EXTENDED FEYNMAN FORMULA FOR HARMONIC OSCILLATOR

P.A. HORVÁTHY [x] [+]

ABSTRACT : A slight modification of Feynman's original method leads to the Maslov correction in the path integral formula of a harmonic oscillator Caustics are treated in a direct geometric way.

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[x] Centre de Physique Théorique , CNRS, Marseille (France)

[+] On leave from Veszprem University of Chemical Engineering (Hungary)

POSTAL ADDRESS : CNRS - Luminy - Case 907
Centre de Physique Théorique
F-13288 MARSEILLE CEDEX 2 (France)

1. INTRODUCTION

As pointed out by Souriau [2], Feynman's formula for a harmonic oscillator ([1], p. 63)

$$K(x_2,t_2|x_1,t_1) = \tag{1}$$
$$\left(\frac{m\omega}{2\pi i\hbar \sin\omega(t_2-t_1)}\right)^{\frac{1}{2}} \cdot \exp\left\{\frac{i m\omega}{2\hbar \sin\omega(t_2-t_1)}\left[(x_1^2+x_2^2)\cos\omega(t_2-t_1) - 2x_1x_2\right]\right\}$$

is valid only for $|t_2-t_1| < T/2$, a half period. The general expression is obtained [2] by introducing the Maslov correction [2], [3], [6] and given as

(2) for $t_2 \neq t_1 + k\frac{T}{2}$, k integer

$$K(x_2,t_2|x_1,t_1) = \left(\frac{m\omega}{2\pi\hbar\,|\sin\omega(t_2-t_1)|}\right)^{\frac{1}{2}} \cdot$$
$$\cdot \exp\left\{-\frac{i\pi}{2}\left[\frac{1}{2} + \mathrm{Ent}\,\frac{\omega(t_2-t_1)}{\pi}\right]\right\} \cdot$$
$$\cdot \exp\left\{\frac{i m\omega}{2\hbar \sin\omega(t_2-t_1)} \cdot \left[(x_1^2+x_2^2)\cos\omega(t_2-t_1) - 2x_1x_2\right]\right\}$$

and for $t_2 = t_1 + k\frac{T}{2}$, k integer (caustics)

$$K(x_2,t_2|x_1,t_1) = \exp\left\{-\frac{i\pi}{2}k\right\} \cdot \delta(x_1 - (-1)^k x_2) \tag{3}$$

The effect of the correction factor $\exp\left\{-\frac{i\pi}{2}\cdot \mathrm{Ent}\,\frac{\omega(t_2-t_1)}{\pi}\right\}$ is a jump in phase at every half-period, observed by Gouy [7] in classical optics and having the consequence of reversing the interference pattern (see [8] for details). A similar phenomenon is observed in electron optics [9] as well as in molecular [12] and nuclear [13] scattering. (2) is more or less well-known ([4] ,[13]) ; it is generally derived by Morse's Theory [14] . At caustics, i.e. for $t_2 = t_1 + k\frac{T}{2}$, k integer, most of the authors are contended to observe that (2) diverges; they study then the corrections due to higher order perturbation. Souriau [2] derives (3) by an indirect way, noting the relation to metaplectic representation.

The aim of this paper is to show how the above results may be obtained by slightly modifying Feynman's original method.

2. FEYNMAN'S METHOD

First, we resume briefly Feynman's original method ([1] , pp. 58-73) in computing the quantum mechanical kernel for a harmonic oscillator.

Suppose $|t_2 - t_1| < T/2$, the half period (assumed implicitly by Feynman). Then, for any pair of points $x_1, x_2 \in \mathbb{R}$ there is a unique classical path $\bar{\gamma} : t \to \bar{\gamma}(t) \in \mathbb{R}$ between (x_1, t_1) and (x_2, t_2) It is useful to write then any path γ in the form $\gamma = \bar{\gamma} + \eta$, where the "varied curves" η vanish at the end points : $\eta(t_1) = \eta(t_2) = 0$

The quantum mechanical kernel, being expressed in terms of a gaussian integral, is a product of two factors [1] :

$$(4) \quad K(x_2, t_2 | x_1, t_1) = \exp\left\{ \frac{i}{\hbar} S(\bar{\gamma}) \right\} \cdot F(t_2 - t_1)$$

$S(\bar{\gamma})$ is here the hamiltonian action along the classical path $\bar{\gamma}$. For a harmonic oscillator

$$(5) \quad S(\bar{\gamma}) = \frac{m\omega}{2 \sin \omega(t_2 - t_1)} \left[(x_1^2 + x_2^2) \cos \omega(t_2 - t_1) - 2 x_1 x_2 \right]$$

The second factor in (4) depends only on $t_2 - t_1$ and is a result of integration over all paths η vanishing at the end points

$$(6) \quad T(t_2 - t_1) = \int \exp\left\{ \frac{i}{\hbar} \int_{t_1}^{t_2} \frac{m}{2}\left[(\dot{\eta}(t))^2 - \omega^2 \eta^2(t)\right] dt \right\} \mathcal{D}\eta$$

In order to assign a precise mathematical meaning and compute (6), Feynman expands the η's in Fourier series

$$(7) \quad \eta(t) = \sum_{j=1}^{\infty} a_j \sin \frac{j\pi}{t_2 - t_1}(t - t_1)$$

and, instead of integrating over the η's, integrates over the space of Fourier coefficients $(a_1, a_2, \ldots)$

$$(8) \quad T(t_2 - t_1) = \lim_{n\to\infty} J \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ \sum_{j=1}^{n} \frac{im}{\hbar}\left[\left(\frac{\pi j}{t_2 - t_1}\right)^2 - \omega^2\right] a_j^2 \right\} \frac{da_1}{A} \cdots \frac{da_n}{A}$$

The difficulty introduced by the infinite-valued Jacobian J is removed by a suitable choice of the normalizing factors A, which symbolize the measure of integration in the space of Fourier coefficients.

Carrying out the integration and fitting the results to the case $\omega = 0$, a free particle, Feynman gets (1). The ambiguity caused by in (1) is physically unimportant, for it gives only an overall phase factor.

3. BEYOND CAUSTICS

Note that beyond caustics, i.e. for $|t_2 - t_1| > T/2$, but $|t_2 - t_1| \neq \neq k\frac{T}{2}$ we have again a well-defined classical path between (x_1, t_1) and (x_2, t_2) and our formulas (4) - (7) are valid. A change to integration over Fourier coefficients is again possible. As to the factors J, A and integration order, note that they are essentially the same, as in (8) : they depend only on the transformation $\eta \to (a_1, a_2, \ldots)$ and are completely independent of "physics", i.e. of the function to be integrated. Thus (8) will be perfectly meaningful as soon as we arrive to remove the ambiguity due to the

$$(9) \qquad k = \mathrm{Ent}\,\frac{\omega(t_2-t_1)}{\pi} = \mathrm{Ent}\,\frac{t_2-t_1}{T/2}$$

number of negative terms in the sum of (8) . This is achieved by an analytic extension of the classical Fresnel integral, possible for $\mathrm{Im}\,\lambda \geq 0$, $\lambda \neq 0$ [5] , [6]

$$(10) \quad F(\lambda) = \int_{-\infty}^{\infty} \exp\left(i\frac{\lambda}{2}x^2\right) dx = \begin{cases} \text{for } \lambda > 0 & \sqrt{\frac{2\pi}{\lambda}} \cdot e^{i\frac{\pi}{4}} \\ \text{for } \lambda < 0 & \sqrt{\frac{2\pi}{-\lambda}} \quad e^{-i\frac{\pi}{4}} \end{cases}$$

Thus, Feynman's formula has to be modified only by taking absolute value in $|\sin\omega(t_2-t_1)|$ and multiplying by

$$(11) \qquad \exp\left(-\frac{i\pi}{2}k\right) = \exp\left(-\frac{i\pi}{2}\cdot \mathrm{Ent}\,\frac{\omega(t_2-t_1)}{\pi}\right)$$

in accordance with (2).

4. AT CAUSTICS

For $t_2 = t_1 + k\frac{T}{2}$, k integer, the situation is radically changed : all classical paths starting from x_1 coalesce to $(-1)^k x_1$. Thus, for any arbitrary pair of points x_1 , x_2 , we have either no classical path at all or an infinity of classical paths between them. Feynman's method breaks down even in this latter case, because the coefficient of a_k^2 in (8) vanishes and the Fresnel integral diverges. In terms of "infinite dimensional manifolds" [8], [10], [14] , (4) is valid if the hamiltonian action, considered as a function defined on the set of all paths between x_1 and x_2 , has only one "critical point", i.e. classical path. At caustics this condition is not satisfied and one has to evaluate the Feynman integral by other means.

The easiest way is to work with operators, rather than with kernels merely. Remember that the time evolution of a system is given as

(12)
$$\psi_{t_2}(x_2) = \left[\mathcal{U}_{t_2-t_1}\psi_{t_1}\right](x_2) = \int_{\mathbb{R}} K(x_2,t_2|x_1,t_1)\,\psi_{t_1}(x_1)\,dx_1$$

By the multiplication law one has

(13)
$$\mathcal{U}_{k\frac{T}{2}} = \left[\mathcal{U}_{\frac{T}{4}}\right]^{2k}$$

By (1) and (12)

(14)
$$\left[\mathcal{U}_{\frac{T}{4}}\psi_{t_1}\right](x_2) = \int_{\mathbb{R}} \left(\frac{m\omega}{2\pi\hbar}\right)^{\frac{1}{2}} \cdot e^{-\frac{i\pi}{4}} \cdot\cdot\; e^{-\frac{im\omega}{\hbar}x_1x_2}\,\psi_{t_1}(x_1)\,dx_1$$

essentially a Fourier-transform and thus, noting that if the Fourier-transform of a function is once more Fourier-transformed, then one obtains the original function reflected with respect to $x = 0$, one gets

(15)
$$\psi_{t_2}(x_2) = e^{-\frac{i\pi}{2}k} \cdot \psi_{t_1}((-1)^k x_2)$$

which is just (3).

Note that (13) could be interpreted as

(16)
$$K(x_2,t_2|x_1,t_1) = \int K^{\alpha}(x_2,t_2|x_1,t_1)\,d\alpha$$

where α is a continuous parameter characterizing the "critical points" (i.e. classical paths) of S. The partial amplitudes K^{α} are composed of the contributions of the corresponding classical path $\bar{\gamma}^{\alpha}$ multiplied by the correction factor due to paths "oscillating around $\bar{\gamma}^{\alpha}$". These "oscillating paths" are exactly those which pass through $\bar{\gamma}^{\alpha}(t_1+\frac{T}{4})$, $\bar{\gamma}^{\alpha}(t_1+3\frac{T}{4}),\ldots\;\bar{\gamma}^{\alpha}(t_1+(2k-1)\frac{T}{4})$

$$(17)\qquad K^{\alpha}(x_2,t_2\,|\,x_1,t_1) = \exp\frac{i}{\hbar}\, S(\bar{\gamma}^{\alpha})\cdot \mathcal{F}^{\alpha}(t_2-t_1)$$

It is easy to see that the dividing points could be substituted by any ordered set $t^{(1)}, t^{(3)}, \dots t^{(2k-1)}$ satisfying

$$(18)\quad t_1 < t^{(1)} < t_1+\frac{T}{2} < \dots < t_1+(k-1)\frac{T}{2} < t^{(2k-1)} < t_1+k\frac{T}{2}$$

(16)-(17) is the substitute to (4) valid for coalescing paths.

5. CONCLUDING REMARKS

In describing the propagator near caustics, one studies generally [8], [9], [12], [13] the effect of higher order corrections due to anharmonicity, which change our δ to a more realistic function. We conjecture however, that the <u>phase</u> of the wave-function will be determined essentially by the pure quadratic part, which we have studied. This would be observable in interference-experiments, supposed we have a kind of <u>structural stability</u> [11] in phase. This problem will be studied elsewhere.

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