{De})^2}\Big[1 + \tau - \Gamma_o(k)\Big(\tau + \frac{\omega_*^{(e)}(\underset{\sim}{k})}{\omega_k}\Big)\Big] \quad , \tag{82}$$

where $\Gamma_o(k) \equiv I_o(b)\exp(-b)$, $b \equiv (k_\perp \rho_i)^2$, we find

$$\omega(\underset{\sim}{k}) = \omega_*^{(e)}(\underset{\sim}{k})\left[\frac{\Gamma_o(k)}{1+\tau[1-\Gamma_o(k)]}\right] , \tag{83}$$

and

$$\frac{\omega_*^{(i)}(\underset{\sim}{k}) - \omega(\underset{\sim}{k})}{\omega(\underset{\sim}{k})} = -\frac{(1+\tau^{-1})}{\Gamma_o(k)} . \tag{84}$$

Equation (80) then becomes

$$\mathrm{Im}\,\chi_k^{(i)} = \frac{(1+\tau^{-1})}{(k\lambda_{Di})^2}\int d\underset{\sim}{v} f_M J_o^2(k)\Big[\bar{\Sigma}_k/\Gamma_o(k)$$

$$- \left(\frac{2}{\omega_k}\right)\sum_q M_{\underset{\sim}{k},\underset{\sim}{q}} J_o^2(q)\, I_q\left(\frac{\bar{\Sigma}_q}{\omega_q}\right)\Big/\Gamma_o(q)\Big] . \tag{85}$$

To proceed, we will write the velocity integral as an average,

$$\int d\underset{\sim}{v} f_M \equiv \langle\!\langle \ldots \rangle\!\rangle$$

and factor

$$\langle\!\langle J_o^2(k) J_o^2(q) \rangle\!\rangle \simeq \langle\!\langle J_o^2(k) \rangle\!\rangle \langle\!\langle J_o^2(q) \rangle\!\rangle = \Gamma_o(k)\Gamma_o(q) . \tag{86}$$

This is in error by an amount of order unity for comparable k and q but is sufficient to elucidate the principal features of the result:

$$\mathrm{Im}\,\chi_k^{(i)} \simeq \frac{(1+\tau^{-1})}{(k\lambda_{Di})^2}\left[\langle\!\langle \bar{\Sigma}_k \rangle\!\rangle - \left(\frac{2\Gamma_o(k)}{\omega_k}\right)\sum_q M_{\underset{\sim}{k},\underset{\sim}{q}} I_q\left(\frac{\bar{\Sigma}_q}{\omega_q}\right)\right]$$

$$= \frac{(1+\tau^{-1})}{(k\lambda_{Di})^2}\left(1 - 2\frac{\Gamma_o(k)}{\Gamma_o(q)}\right)\langle\!\langle \bar{\Sigma}_k \rangle\!\rangle , \tag{87}$$

where $\bar{q}$ is an appropriate mean wavenumber and we have taken note of Eq. (79). Finally, we write $\ll\bar{\Sigma}_k\gg \equiv k_\perp^2 D/\omega_k$ and use Eqs. (28) and (82) to find

$$\frac{\gamma_k}{\omega(\underset{\sim}{k})} = (1+\tau^{-1})\left[\frac{2}{\Gamma_o(\bar{q})} - \frac{1}{\Gamma_o(k)}\right]\frac{k_\perp^2 D}{\omega_*^{(e)}(\underset{\sim}{k})}$$

$$> 0 \tag{88}$$

since $\bar{q} \gg k$. In the limit of zero gyroradius,

$$\gamma_k \to (1+\tau^{-1})k_\perp^2 D \quad .$$

In this limit, the approximations (67) and (86) become exact.

Let us consider the total change of plasmon energy due to the ion interaction. We write

$$\frac{\partial E^{(i)}}{\partial t} = \sum_{\underset{\sim}{k}} 2\omega(\underset{\sim}{k})\gamma_{\underset{\sim}{k}}^{(n)} N_k$$

$$= -\frac{1}{4\pi}\sum_{\underset{\sim}{k}} \omega(\underset{\sim}{k})\varepsilon_k^{(n)}|\underset{\sim}{k}|^2 I_{\underset{\sim}{k}} \quad . \tag{89}$$

If we use Eq. (44) in Eq. (80), minor algebra shows that Eq. (89) is just the negative of Eq. (68). Thus, the ion energy budget is balanced; ions heat at the same rate as energy flows from the turbulent plasmons due to the resonant ion interaction. Notice that it is plasmon energy, rather than electric field energy, which is being discussed, and that expression (24) of DT is missing a factor of the dielectric. The second velocity moment of the Vlasov equation leads to

$$\frac{\partial \langle K\rangle^{(i)}}{\partial t} = -\underset{\sim}{\nabla}\cdot\langle \underset{\sim}{q}\rangle^{(i)} + \langle \delta \underset{\sim}{j}^{(i)}\cdot\delta \underset{\sim}{E}\rangle \quad ,$$

where $\underset{\sim}{q}$ is the heat flux, or

$$\frac{\partial}{\partial t}\left[\langle K\rangle^{(i)} + E^{(i)}\right] = -\underset{\sim}{\nabla}\cdot\langle \underset{\sim}{q}^{(i)}\rangle \quad .$$

Thus, the renormalized theory behaves in the same fashion as quasilinear theory: the mean distribution describes the evolution of the resonant particles while $\langle\delta f\delta f\rangle$ describes the turbulent waves, including both electric field fluctuations and particle "sloshing". The total energy balance (4) follows by summing over species and by assuming that since the electrons are linear, the quasilinear conservation laws hold for the electrons.

We now wish to discuss briefly how these results are modified if the $\delta\bar{\bar{f}}_{-q}^{(d)}$ term is not neglected in Eq. (32). In this case we have a complicated integral equation to solve for $\delta\bar{\bar{f}}^{(d)}$:

$$\left(\frac{T}{e}\right)\left[\underset{\sim}{k}\cdot\underset{\sim}{\partial}\delta\bar{\bar{f}}_{k}^{(d)} - \sum_{q} M_{\underset{\sim}{k},\underset{\sim}{q}}\, g_{k+q} I_q g_q \left(\underset{\sim}{q}\cdot\underset{\sim}{\partial}\delta\bar{\bar{f}}_{q}^{(d)}\right)\right]$$

$$= \sum_{q} M_{\underset{\sim}{k},\underset{\sim}{q}}\, g_{k+q} I_q g_q [\omega_*(\underset{\sim}{q}) - \underset{\sim}{q}\cdot\underset{\sim}{v}] f_M \quad . \tag{90}$$

If we pass to the Markovian limit and approximate

$$M_{\underset{\sim}{k},\underset{\sim}{q}} \simeq k_\perp^2 m_q \quad ,$$

$$m_q \equiv \tfrac{1}{2}(c/B)^2 q_\perp^2 \quad , \tag{91}$$

Eq. (90) can be solved by multiplying by $m_q g_q I_q g_q$ and summing over q :

$$\left(\frac{T}{e}\right) \underset{\sim}{k} \cdot \underset{\sim}{\partial} \delta \bar{f}_k^{(d)} \simeq (1+a)^{-1} k_\perp^2 \sum_q M_q g_q I_q g_q [\omega_*(\underset{\sim}{q}) - \underset{\sim}{q} \cdot \underset{\sim}{v}] f_M \quad , \tag{92}$$

where

$$a \equiv \sum_q q_\perp^2 m_q g_q I_q g_q \quad . \tag{93}$$

It is easy to verify that a is real.

One can think of the correction a as a turbulent renormalization of the temperature:

$$a \equiv \delta T/T \quad . \tag{94}$$

To see that this is plausible, define a turbulent temperature by

$$\bar{T} \equiv (1+a)T = T + \delta T \quad , \tag{95}$$

and extract from Eq. (70) a factor of $(1+a)^{-1}$ so that in the Markovian limit

$$\begin{aligned} \mathrm{Im}\,\chi_k^{(i)} = \frac{1}{(k\bar{\lambda}_{Di})^2} \mathrm{Re} \int d\underset{\sim}{v}\, g_k \Big\{ \omega_*^{(i)}(\underset{\sim}{k}) - \omega_k \\ + k_\perp^2 \sum_q m_q g_q I_q g_q \left(\omega_*^{(i)}(\underset{\sim}{q}) - \omega_q\right) + \Big[a\left(\omega_*^{(i)}(\underset{\sim}{k}) - \omega_k - i\,\Sigma_k\right) \\ - i\, k_\perp^2 \sum_q m_q g_q I_q g_q\, \Sigma_k \Big] \Big\} \quad , \end{aligned} \tag{96}$$

where $\bar{\lambda}_{Di}$ is the Debye length defined with $\bar{T}$ instead of T. In Eq. (96), the terms in brackets are both of order a. If they are neglected, Eq. (96) reduces correctly to the Markovian version of Eq. (74). This is correct sufficiently near the stochasticity threshold, where $a \ll 1$. For finite a, little can be said at present. Not only are the last terms of Eq. (96) important, so are the polarization and mode coupling terms which have been completely neglected here. Much further work is called for.

VI. Discussion and Conclusion

In summary, we have considered a renormalized theory of collisionless drift waves in shear-free geometry which begins with the so-called Coherent Approximation to the Direct Interaction Approximation. This theory is basically a description of dielectric response in a turbulent medium; it reduces correctly to weak turbulence theory. A central feature of the theory is that not only the propagator, but also the mean distribution function are renormalized by the turbulence. This is necessary for consistent energetics; simpler theories renormalize only the propagator and violate energy conservation. We retained from the DIAC only the so-called diffusion terms, thus making the theory a renormalized version of Compton scattering and the systematic kinetic generalization of the Dupree-Tetreault calculation. We find that the long wavelength limit of the nonlinear growth rate $\gamma_k^{(n)}$ is proportional to $k_\perp^2 D$, in disagreement with DT. However, when $\gamma_k^{(n)}$ is appropriately summed over all modes, the net resonant particle heating is proportional to the square of a typical parallel wavenumber, as DT argued. The sum of the mean particle kinetic energy and the turbulent energy in all waves (including both electric energy and turbulent "sloshing") is conserved.

The turbulent renormalization $\delta\bar{f}$ of the distribution, which plays a central role in the theory, is the generalization of the so-called β term of DT. Though they and we considered only ions, a similar correction exists for the electrons. If we followed in the spirit of the approximations leading to Eq. (53), we would find

$$\underset{\sim}{k}\cdot\underset{\sim}{\partial}\delta\bar{f}_k^{(d,e)} \simeq + i(k_\perp^2 D)\left(\frac{e}{T_e}\right) f_M^{(e)} \quad , \tag{97}$$

a cancellation would occur in the electron version of Eq. (50), and the electron susceptibility would become

$$\chi_k^{(e)} \simeq \frac{1}{(k\lambda_{De})^2} + \frac{1}{(k\lambda_{De})^2} \frac{\left(\omega_k - \omega_*^{(e)}\right)}{\sqrt{2}\, k_\parallel v_{te}} Z\left(\frac{\omega + ik_\perp^2 D}{\sqrt{2}\, k_\parallel v_{te}}\right) \quad . \tag{98}$$

This is also the result which follows from Catto's prescription. Hirshman and Molvig used the shear-modified version of Eq. (98) in their recent discussion of electron nonlinearities.[20] Since we have argued that Eq. (53) is inconsistent on the ions, we expect similar problems on the electrons, although Hirshman and Molvig claim that the error is small. In any case, Eq. (98) is correct only in the long-wavelength limit.

In closing, we remark that Kleva and Krommes have applied techniques similar to those discussed in the present paper to a certain problem of collisionless tearing turbulence in the presence of stochastic magnetic fields.[21] Finally, before a quantitative renormalized theory of drift or tearing turbulence can be presented, the effects of the polarization terms must be assessed. Work on this problem is underway.[22]

ACKNOWLEDGMENTS

We are grateful to Bob Kleva, Rick Jensen, Carl Oberman, Harry Mynick, Philippe Similon, Tom Dupree, Steve Hirshman, and Kim Molvig for stimulating discussions on related matters.

This work was jointly supported by the United States Department of Energy Contract No. EY-76-C-02-3073 and by the United States Air Force Office of Scientific Research Contract No. F 44620-75-C-0037.

APPENDIX: Compton Scattering for Collisionless Drift Waves

The equations discussed in the text are the renormalized version of Compton scattering in weak turbulence theory. We present here the technical details of the mapping from the renormalized to the weak turbulence limit.

As is well-known,[23] Compton scattering arises from the so-called "third-order dielectric function" $\varepsilon^{(3)}$ and contributes to the $O(E^2)$ nonlinear dielectric a term

$$\varepsilon^{(n)}(k) = 2\sum_q \varepsilon^{(3)}(k|q,k,-q)I_q \quad . \tag{A1}$$

We shall consistently ignore polarization effects. The third-order dielectric is defined in Ref. 19:

$$\varepsilon^{(3)}(k|q,k,-q) = \tfrac{1}{2} i\Phi_k g_k \underset{\sim}{q}\cdot\partial g_{k-q}[(-\underset{\sim}{q})\cdot\partial g_k \underset{\sim}{k}\cdot\partial\langle f\rangle + (k \leftrightarrow -q)] \quad . \tag{A2}$$

The velocity and species arguments of the propagators and distribution are obvious, and we omit them for compactness. The first task is to reduce the renormalized dielectric (8) to Eq. (A1). This is readily accomplished by treating the turbulent collision operator Σ as small and nonresonant, and expanding according to Eq. (11). The $O(E^2)$ contributions to Eq. (8) are then

$$\varepsilon_k^{(n)} = i\Phi_k g_k^{(o)}\left(\Sigma_k\, g_k^{(o)} \underset{\sim}{k}\cdot\partial\langle f\rangle - \underset{\sim}{k}\cdot\partial\delta\bar{f}_k\right) \quad . \tag{A3}$$

If we insert the diffusion part of Σ [Eq. (31)] and of $\delta\bar{f}$ [Eq. (32)] correct to zeroth order in E^2, it is easy to see that we obtain precisely Eq. (A2); the first term corresponds to $\Sigma^{(d)}$, the second to $\delta\bar{f}^{(d)}$.

We now reduce Eq. (A1) to the form specific to the drift wave problem. The form of the propagator is given in Eq. (62), which can be written as

$$g_k^{(o)}(\phi,\phi') = \int_0^\infty d\tau \exp(i\tilde{\omega}_k\tau) \sum_{n,n'} a_{n'}^*(\underset{\sim}{k}) a_n(\underset{\sim}{k}) \exp[i(n'-n)\phi - in\Omega\tau]$$
$$\times \Delta(\phi + \Omega\tau - \phi') \quad , \tag{A4}$$

where

$$a_n(\underset{\sim}{k}) \equiv J_n(k)\exp(in\alpha_{\underset{\sim}{k}}) \quad . \tag{A5}$$

The a_n's obey the addition formula

$$\sum_n a_n(\underset{\sim}{k}) a_n(\underset{\sim}{k}') = J_o(|\underset{\sim}{k} - \underset{\sim}{k}'|) \quad . \tag{A6}$$

It is useful to employ the identity (72) and to recall that $g_{-q} = g_q^*$. The integrations can now be performed straightforwardly by integrating first over polar angle, then over time. Retaining only terms of dominant order in ω/Ω and using formula (A6) twice, one finds that

$$\varepsilon_k^{(n)} = i \sum_s (k\lambda_{Ds})^{-2} \int d\underset{\sim}{v} f_M J_o^2(k) \bar{g}_k^{(o)} \sum_q M_{\underset{\sim}{k},\underset{\sim}{q}} J_o^2(q) \bar{g}_{k-q}^{(o)} I_q$$
$$\times \{\bar{g}_k^{(o)} [\omega_*(\underset{\sim}{k}) - \omega_k] + \bar{g}_{-q}^{(o)} [\omega_*(\underset{\sim}{q}) - \omega_q]\} \quad . \tag{A7}$$

One now assumes that ω_k and ω_q are nonresonant but that $\omega_k - \omega_q \simeq (\underset{\sim}{k}-\underset{\sim}{q})_{||} v_{||}$, so that

$$\bar{g}_k^{(o)} \simeq (-i\tilde{\omega}_k)^{-1} \quad , \quad \bar{g}_{k-q}^{(o)} \simeq \pi\delta(\tilde{\omega}_k - \tilde{\omega}_q) \quad . \tag{A8}$$

Restricting attention to the ions, one finds the nonlinear ion susceptibility to be

$$\varepsilon_k^{(n,i)} = -i\pi \frac{1}{(k\lambda_{Di})^2} \int d\underset{\sim}{v} f_M J_o^2(k) \frac{1}{\tilde{\omega}_k} \sum_q M_{\underset{\sim}{k},\underset{\sim}{q}} J_o^2(q)\,\delta(\tilde{\omega}_k - \tilde{\omega}_q) I_q \times \left(\frac{[\omega_*^{(i)}(\underset{\sim}{k}) - \omega_k] - [\omega_*^{(i)}(\underset{\sim}{q}) - \omega_q]}{\tilde{\omega}_k} \right) \quad . \tag{A9}$$

This formula is to be compared to expression (80). In the limit of small $\bar{\Sigma}$ and nonresonant k and q, only the first term of Eq. (80) and the $\bar{\Sigma}_{k-q}$ part of the second term survive. If one writes $\bar{\Sigma}_k$ in terms of its definition (78) and uses

$$\bar{\Sigma}_{k-q}/\hat{\omega}_{k-q} \simeq \pi\delta(\tilde{\omega}_k - \tilde{\omega}_q) \quad , \tag{A10}$$

one finds that Eq. (A9) agrees with the limit of Eq. (80) except for the factor $J_o^2(|\underset{\sim}{k}-\underset{\sim}{q}|)$ in Eq. (78), which should be replaced by $J_o^2(q)$. This is an artifact of the approximate, angle-averaged treatment we have given of the renormalized propagator; it does not affect our qualitative discussion of the renormalized theory.

Because of the induced scattering resonance condition $\omega_k \simeq \omega_q$ and the fact that $k_{\parallel} \ll k_{\perp}$, contributions to the induced scattering growth rate arise only from modes $|\underset{\sim}{k}| \sim |\underset{\sim}{q}|$. What we have shown in Sec. V is that, in the renormalized theory and for small $|\underset{\sim}{k}|$, substantial contributions to the nonlinear growth rate arise from modes satisfying $|\underset{\sim}{k}| \ll |\underset{\sim}{q}|$.

From the evolution equation for plasmon number,

$$\frac{\partial N_{\underset{\sim}{k}}}{\partial t} = 2\gamma_{\underset{\sim}{k}} N_{\underset{\sim}{k}} \quad , \tag{A11}$$

one finds the rate of change of total mode energy $E \equiv \sum_{\underset{\sim}{k}} \omega(\underset{\sim}{k}) N_{\underset{\sim}{k}}$:

$$\begin{aligned}\frac{\partial E}{\partial t} &= \frac{1}{4\pi} \sum_{\underset{\sim}{k}} \omega(\underset{\sim}{k}) \epsilon_k^{(n)} |\underset{\sim}{k}|^2 I_k \\ &= \frac{1}{8\lambda_{Di}^2} \int d\underset{\sim}{v} f_M \sum_{k,q} M_{\underset{\sim}{k},\underset{\sim}{q}} J_o^2(k) J_o^2(q) I_k I_q \left(\frac{\omega_k}{\tilde{\omega}_k} - \frac{\omega_q}{\tilde{\omega}_q}\right) \delta(\tilde{\omega}_k - \tilde{\omega}_q) \\ &\quad \times \left(\frac{[\omega_*^{(i)}(\underset{\sim}{k}) - \omega_k] - [\omega_*^{(i)}(\underset{\sim}{q}) - \omega_q]}{\tilde{\omega}_k}\right) \quad .\end{aligned} \tag{A12}$$

This is just minus the induced scattering limit of Eq. (66), so the proper energy balance

$$\frac{\partial E^{(i)}}{\partial t} = - \langle \delta \underset{\sim}{j}^{(i)} \cdot \delta \underset{\sim}{E} \rangle$$

is maintained. When thermal corrections are neglected, Eq. (A9) reduces to the form given by Sagdeev and Galeev,[24] plasmon energy is conserved, and there is no net particle heating arising from the ion interaction.

REFERENCES

[1]T.H. Dupree, Phys. Fluids 9, 1773 (1966).

[2]A.A. Galeev, Phys. Fluids 10, 1041 (1967).

[3]T.H. Dupree, Phys. Fluids 10, 1049 (1967); C.T. Dum and T.H. Dupree, Phys. Fluids 13, 2064 (1970); G. Benford and J.J. Thomson, Phys. Fluids 15, 1496 (1972).

[4]A.A. Galeev, Zh. Eksp. Teor. Fiz. 57, 1361 (1969) [Sov. Phys.-JETP 30, 737 (1970)].

[5]L.I. Rudakov and V.N. Tsytovich, Plasma Phys. 13, 213 (1971).

[6]P.J. Catto, Phys. Fluids 21, 147 (1978).

[7]See also T.H. Dupree, Phys. Fluids 11, 2680 (1968).

[8]T.H. Dupree and D.J. Tetreault, Phys. Fluids 21, 425 (1978).

[9]L.M. Al'tshul' and V.I. Karpman, Zh. Eksp. Teor. Fiz 49, 515 (1965) [Sov. Phys.-JETP 22, 361 (1966)].

[10]P.C. Martin, E.D. Siggia, and H.A. Rose, Phys. Rev. A8, 423 (1973).

[11]H.A. Rose, Ph.D. thesis (Harvard, 1974)(unpublished); H.A. Rose, LA-UR-78-1902 (1978); J. Stat. Phys. (submitted).

[12]R.H. Kraichnan, J. Math. Phys. 2, 124 (1961).

[13]R.H. Kraichnan, Fluid Mech. 5, 497 (1959); R.H. Kraichnan, Adv. in Math. 16, 305 (1975).

[14]S.A. Orszag and R.H. Kraichnan, Phys. Fluids 10, 1720 (1970).

[15]See also B.B. Kadomtsev, *Plasma Turbulence* (Academic Press, New York, 1965), Chap. III.

[16]J.A. Krommes, "Turbulence, Clumps, and the Bethe-Salpeter Equation," Proc. 3rd Annual Kiev Theory Meeting (Trieste, 1977) (to be published).

[17]T.H. Dupree, Phys. Fluids **15**, 334 (1972).

[18]D.F. Dubois and M. Espedal, Plasma Phys. **20**, 1209 (1978).

[19]J.A. Krommes and R.G. Kleva, Princeton Plasma Phys. Lab. Rept. PPPL-1522 (1979); Phys. Fluids (submitted)

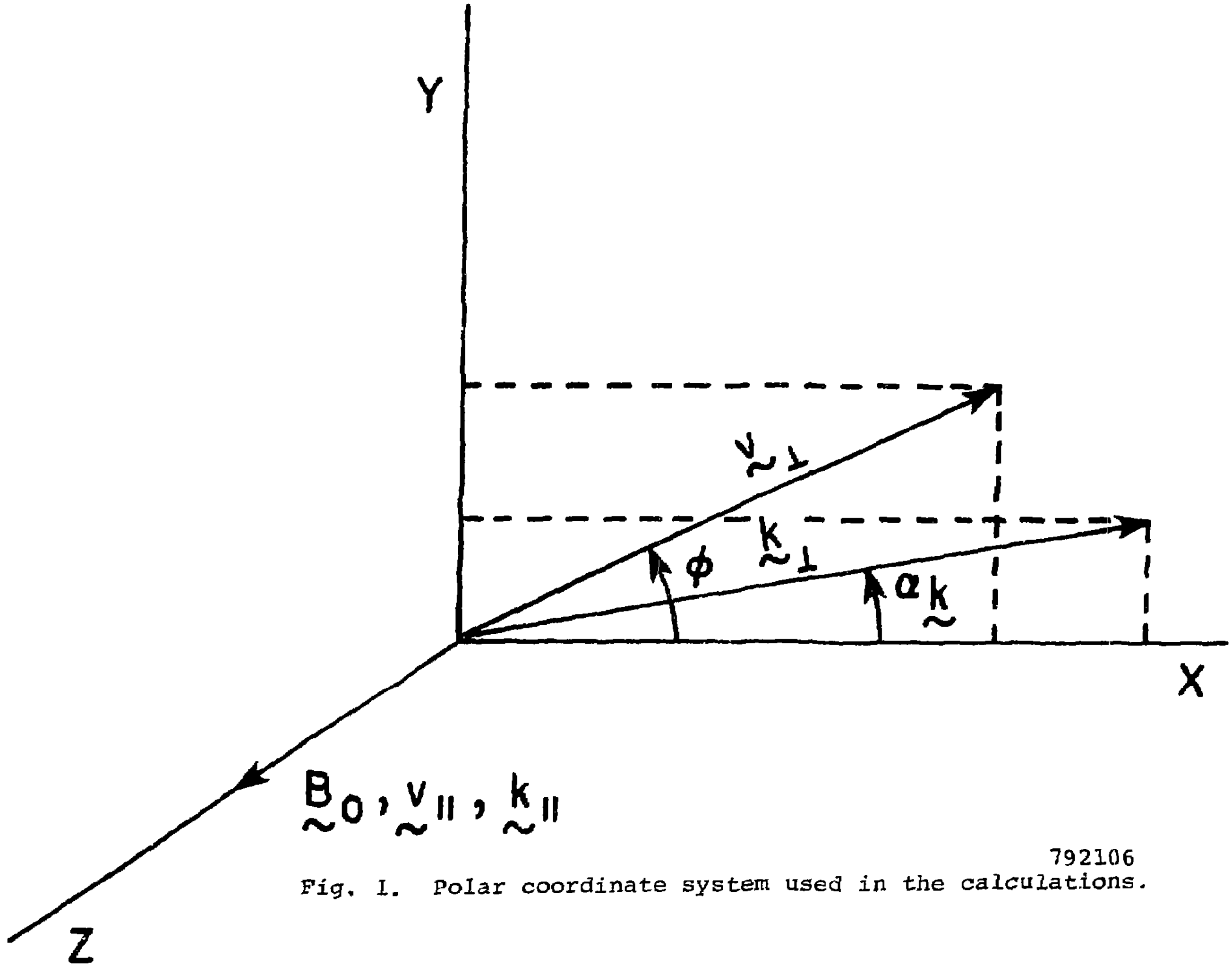
[20]S. Hirshman and K. Molvig, Phys. Rev. Lett. **42**, 648 (1979).

[21]R.G. Kleva and J.A. Krommes (in preparation).

[22]D.F. Dubois and M. Espedal, Bull. Am. Phys. Soc. **23**, 892 (1978), abstract 9E12; P. Similon (private communication).

[23]R.Z. Sagdeev and A.A. Galeev, Nonlinear Plasma Theory, edited by T.M. O'Neil and D.L. Book (Benjamin, New York, 1969), Chap. 2.

[24]*Ibid.*, p. 106.



792106

Fig. 1. Polar coordinate system used in the calculations.