INSTITUTE OF THEORETICAL
AND EXPERIMENTAL PHYSICS

ITEP - 176

M.B.Voloshin

SUM RULES FOR Υ-FAMILY FORMATION IN e^+e^--ANNIHILATION

M O S C O W 1 9 7 8

УДК 530.145:538.3 М-16

A b s t r a c t

Predictions are derived for the integral $\mu^{-2}\int R_{b}(s)\exp\left(\frac{M_{\Upsilon}^{2}-s}{\mu^{2}}\right)ds$ with $R_{b}(s)$ being the contribution of states containing $b\bar{b}$ quark pair to the ratio R measured in $e^{+}e^{-}$-annihilation. The theoretical results refer to the region of the parameter μ : $\mu = 2 - 3{,}7$ Gev where the integral is saturated by contribution of Υ and Υ' resonances. An agreement of theoretical prediction with experimental data is found for the mass of b-quark $m = 4{,}65 \pm 0{,}05$ Gev. In the region of the μ variable considered the effects of short distance gluon exchange enhance the free quark result by a μ-dependent factor 2 - 8. Therefore the agreement with experimental data is by no means trivial and seems to be a convincing illustration of reality of gluon exchange at short distances.

1. Introduction

It is quite evident by now that Υ resonances observed both in hadroproduction [1] and in e^+e^--annihilation [2] are S-wave bound states of quarkonium made from new heavy quarks $b\bar{b}$. Thus the Υ family is one more copy of a quarkonium picture which has become more or less familiar from study of charmonium levels. The difference between these two systems is that b quark is roughly three times heavier than the charmed one and that its electric charge is $Q_b = -1/3$.

On the other hand a quite predictive approach to description of charmonium and charmed particles has been developed [3,4] relying on heaviness of the charmed quark and exploiting the very first principles of QCD. In particular within this approach the mass of c-quark was estimated from comparison of sum rules for $e^+e^- \to$ charm with experimental data. This mass parameter is essentially the only one required to make predictions about charm photo- and electroproduction total cross sections [5].

The main purpose of this paper is to extend the approach developed in Refs. 3,4 to description of the Υ resonances. For this we pursue the general pattern of those papers and consider vacuum polarization by vector (electromagnetic) current of b quarks:

$$(q_\mu q_\nu - g_{\mu\nu} q^2) P(q^2) = \qquad (1)$$
$$= i\int d^4x\, e^{iqx} \langle 0 | T\{\bar{b}(x)\gamma_\mu b(x), \bar{b}(0)\gamma_\nu b(0)\} | 0\rangle$$

For q^2 real and below $b\bar{b}$ threshold

$$q^2 = 4m^2 - 4k^2 \qquad (k^2 > 0) \qquad (2)$$

the amplitude $P(q^2)$ is determined by distances $r \leq (2k)^{-1}$ so that it can be calculated by means of short-distance QCD provided that k is sufficiently large.

On the other hand via a dispersion relation one can express $P(q^2)$ below threshold in terms of its absorptive part in the physical region measured by e^+e^--annihilation cross section into physical states containing $b\bar{b}$ quark pair:

$$P(q^2) = \frac{q^2}{12\pi^2 Q_b^2} \int \frac{R_b(s)\, ds}{s(s-q^2)} \tag{3}$$

where $R_b(s)$ is b-quark contribution to the ratio R measured in e^+e^--annihilation and the integral runs over all values of s for which $R_b(s) \neq 0$. In particular the lowest state contributing to the integral in eq. (3) is the Υ-resonance. In case the amplitude $P(q^2)$ is calculated theoretically eq. (3) can be considered as a sum rule for the experimentally measurable quantity $R_b(s)$.

Eq. (3) can be studied for arbitrary values of q^2 of the form (2) provided that k is large enough to asure short-distance QCD calculation of $P(q^2)$. However in the rest of this paper we shall be interested in considering relations like eq. (3) for values of q^2 as close to the physical region as possible, since in this case Υ and Υ' dominate the r.h.s. of eq. (3) and one can make predictions which can be checked without knowledge of detailed structure of $R_b(s)$ above the Υ' resonance. Therefore we must inspect what low values of k one can reach in eq. (2) not to spoil the applicability of short-distance QCD with its asymptotic freedom.

It may seem to the first sight that the region of asymptotic freedom is limited by the growth of the effective coupling $\alpha_s(2k)$. It turns out however that before the coupling becomes large other effects come into play. These effects are the so-called power terms which arise (for heavy quarks) from interaction of quarks with nonperturbative fluctuations of gluonic field (instantons [6] etc.) present in the true vacuum of QCD. In the charmonium theory these terms were first introduced in Ref.7 and it was also found [8] that for vacuum polarization in the light quark sector the power terms rather than the growth of α_s first violate the asymptotic freedom calculations at relatively long distances.

For heavy quarks the leading power correction (e.g. to the amplitude $P(q^2)$) which shows up first when k^2 goes down is expressed in terms of the vacuum mean value of the square of gluonic field tensor $\langle 0|\pi\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle$. This value was estimated [7] from charmonium sum rules

$$\langle 0|\pi\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle \simeq 0.12 \text{ Gev}^4 \qquad (4)$$

with a possible (20–40)% uncertainty. Implications of nonperturbative vacuum fluctuations of gluonic field characterized by the mean value (4) were analized in Ref.9 where it was shown in particular that the relative magnitude of the correction to $P(q^2)$ due to the mean value (4) is proportional (for $k \ll m$) to the quantity

$$\frac{1}{144} \langle 0|\pi\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle \frac{m^2}{k^6}$$

(m: mass of the (b)quark). In sec. 4 we shall quantify this statement and find that for the case of the Υ-family this correction is small enough if $k \gtrsim$ 1 Gev. Note that for such k the effective coupling is still small $\alpha_s(2k) \lesssim \alpha_s(2\,\mathrm{Gev}) \simeq 0.21$. (In this paper the normalization [3-5,7,8] is adopted $\alpha_s(2m_c \simeq 2.5\,\mathrm{Gev}) \simeq 0.2$. Our results are rather sensitive to the value of α_s and in principle enable to fit it. However we find that the adopted value fits quite well. We shall return to discussion of this matter in sec. 5. It is also worth mentioning that the adopted value is essentially lower than that estimated from fits [10] to leptoproduction data. But it is the above normalization of α_s which provides an agreement with experimental data and selfconsistency of various predictions of the charmonium theory).

So, we can apply short distance QCD when calculating $P(4m^2 - 4k^2)$ down to $k \simeq$ 1 Gev. From a rough estimate $m \simeq M_\Upsilon/2 \simeq 4.7\,\mathrm{Gev}$ it is clear that such values of the momentum k are nonrelativistic for b quarks: $k^2/m^2 \ll 1$. In the nonrelativistic region however there appear threshold singularities the leading of which are given by graphs with parallel exchange of "Coulomb" gluons (see Fig. 1). These graphs provide a series in powers of $\alpha_s(2k)\, m/k$ rather than in powers of $\alpha_s(2k)$ and in the region of our interest we must sum up all such terms. This summation can be performed [4,9] in terms of the Schrödinger equation in the potential $-\frac{4\alpha_s}{3r}$. In sec. 2 we shall follow the method of Green's function [9] which enables to sum up all $(\alpha_s m/k)^n$ singularities in the vacuum polarization amplitude.

In sec. 3 we shall perform the so-called L-transform [8] of the sum rule (3). The L-transform is defined as follows

$$F(\mu^2) \equiv L_\mu\, P(4m^2 - 4k^2) =$$

$$= \lim_{\substack{k^2 \to \infty,\, n \to \infty, \\ k^2/n = \mu^2/4}} \frac{(k^2)^n}{(n-1)!} \left(-\frac{d}{dk^2}\right)^n P(4m^2 - 4k^2) \quad (5)$$

As a result of this transformation the $(s-q^2)^{-1}$ weight factor in the integral in eq. (3) transforms into an exponential weight factor

$$F(\mu^2) = (12\pi^2 Q_q^2 \mu^2)^{-1} \int \mathcal{R}_q(s) \exp\left(\frac{4m^2 - s}{\mu^2}\right) ds \quad (6)$$

And for small enough μ the contribution of higher states to this integral is sharply cut off.

One can note also that in definition of the L -transform formally k is going to infinity. In fact however the quantity $F(\mu^2)$ is determined by behaviour of $P(4m^2 - 4k^2)$ at $2k \simeq \mu$ since simultaneously with increasing k^2 one must differentiate the amplitude P $n = 4k^2/\mu^2$ times. Therefore the region $k \gtrsim$ 1 Gev corresponds to $\mu \gtrsim$ 2 Gev and all the above remarks concerning the relevance of asymptotic freedom in terms of k hold also in terms of μ (more pr cisely $\mu/2$). The quantity $F(\mu^2)$ will be estimated in sec. 3.

In sec. 4 we shall discuss corrections to the results of sections 2 and 3 and in sec. 5 a comparison of theoretical predictions with data on Υ and Υ' formation in e^+e^--annihilation will be given for $\mu = 2 - 3.7$ Gev. (The values of μ under consideration are bound from above by relativistic corrections to our essentially nonrelativistic approach. The relative magnitude of these corrections is $\mu^2/4m^2$). As a result we estimate the mass of the b quark $m = 4{,}65 \pm 0{,}05$ Gev and make some judgements on the value of the coupling constant α_s.

2. Calculation of the Vacuum Polarization in the Near Threshold Region

In this section a calculation of $P(4m^2-4k^2)$ will be given summing up all $(\alpha_s m/k)^n$ terms for $k^2 \ll m^2$ and neglecting all corrections proportional to k^2/m^2. The calculation is based on a relation [9] between the vacuum polarization amplitude and the nonrelativistic Green's function of the relative motion in quarkonium $G(\vec{x},\vec{y},\varepsilon)$. The relation has the form

$$P(4m^2-4k^2) = \frac{3}{2m^2} \lim_{\substack{\vec{x}\to 0\\ \vec{y}\to 0}} G(\vec{x},\vec{y},-k^2/m) \qquad (k^2 \ll m^2) \tag{7}$$

which is valid up to an unimportant (though infinite) additive constant independent of k. (In Ref. 9 this constant was eliminated by differentiating eq. (7) over k). For the Green's function $G(\vec{x},\vec{y},\varepsilon)$ in Ref. 9 was obtained the following expansion

$$G(\vec{x},\vec{y},\varepsilon) = G_{(0)}(\vec{x},\vec{y},\varepsilon) - \frac{\langle 0|\pi\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle}{18} \cdot \tag{8}$$

$$\cdot \int d^3z\, d^3z' (\vec{z}\,\vec{z}')\, G_{(0)}(\vec{x},\vec{z},\varepsilon)\, G_{(1)}(\vec{z},\vec{z}',\varepsilon)\, G_{(0)}(\vec{z}',\vec{y},\varepsilon) +$$

$$+ \ldots$$

which accounts both for perturbative contribution (Fig.1) (the first term on the r.h.s.) and for the leading nonperturbative power correction due to the vacuum mean value (4) (the second term on the r.h.s.). Here $G_{(0)}$ and $G_{(8)}$ denote the "Coulomb" Green's functions for color singlet and color octet states of $b\bar{b}$ pair respectively. They obey the equations

$$\left(-\frac{1}{m}\left(\frac{\partial}{\partial \vec{x}}\right)^2 + V_{(0,8)}(|\vec{x}|) - \varepsilon\right) G_{(0,8)}(\vec{x},\vec{y},\varepsilon) = \delta(\vec{x}-\vec{y}) \qquad (9)$$

with the condition of regularity at infinity and the potentials

$$V_{(0)}(r) = -\frac{4}{3}\frac{\alpha_s}{r}, \quad V_{(8)}(r) = +\frac{2}{3}\frac{\alpha_s}{r} \qquad (10)$$

The three dots in eq. (8) refer to contribution of vacuum mean values of gluonic operators of dimension $d > 4$.

We start with considering the perturbative contribution to the r.h.s. of eq. (7) which is described according to eq. (8) by the Green's function $G_{(0)}(\vec{x},\vec{y},-k^2/m)$. For b quarks this contribution dominates $P(4m^2-4k^2)$ for $k \gtrsim 1$ Gev. Keeping in mind that in eq. (7) one needs only the $\vec{x}\to 0$, $\vec{y}\to 0$ limit of the Green's function we can take $\vec{y}=0$ from the very beginning, and denote $G_{(0)}(\vec{x},0,-k^2/m) = G(r,k^2)$ since for $\vec{y}=0$ the Green's function depends only on $r = |\vec{x}|$. Then from eqs. (9) and (10) one obtains the equation ($r \neq 0$)

$$\left(-\frac{1}{mr^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{4}{3}\frac{\alpha_s}{r} + \frac{k^2}{m}\right) G(r,k^2) = 0 \qquad (11)$$

Its solution regular at $r \to \infty$ is given by

$$G(r,k^2) = C e^{-kr}\, \Psi\left(1-\frac{2m\alpha_s}{3k}, 2, 2kr\right) \qquad (12)$$

where $\Psi(a,c,z)$ is the standard notation for the confluent hypergeometric function which has no exponential growth at $z \to \infty$ (see e.g. 11). The normalization constant C in eq. (12) is determined by the δ-function singularity in the original equation (9), which implies that when $r \to 0$ the leading singularity of $G(r, k^2)$ must have the form

$$G(r, k^2) \underset{r \to 0}{\simeq} \frac{m}{4\pi r} \tag{13}$$

(it is this k-independent singularity that gives rise to the infinite additive constant in eq. (7)). The expansion of the function Ψ for small kr is given by [11]

$$\Psi\left(1 - \frac{2m\alpha_s}{3k}, 2, 2kr\right) = \frac{-1}{\Gamma\left(-\frac{2m\alpha_s}{3k}\right)} \left\{ \frac{3}{4m\alpha_s r} - \right.$$

$$\left. - \ln 2kr - \psi\left(1 - \frac{2m\alpha_s}{3k}\right) + \psi(1) + \psi(2) + O(r \ln r) \right\}, \tag{14}$$

$\psi(z) = d \ln \Gamma(z) / dz$. From this one readily finds that the condition (13) is satisfied if the normalization in eq. (12) is chosen of the form

$$C = - \frac{m^2 \alpha_s}{3\pi} \Gamma\left(- \frac{2m\alpha_s}{3k}\right) \tag{15}$$

With this value of C and again using the expansion (14) one readily calculates the limit involved in eq. (7) and finally finds

$$P(4m^2 - 4k^2) \underset{k^2 \ll m^2}{=} - \frac{3}{8\pi m} \left\{ k + \frac{4}{3} m\alpha_s \ln k + \right.$$

$$\left. + \frac{4}{3} m\alpha_s \psi\left(1 - \frac{2m\alpha_s}{3k}\right) \right\} + \text{const.} \tag{16}$$

The terms in this expression have simple physical interpretation. The first one in the curly bracket is the free theory result, the second one is the result of first iteration of the "Coulomb" interaction $V_{(0)}$, and the third represents the contribution of Coulomb S-wave poles at $k = \frac{2}{3} m \alpha_s / n$, $n = 1, 2, \ldots$ It can be also noted that eq. (16) corresponds to QED results [12] obtained in a slightly different manner than ours.

Two more remarks are in order. First is that in presenting consideration of this section we did not specify the normalization point for the coupling α_s. One can readily convince oneself that when calculating the Green's function from eq. (9) for $|\vec{x}|, |\vec{y}| \lesssim k^{-1}$ by iterations of the interaction $V_{(0)}$ the latter is integrated over r with the weight factor e^{-2kr}. Therefore the distances relevant are $r \lesssim (2k)^{-1}$ and therefore in eq. (16) one should take $\alpha_s(2k)$.

The second remark concerning eq. (16) is that in QCD this expression is relevant only in the region of asymptotic freedom. For the case of b-quarks ($k \gtrsim 1$ Gev) only tails of Coulomb poles (given by $\psi(1 - \frac{2m\alpha_s}{3k})$) are seen in this region. For lower values of k power terms come into play and destroy the validity of eq. (16). For more massive quarks however, e.g. for quarks with mass $m \gtrsim 15$ Gev the asymptotic freedom extends to the region $k < \frac{2}{3} m \alpha_s(2k)$ so that lowest bound states of such superheavy quarkonium must be essentially "Coulomb"-like. In this case eq. (16) gives their dominant "Coulomb" parameters (the binding energy, the width Γ_{ee}) while the second term on the r.h.s. of eq. (8) describes corrections due to v.e.v. (4) to these parameters. A detailed analisys of properties of superheavy quarkonium within this approach will be given elsewhere. For the case of our interest (b quarks) the second term on the r.h.s. of eq. (8) only imposes a bound in k on applicability of eq. (16) and we postpone a discussion of this bound till sec. 4.

3. $\mathcal{L}$-transform of the Vacuum Polarization Amplitude

In this section we calculate the $\mathcal{L}$-transform of the amplitude $P(4m^2-4k^2)$ given by eq. (16) and also here will be discussed a connection of the $\mathcal{L}$-transform with the moments of e^+e^--annihilation cross section [3,4,7].

The $\mathcal{L}$-operator is defined in eq. (15)

$$\mathcal{L}_\mu = \lim_{\substack{k^2\to\infty,\ n\to\infty \\ k^2/n=\mu^2/4}} \frac{(k^2)^n}{(n-1)!}\left(-\frac{d}{dk^2}\right)^n$$

It has the following simple properties

$$\mathcal{L}_\mu\, k^{-p} = \left[\Gamma\left(\frac{p}{2}\right)\left(\frac{\mu}{2}\right)^p\right]^{-1} \tag{17}$$

$$\mathcal{L}_\mu \ln k = -1/2 \tag{18}$$

Using these one finds from eq. (16)

$$\begin{aligned}\mathcal{L}_\mu P(4m^2-4k^2) &= -\frac{3}{8\pi m}\left[\frac{\mu}{2\Gamma(-\frac{1}{2})} - \frac{4}{3}m\alpha_s\frac{1}{2} + \right.\\ &\left. + \mathcal{L}_\mu \frac{4}{3}m\alpha_s\, \psi\left(1-\frac{2m\alpha_s}{3k}\right)\right] = \\ &= \frac{3\mu}{32\pi^{3/2}m}\left[1+2\sqrt{\pi}\,\frac{4}{3}\,\frac{m\alpha_s}{\mu} - \frac{4\sqrt{\pi}}{\mu}\mathcal{L}_\mu \frac{4}{3}m\alpha_s\,\psi\left(1-\frac{2m\alpha_s}{3k}\right)\right].\end{aligned} \tag{19}$$

To calculate $\mathcal{L}$ -transform of the function $\psi(1-\frac{2m\alpha_s}{3k})$ one can use the representation

$$\psi\left(1-\frac{2m\alpha_s}{3k}\right)=-C-\sum_{s=1}^{\infty}\left[-\frac{1}{s}+\frac{1}{s-\frac{2m\alpha_s}{3k}}\right] \tag{20}$$

(C is the Bernouilli constant) and find

$$\mathcal{L}_\mu\left(s-\frac{2m\alpha_s}{3k}\right)^{-1}=\frac{1}{s}\sum_{p=0}^{\infty}\mathcal{L}_\mu\left(\frac{2m\alpha_s}{3sk}\right)^p=\frac{1}{s}\sum_{p=1}^{\infty}\frac{(\beta/s)^p}{\Gamma(\frac{p}{2})} \tag{21}$$

where

$$\beta=\frac{4m\alpha_s(\mu)}{3\mu} \tag{22}$$

(the term with $p=0$ in eq. (21) vanishes).

The summation in eq. (21) can be performed using Laplace transform to obtain the result

$$\sum_{p=1}^{\infty}[(\beta/s)^p/\Gamma(\tfrac{p}{2})]=\frac{\beta/s}{\sqrt{\pi}}+\left(\frac{\beta}{s}\right)^2 e^{\beta^2/s^2}\left[1+erf(\beta/s)\right]$$

where $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. Keeping track of all terms one obtains from eqs. (19), (20) and (21)

$$F(\mu^2)\equiv\mathcal{L}_\mu\,P(4m^2-4k^2)=\frac{3\mu}{32\pi^{3/2}m}\,\Phi(\beta) \tag{23}$$

and

$$\Phi(\beta)=1+2\sqrt{\pi}\beta+\frac{2\pi^2}{3}\beta^2+4\sqrt{\pi}\sum_{s=1}^{\infty}\left(\frac{\beta}{s}\right)^3 e^{\beta^2/s^2}\left[1+erf\left(\frac{\beta}{s}\right)\right] \tag{24}$$

The plot of the function $\Phi(\beta)$ is given in Fig. 2.

Let us discuss now a relation between the $\mathcal{L}$ -transform of the vacuum polarization amplitude and the moments of the e^+e^--annihil ion c ss section used in Refs. 3,4,7. In those papers wer consi red theoretical predictions for the moments

$$A_n = (4m_f^2)^n \int \frac{R_f(s)\,ds}{s^{n+1}} \qquad (25)$$

where m_f is the mass of heavy quark of a given flavour f and $R_f(s)$ is the f-flavour contribution to the ratio R (more specifically c-quarks were considered).

Consider now the r.h.s. of eq. (25) for $n = 4m^2/\mu^2$ and $\mu^2 \ll 4m^2$ (i.e. $n \gg 1$). In this case one can write

$$(4m^2)^n/s^{n+1} = \frac{1}{\mu^2 n}\left(\frac{4m^2}{s}\right)^{n+1} =$$

$$= \frac{1}{\mu^2 n} \exp\left(\frac{4m^2 - s}{\mu^2}\right)\left(1 + O(n^{-1})\right).$$

Comparing this with eq. (6) one sees that the quantity $F(\mu^2)$ for $\mu^2 = 4m^2/n \ll 4m^2$ coincides up to a normalization factor with the asymptotic expression for nA_n when $n \gg 1$. It can be also noted that in the sense of this relation between $F(\mu^2)$ and nA_n four first terms in expansion of eq. (24) in powers of β correspond to the results of Ref. 4 (see eqs. (7.11) - (7.13) of Ref. 4) while the rest terms have not been calculated there.

4. Corrections to Eqs. (16) and (23)

In this section some corrections to eqs (16) and (23) will be discussed. As a result these expressions will be slightly modified before confronting with experimental data, and also the region of applicability of our results will be clarified.

Eqs. (16) and (23) give the result of summation of $(\alpha_s(2k)m/k)^n$ terms in the vacuum polarization amplitude when $k^2 \ll \mu^2$. It is interesting to note that one can readily find also all the terms of order $\alpha_s(m)(\alpha_s(2k)m/k)^n$. According to Schwinger[13] these terms arise from nonsingular in k when $k \to 0$ part of the electromagnetic form-factor of b-quark calculated to the first order in α_s (see Fig. 3). Adapting the QED result[13] to the case of QCD one finds that this effect results in the overall factor

$$1 - \frac{4}{\pi}\left(\frac{4}{3}\alpha_s(m)\right) \tag{26}$$

which renormalizes the expression obtained by using Schrödinger equation. Note also that this factor comes from distances $r \sim m^{-1}$. This factor is quite familiar to those who followed the literature on J/ψ width into e^+e^-[14]. In terms of our consideration the factor (26) must be introduced into the r.h.s. of eq. (7).

As to the corrections comming from exchange of transversal ("non Coulomb") gluons and from the three gluon vertices these are proportional to either $(k/m)^2$ or α_s^2 or $(k/m)^2\alpha_s$. For the values of k considered these corrections seem to be unimportant.

Thus accounting for the factor (26) one finds from eq. (23)

$$f(\mu^2) = \frac{3\mu}{32\pi^{3/2}m}\left(1 - \frac{16}{3\pi}\alpha_s(m)\right)\Phi(\beta) \tag{27}$$

This is in fact the expression which will be compared with experimental data in the region μ = 2 - 3.7 Gev. For larger μ relativistic effects become essential, while for $\mu \lesssim$ 2 Gev power terms rapidly grow up. Now we proceed to a discussion of the latter to argue that for $\mu >$ 2 Gev they can be neglected.

The leading power correction to the vacuum polarization is given by the second term on the r.h.s. of eq. (8). In case $k/m \gg \alpha_s$ (but still $k^2/m^2 \ll 1$) so that $\beta \ll 1$ one can neglect to a first approximation the Coulomb interaction and write

$$G_{(0)}(\vec{x},\vec{y},-k^2/m) \simeq G_{(8)}(\vec{x},\vec{y},-k^2/m) \simeq \frac{m}{4\pi r} e^{-kr}. \quad (28)$$

$(r = |\vec{x}-\vec{y}|)$.

For this case the result [9] is

$$P(4m^2-4k^2) = -\frac{3k}{8\pi m}\left(1+\frac{\langle 0|\pi\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle}{192}\frac{m^2}{k^6}\right)$$

and

$$F(\mu^2) = \frac{3\mu}{32\pi^{3/2} m}\left(1-\frac{8}{9}\langle 0|\pi\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle \frac{m^2}{\mu^6}\right) \quad (29)$$

$(\beta \ll 1)$.

For arbitrary β the function $F(\mu^2)$ must be of the form

$$F(\mu^2) = \frac{3\mu}{32\pi^{3/2} m}\left(1-\frac{16}{3\pi}\alpha_s(m)\right)\Big[\Phi(\beta) - \quad (30)$$

$$-\frac{8}{9}\langle 0|\pi\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle X(\beta)\Big],$$

where the function $X(\beta)$ can in principle be found by integrating well known Coulomb functions in eq. (8). However this involves a calculation of rather complicated convolutions of confluent hypergeometric functions. So far I failed to find $X(\beta)$ in a closed form. The first nontrivial term in the expansion of $X(\beta)$ can be obtained by considering the interaction $V_{(0,8)}$ in eqs. (9) perturbatively. The result is

$$X(\beta) = 1 + \frac{11\sqrt{\pi}}{6}\beta + O(\beta^2)$$

By considering the singularities of the terms on the r.h.s. of eq. (8) in the k^2-plane one can also find that apart from preexponential factors $\Phi(\beta)$ and $X(\beta)$ develope the same exponential asymptotic behavior when $\beta \to \infty$

$$\Phi(\beta) \backsim X(\beta) \backsim e^{\beta^2}$$

(Surely, the limit $\beta \to \infty$ has no physical sense since for $\mu \to 0$ all power terms are important not only the leading one. But as a mathematical poblem the asymptotic behavior of $\Phi(\beta)$ and $X(\beta)$ can be considered). Thus, no special reasons are seen for which the function $X(\beta)$ can be enhanced with respect to $\Phi(\beta)$ at least for moderate values of $\beta \lesssim 1$. (In fact we need $\beta \lesssim 0,7$). Therefore allowing for a possible numerical factor between $X(\beta)$ and $\Phi(\beta)$ we restrict ourselves to the values of μ for which the quantity $\langle 0 | \pi\alpha_s G^2_{\mu\nu} G^2_{\mu\nu} | 0 \rangle \frac{m^2}{\mu^6}$ does not exceed $\sim 5\%$, which correspond to $\mu \gtrsim 2$ Gev. In what follows the power correction will be completely neglected for $\mu > 2$ Gev.

5. Comparison with Experimental Data and Concluding Remarks

Summarising the preceeding discussion and using eq.(27) we conclude that for $\mu = 2 - 3{,}7$ Gev there is the sum rule for hidden b-flavour formation in e^+e^--annihilation

$$\mu^{-2}\int R_b(s)\exp\left(\frac{M_\Upsilon^2-s}{\mu^2}\right)ds =$$

$$= 12\pi^2 Q_b^2 F(\mu^2)\exp\left(\frac{M_\Upsilon^2-4m^2}{\mu^2}\right) = \tag{31}$$

$$= \exp\left(\frac{M_\Upsilon^2-4m^2}{\mu^2}\right)\frac{\sqrt{\pi}\mu}{8m}\left(1-\frac{16}{3\pi}\alpha_s(m)\right)\Phi(\beta); \quad (Q_b=-1/3).$$

This sum rule must be valid up to a (10-15)% accuracy for the above mentioned values of μ. (In eq. (31) both hands of eq. (6) are multiplied by exp. $[(M_\Upsilon^2-4m^2)/\mu^2]$ so that besides the running parameter μ the first expression in eq. (31) contains no theoretical quantities.)

In Fig. 4 is given a comparison of eq. (31) with experimental data on Υ and Υ' formation in e^+e^--annihilation. The experimental inputs [2] are M_Υ = 9,46 Gev, $M_{\Upsilon'}$ = = 10,015 Gev, $\Gamma(\Upsilon \to e^+e^-)$ = 1,2 Kev, $\Gamma(\Upsilon' \to e^+e^-)$ = = 0,33 Kev. (We recall that a narrow resonance contribution to $R(s)$ is given by

$$R_{Res}(s) = \frac{9\pi}{\alpha^2} M_{Res}\,\Gamma(Res \to e^+e^-)\,\delta(s-M_{Res}^2)).$$

Note that contribution of Υ' to the experimental integral in eq. (31) is at most 12% (for μ = 3,7 Gev). The contribution of still higher $b\bar{b}$ states is completely negligible due to exponential cutoff in the integral. The experimental uncertainties result in that the "experimental" curve in Fig. 4 (heavy line) can be multiplated as a whole by a factor (1±0,2).

The theoretical curves are sensitive both in normalization and in shape to values of α_s and m. In case one fixes α_s(2,5 Gev) = 0,2 as it has been done in charmonium theory [3-5,7] then the best fit for the mass of b quark is

. $m = 4.64$ Gev (the curve a in Fig. 4). By considering theoretical curves with m close to this value one can convince oneself that a mass parameter out of the interval $m = 4.65 \pm 0.05$ Gev can not fit the experimental curve even allowing for theoretical and experimental uncertainties. We illustrate this by drawing the curve b corresponding to $m = 4.60$ Gev.

If one also allows to vary α_s, then a somewhat better fit to experimental data is found in the region $\mu \gtrsim 2.3$ Gev (where theoretical uncertainties seem to be minimal) if $\alpha_s(2.5 \text{ Gev}) = 0.22$ and $m = 4.67$ Gev (the curve c in Fig. 4). However with our accuracy we cannot assert for sure that this value of α_s is much more preferable than $\alpha_s(2.5 \text{ Gev}) = 0.2$.

It can be also noted that for $\mu \lesssim 2$ Gev the theoretical curves are much more steeper than the experimental one (this trend is already seen from Fig. 4 at $\mu < 2.3$Gev). According to eq. (30) this deviation of theoretical curves must be corrected by power terms. Therefore qualitatively the behaviour of theoretical and experimental curves in this region is in agreement with the picture described above. However a detailed quantitative description in this region requires a calculation of the function $X(\beta)$.

Anyhow the agreement of theoretical prediction with experimental data on Υ and Υ' resonances seems to be by no means trivial and can be considered as one of most convincing illustrations of reality of gluonic exchange at short distances. Indeed the gluonic exchange brings to eq. (31) the factor $(1 - \frac{16}{3\pi}\alpha_s(m))\,\Phi(\beta)$ which for the values of β considered ($\beta = 0.3 - 0.7$) varies roughly from 2 to 8. One can readily verify that the free quark prediction (i.e. for $\alpha_s = 0$) gives nothing like the experimental curve with any value of the b quark mass. (See for example the curve d in Fig.4 which corresponds to free quarks with $m = 4.65$ Gev).

Probably an improvement of experimental data and theoretical predictions (a calculation of the function $\chi(\beta)$ and an account of relativistic corrections) will make it possible to fit from the sum rules like (31) not only the b quark mass but also precise values of α_s and of vacuum expectation value (4).

I am thankful to V.A.Novikov and L.B.Okun for useful discussions and comments.

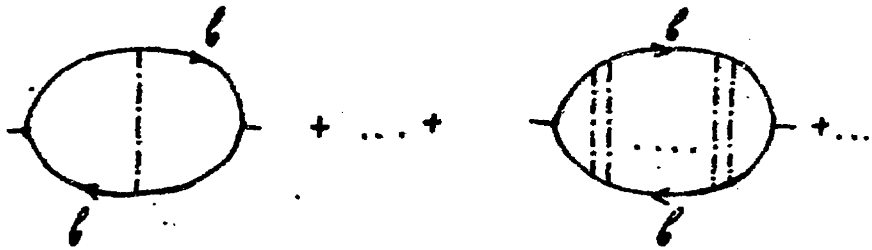

Fig.1. Diagrams with "Coulomb" gluon exchange which give the leading threshold singularities. The dotted lines correspond to "Coulomb" gluons.

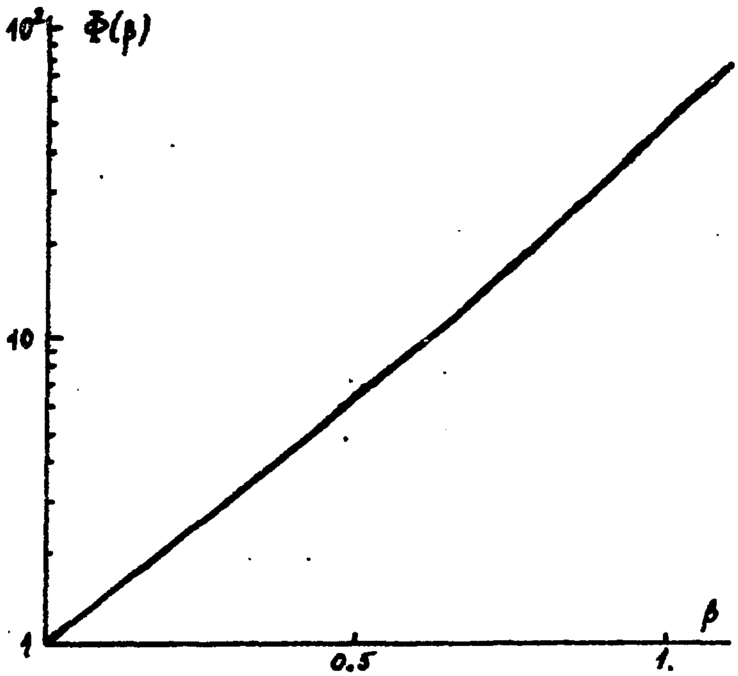

Fig. 2. The plot of the function $\Phi(\beta)$.

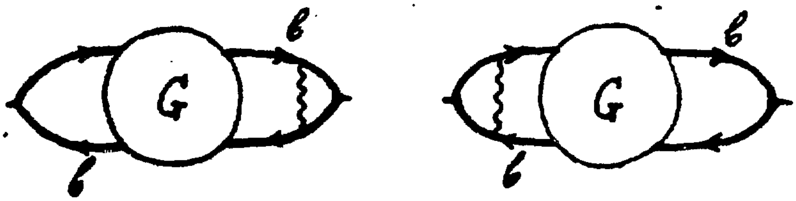

Fig. 3. The graphs for formfactor correction to vacuum polarization amplitude.

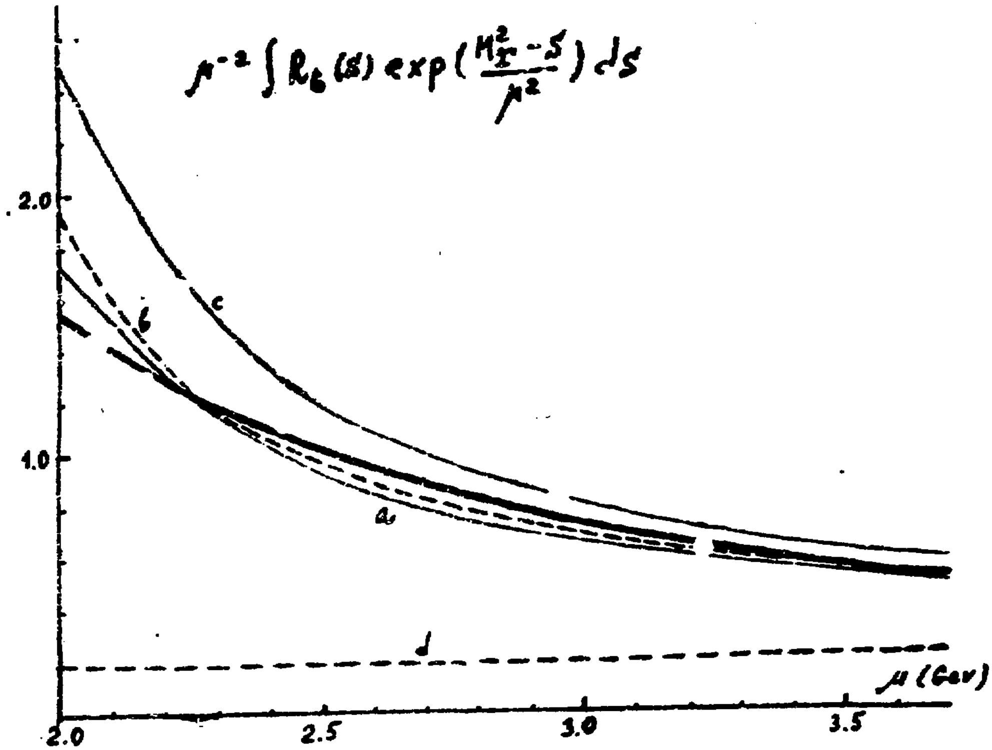

Fig. 4. The quantity $\mu^{-2}\int R_b(s)\exp\left(\frac{M_\Upsilon^2-s}{\mu^2}\right)ds$.
The heavy line is calculated from experimental masses and e^+e^- widths of Υ and Υ'. (Experimental errors correspond to that this curve can be multiplied as a whole by a factor (1 ± 0.2)). The curves a, b and c are theoretical predictions for various values of α_s and m :

a) $\alpha_s(2.5\ \text{Gev}) = 0.2$, $m = 4.64$ Gev,

b) $\alpha_s(2.5\ \text{Gev}) = 0.2$, $m = 4.60$ Gev,

c) $\alpha_s(2.5\ \text{Gev}) = 0.22$, $m = 4.67$ Gev.

The curve d is the free quark prediction for $m = 4.65$ Gev.

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Волошин М.Б.
Правила сумм для рождения Υ -частиц в e^+e^--аннигиляции
Работа поступила в ОНТИ I2/XII-I978г.

Подписано к печати 26/XII-78г. Т-2360I.Формат 70xI08 I/I6.
Печ.л.I,5.Тираж 290 экз.Заказ I76.Цена IIкоп.Индекс 3624

Отдел научно-технической информации ИТЭФ,II7259,Москва

ИНДЕКС 3624