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O(3) SYMMETRIC MERONS IN SU(N) GAUGE THEORY

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ABSTRACT

We investigate non-reducible $O(3)$ symmetric meron solutions to classical $SU(N+1)$ Yang-Mills theory in four dimensional Euclidean space. For even N the solutions have topological charge densities equal to a sum of delta-functions with integer coefficients while for odd N /$N>1$/ these coefficients can be both integer and half-integer. In all cases they correspond to solutions of a system of N coupled singular elliptic equations. We discuss the existence of two meron solutions of this system and for $N=3,4$ give some numerical solutions too.

I. Introduction

Particular solutions to classical gauge theories have been the focus of many recent investigations. Part of these solutions - called merons - are infinite action solutions of Euclidean Yang-Mills equations with topological charge density concentrated at points. The interest in merons stems from the fact that they can be used in models for quark confinement[1,2]. The first meron solution was obtained in $SU(2)$[3], later it was generalized to multiple merons on a line[4]. These $SU(2)$ configurations of course can always be embedded into any $SU(M)$ theory believed to describe the interaction among quarks. /The general belief is that for the physical world $M=3$, however, with integer charged quarks it may be 4 as well[5]. The $SU(5)$ merons may play a role in a so called grand-unification scheme[6]/. However, in these cases there is a possibility that genuine $SU(M)$ configurations exist which cannot be obtained from an $SU(2)$ embedding. We call these non-reducible. It is our intention in this paper to exhibit the existence and discuss the properties of such non-reducible merons in an $SU(N+1)$ gauge theory.

So far this kind of merons has been discussed only for $SU(3)$[7]. There it was shown that non-reducible $SU(3)$ merons have their topological charge quantized in integers instead of in halfintegers, as in the case for $SU(2)$ merons. Here we derive the generalization of this fact for any N /Sect.III./.

To obtain non-reducible merons in $SU(N+1)$ we use an $O(3)$

symmetric ansatz of ref.[8] used to obtain non-reducible instantons. In terms of this ansatz the original four dimensional gauge theory gets replaced by N mutually coupled, two dimensional Abelian Higgs models with a residual $U(1) \times \ldots \times U(1)$ /N times/ gauge group. In the case $N > 2$ we find a novel thing: the possible meron charges in some different $U(1)$ components are different. For even N they all are integers while for odd N they can be both integers and half-integers.

We reduce the problem of finding $SU(N+1)$ merons to obtaining solutions with appropriate boundary conditions to a coupled system of elliptic differential equations for N real functions. We outline a general proof of existence /for any N/ of two meron solutions of this system using upper and lower bounds, that we give explicitely for $N=3$ /$SU(4)$/ and $N=4$ /$SU(5)$/. In these cases we also give numerical solutions for all possible two merons.

The paper is organised as follows. In sect. II. we describe the $O(3)$ symmetric ansatz and derive the $SU(N+1)$ gauge field equations in terms of this ansatz. In sect. III. we look for solutions to these equations that have topological charge density appropriate to merons and derive the possible values of meron charges. Sect. IV. contains the discussion of two meron solutions together with the proof of their existence and the numerical solutions for $SU(4)$ and $SU(5)$, while in sect. V. we discuss some general properties of our two merons.

II. O(3) Symmetric Ansatz for SU(N+1) Gauge Theory

To find O(3) symmetric merons in an SU(N+1) gauge theory we use the ansatz of ref.[8] exhibiting in a manifest way the symmetry under the mixed angular momentum operator

$$\vec{J} = -i\,\vec{r} \times \vec{\nabla} + \vec{T} \tag{1}$$

where $\vec{T}$ is a matrix representation of SO(3). The ansatz which is a generalisation for SU(N+1) of the expressions used in refs.[4,7] for cylindrically symmetric merons in SU(2) and SU(3) respectively takes the form

$$\begin{aligned} \vec{W} &= \left(\vec{M}(r,\hat{r},t) - \vec{T}\right) \times \frac{\vec{r}}{r^2} + A_1(r,\hat{r},t)\frac{\vec{r}}{r} \\ W_0 &= A_0(r,\hat{r},t) \end{aligned} \tag{2}$$

Here $\vec{M}$ is a vector and A_0, A_1 are scalars under (1) and $\hat{r}$ is the radial unit vector. Using the most recent results[9] for constructing gauge fields with given symmetry properties one can prove in a straightforward way that the expressions above are really the most general SO(3) symmetric ones for an SU(N+1) gauge theory.

Though $\vec{M}$, A_1 and A_0 do depend on the spherical angles via $\hat{r}$ the field strengths

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - i\,[W_\mu, W_\nu]$$

depend only on derivatives with respect to r and t because the angular derivatives in

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{r} \times (\hat{r} \times \vec{\nabla})$$

combine with the $\vec{\tau}\times\vec{T}$ in (2) to yield commutators with $\vec{\tau}\times\vec{\xi}$ that are determined by the SO(3) symmetry alone[8]. Therefore from now on we shall evaluate all fields on the z axis. The SO(3) symmetry then enables us to evaluate them along any other axis[10]. As we want to have non-reducible SU(N+1) merons we shall consider the case when $\vec{T}$ is the maximal embedding of SO(3) in SU(N+1) with

$$T_3 = \mathrm{diag}\left(\frac{N}{2}, \frac{N}{2}-1, \dots, 1-\frac{N}{2}, -\frac{N}{2}\right) \tag{3a}$$

Finally our ansatz for A_0, A_1, $M_\pm$ /along the z axis/ is

$$A_\mu = \frac{1}{2}\begin{pmatrix} A_\mu^1 & & & & \\ & A_\mu^2 - A_\mu^1 & & & \\ & & \ddots & & \\ & & & A_\mu^N - A_\mu^{N-1} & \\ & & & & -A_\mu^N \end{pmatrix} \qquad M_+ = \begin{pmatrix} 0 & \varphi_1 & & & \\ & 0 & \varphi_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & \varphi_N \\ & & & & 0 \end{pmatrix}$$

where we introduced a two dimensional notation with $x^0 = t$, $x^1 = r$ and the functions $A_\mu^m(r,t)$ are real and $\varphi_m(r,t)$ complex with $M_- = (M_+)^+$. /Note that the matrices are N+1 dimensional but each contains N functions only./ This ansatz is invariant with respect to Abelian gauge transformations from a residual U(1) x U(1)...x U(1) /N times/ subgroup. To display this we define

$$D_\mu \varphi_m = \partial_\mu \varphi_m + i a_\mu^m \varphi_m \tag{4}$$

$$a_\mu^m = \frac{1}{2} A_\mu^{m-1} - A_\mu^m + \frac{1}{2} A_\mu^{m+1} \tag{5}$$

where $A_\mu^0 = A_\mu^{N+1} \equiv 0$. The (5) equations can be inverted using the $C^{m,k}$ matrix

$$A_\mu^m = - C^{m,k} a_\mu^k \tag{6}$$

with $C^{m,k} = \frac{2k(N+1-m)}{N+1} = \frac{2k\bar{m}}{N+1}$ for $k \leq m$ and $C^{m,k} = \frac{2m(N+1-k)}{N+1} = \frac{2m\bar{k}}{N+1}$ for $k \geq m$. $C^{m,k}$ satisfy two important identities that the we shall frequently use:

$$\frac{1}{2} C^{m-1,k} - C^{m,k} + \frac{1}{2} C^{m+1,k} = -\delta^{m,k} \quad ; \quad \sum_m C^{m,k} = k\bar{k} \tag{7}$$

The SU(N+1) field strengths evaluated on the z axis take the following form in terms of a_μ^m and φ_m:

$$(B_3)_{m,m'} = \delta_{m,m'} \frac{1}{r^2} \left(\frac{1}{2} |\varphi_m|^2 - \frac{1}{2} |\varphi_{m-1}|^2 - \left(\frac{1}{2} N + 1 - m \right) \right)$$

$$(B_+)_{m,m'} = \delta_{m+1,m'} \frac{1}{r} D_1 \varphi_m$$

$$(E_3)_{m,m'} = \delta_{m,m'} \frac{1}{2} \left(F_{01}^m - F_{01}^{m-1} \right), (E_+)_{m,m'} = \delta_{m+1,m'} \left(-\frac{i}{r} \right) D_0 \varphi_m$$

with $F_{\mu\nu}^m = \partial_\mu A_\nu^m - \partial_\nu A_\mu^m = C^{m,k} (\partial_\nu a_\mu^k - \partial_\mu a_\nu^k) = C^{m,k} f_{\mu\nu}^k$ and $\varphi_0 = \varphi_{N+1} \equiv 0$. These field strengths transform covariantly under the Abelian gauge transformation

$$a_\mu^m \to a_\mu^m + \partial_\mu \Lambda^m \qquad \varphi_m \to e^{-i\Lambda^m} \varphi_m$$

for any real function $\Lambda^m(r,t)$.

Now it is easy to express the action of the SU(N+1) gauge theory in terms of $a_\mu^m(r,t)$ and $\varphi_m(r,t)$; after integrating over the angular variables we obtain an effective Lagrangian for the a_μ^m and φ_m in the $R_+^2 = \{t, r \mid r > 0\}$ half plane:

$$L = 2\pi \sum_{m=1}^{N+1} \left\{ \frac{1}{r^2}\left(\frac{1}{2}|\varphi_m|^2 - \frac{1}{2}|\varphi_{m-1}|^2 - \left(\frac{1}{2}N+1-m\right)\right)^2 + \right. \tag{8}$$
$$\left. + |D_1\varphi_m|^2 + |D_0\varphi_m|^2 + \frac{r^2}{4}\left(F_{01}^m - F_{01}^{m-1}\right)^2 \right\}$$

This is the Lagrangian for N Abelian Higgs fields φ_m interacting with each other as well as the N abelian gauge fields a_μ^m of the residual $U(1) \times U(1) \times \ldots \times U(1)$ group in the curved two dimensional half plane R_+^2 with metric $g^{\mu\nu} = r^2 \delta^{\mu\nu}$. The Higgs potential is of the symmetry breaking type with several minima. A particular feature of L is the coupling between the field strengths of the different $U(1)$ components of the direct product group.

The field equations of the $SU(N+1)$ gauge theory expressed in terms of a_μ^m and $\varphi^m = \Phi_1^m + i\Phi_0^m$ are simply given by the variations of L with respect to these functions. For a_μ^m we find

$$\partial_0 (r^2 C^{m,k} f_{01}^k) = -2 I_1^m$$
$$\partial_1 (r^2 C^{m,k} f_{10}^k) = -2 I_0^m \qquad m = 1, \ldots N \tag{9}$$

Here we introduced the vector $I_\mu^m = \Phi_1^m \nabla_\mu \Phi_0^m - \Phi_0^m \nabla_\mu \Phi_1^m$ where the new covariant derivative ∇_μ of the two component vector $\vec{\Phi}_m$ is defined as

$$\nabla_\mu \Phi_i^m = \partial_\mu \Phi_i^m + \epsilon_{ij} a_\mu^m \Phi_j^m$$

The (8) equations can be written in a more compact, coordinate free form

$$* d * C^{m,k} f^k = -2 I^m \qquad m = 1, \ldots N \tag{9a}$$

where $I^m = I^m_\mu dx^\mu$ stands for the external derivative and $*$ denotes the Hodge adjoint. We note the appearence of the r^2 terms in the equ. (9) as a direct consequence of the curved structure of R^2_+.

For the scalars $\vec{\Phi}^m$ we obtain

$$r^2 \nabla_\mu (\nabla_\mu \phi^m_i) - \phi^m_i (|\vec{\Phi}^m|^2 - \tfrac{1}{2}|\vec{\Phi}^{m-1}|^2 - \tfrac{1}{2}|\vec{\Phi}^{m+1}|^2 - 1) = 0 \qquad (10) \quad \begin{matrix} m=1,\dots N \\ i=0,1 \end{matrix}$$

where $|\vec{\Phi}^m|^2 = (\phi^m_1)^2 + (\phi^m_0)^2$

Equ.s (9) and (10) are the field equations for a general SO(3) symmetric SU(N+1) gauge theory. In what follows we shall find solutions to these equations with topological charge density concentrated in different points on the r=0 line.

III. System of Equations for Non-reducible SU (N+1) Merons

The topological charge /or Pontryagin/ density of the four dimensional SU (N+1) gauge theory after some algebraic manipulation can be expressed in terms of a^m_μ and ϕ^m_i as

$$Q(\vec{r},t) = \frac{1}{8\pi^2} E^a_i B^a_i = \frac{1}{8\pi^2 r^2} \left\{ \sum_{m=1}^{N} \left(k^m_{01} + m\bar{m}\, f^m_{01} \right) \right\} \qquad (11)$$

where $k^m_{\mu\nu}$ is the curl of the vector I^m_μ : $k^m_{\mu\nu} = \partial_\mu I^m_\nu - \partial_\nu I^m_\mu$ i.e. $k^m = dI^m$. Note that in (11) in contrast to (8) there is no coupling between the different U(1) .ectors. Integrating $Q(\vec{r},t)$ over the angular variables we obtain the charge density in R^2_+ :

$$q(r,t) = \frac{1}{2\pi} \sum_{m=1}^{N} \left(k^m_{01} + m\bar{m}\, f^m_{01} \right)$$

Now as we want to obtain SO(3) symmetric merons, i.e. merons located on the line r=0 at different t_i we look for solutions to equ.s (9-10) such that

$$Q(\vec{r},t) = \sum_i \alpha_i\, \delta^{(3)}(\vec{r})\, \delta(t-t_i)$$

This means that in R^2_+ we must have

$$\frac{1}{2\pi} \left\{ \sum_{m=1}^{N} \left(k^m_{01} + m\bar{m}\, f^m_{01} \right) \right\} = \sum_i 2\alpha_i\, \delta(r)\, \delta(t-t_i) \tag{12}$$

Using the fact that $k^m_{01} = (dI^m)_{01}$ and that the field equations (9) imply

$$k^m_{01} = -\frac{C^{m,k}}{2} \left(d * d * f^k \right)_{01}$$

we find that the requirement (12) is equivalent to

$$\frac{1}{2\pi} \left\{ \left[-\frac{1}{2} d*d* + 1 \right] \left(\sum_{m=1}^{N} m\bar{m}\, f^m \right) \right\}_{01} = \sum_i 2\alpha_i\, \delta(r)\, \delta(t-t_i) \tag{13}$$

Because of the r^2 in $*d*f$ a solution to (13) and (9) is given by

$$f^m_{01} = \sum_i \frac{4\pi}{m\bar{m}}\, \beta^m_i\, \delta(r)\, \delta(t-t_i) \tag{14}$$

implying $I^m_\mu = 0$ and $\alpha_i = \sum_{m=1}^{N} \beta^m_i$. In what follows we take this solution and reduce the remaining gauge field equations (10) to N coupled elliptic equations for N real functions of r and t. It will also turn out that - within a class of gauge transformations - β^m_i must be an integer multiple of $\frac{1}{2} m\bar{m}$

in order for W_μ be regular away from the locations of merons.

Following refs (4,7) we find an a_μ^m such that $f_{\mu\nu}^m = \partial_\mu a_\nu^m - \partial_\nu a_\mu^m$ where $f_{\mu\nu}^m$ is given by (14):

$$a_\mu^m = \partial_\mu \vartheta^m \qquad\qquad \vartheta^m = \sum_i \frac{2\beta_i^m}{m\bar{m}} \arg(-t + t_i + i\tau) \tag{15}$$

With the aid of these ϑ^m we carry out a rotation on the $\vec{\Phi}^m$ vectors

$$\psi_i^m = U_{ij}^m \phi_j^m \qquad\qquad U^m = \begin{pmatrix} \cos\vartheta^m & \sin\vartheta^m \\ -\sin\vartheta^m & \cos\vartheta^m \end{pmatrix} \tag{16}$$

These U^m rotate the covariant derivatives ∇_μ into ordinary ones as $\partial_\mu U_{ij}^m = a_\mu^m \epsilon_{il} U_{lj}^m$ and $U^m \epsilon (U^m)^{-1} = \epsilon$ so that

$$U^m \nabla_\mu (U^m)^{-1} = U^m (\partial_\mu + a_\mu^m \epsilon_{ij})(U^m)^{-1} = \partial_\mu \tag{17}$$

Therefore equ.s (10), the second set of the gauge field equations, become

$$\tau^2 \Delta \psi_i^m - \psi_i^m \left(|\vec{\psi}^m|^2 - \tfrac{1}{2} |\vec{\psi}^{m-1}|^2 - \tfrac{1}{2} |\vec{\psi}^{m+1}|^2 - 1 \right) = 0 \qquad (18) \quad \begin{matrix} i = 0,1 \\ m = 1,\dots N \end{matrix}$$

As a consequence of (17) the circulation density of $\vec{\psi}^m$

$$\epsilon_{ij} \psi_i^m \partial_\mu \psi_j^m = \epsilon_{ij} \phi_i^m \nabla_\mu \phi_j^m = -I_\mu^m = 0$$

therefore $\vec{\psi}^m$ maintains a constant direction in those connected domains where $\vec{\psi}^m \neq 0$; i.e. by U^m we rotated $\vec{\Phi}^m$ into a single direction. If $\vec{\psi}^m$ solves (18), it can, at worst, change direction by π across curves where $\vec{\psi}^m = 0$. This means that (18) is equivalent to the following system

$$r^2\Delta\psi^m - \psi^m\left((\psi^m)^2 - \tfrac{1}{2}(\psi^{m-1})^2 - \tfrac{1}{2}(\psi^{m+1})^2 - 1\right) = 0 \qquad {}_{m=1,\dots N} \tag{19}$$

where ψ^m is the component of $\vec{\psi}^m$ along a fixed direction which is one of the two possible directions of $\vec{\psi}^m$. This is the system of coupled elliptic differential equations we mentioned above, these equations are the equivalents in SU(N+1) of the equations describing non reducible merons in SU(2) and SU(3) in refs (4,7) respectively. Of course only those solutions of (19) can describe SU(N+1) merons that meet some boundary conditions.

We require for merons that the SU(N+1) gauge field strengths be regular at r=0 /away from the merons, i.e. for all $t \neq t_i$/. This requirement implies that

$$\lim_{r\to 0}|\varphi_m| = \lim_{r\to 0}\left((\varphi_0^m)^2 + (\varphi_1^m)^2\right)^{1/2} = \lim_{r\to 0}|\psi^m| = \sqrt{m\bar{m}} \qquad {}_{m=1,\dots N} \tag{20}$$

as in this case - possibily after a gauge transformation - $\lim_{r\to 0} M_+ = T_+$. Note that $|\varphi_m| = \sqrt{m\bar{m}}$ constitute the minima of the scalar potential in (8). As long as (20) is satisfied there is always a gauge transformation such that applying this transformation we obtain $\lim_{r\to 0}\varphi_1^m = \sqrt{m\bar{m}}$; $\lim_{r\to 0}\varphi_0^m = 0$ for $t_i < t < t_{i+1}$ $i = 1,\dots$ Therefore when we move across t_i at r=0 from (15,16) we see that $\vec{\psi}^m$ changes direction by $2\beta_i^m\pi/m\bar{m}$. On the other hand as we have shown above $\vec{\psi}^m$ can change direction at worst by π . Hence we conclude that β_i^m must be an integer multiple of $\frac{1}{2}m\bar{m}$.

However, β_i^m can be changed by any amount by a gauge transformation unless we restrict our attention to a class of gauge transformations that are continuous across t_i at r=0. The gauge transformation changing β_i^k has the form:

$$\Lambda^m = \delta^{k,m} \sum_i \lambda_i \arg(-t+t_i+ir)$$

Clearly under this transformation $\beta_i^m \to \beta_i^m$ $m \neq k$ and $\beta_i^k \to \beta_i^k + \frac{1}{2} k\bar{k}\lambda_i$, while $\vec{\Phi}^m$-s undergo a rotation by

$$U(\Lambda^m) = \begin{pmatrix} \cos\Lambda^m & \sin\Lambda^m \\ -\sin\Lambda^m & \cos\Lambda^m \end{pmatrix}$$

This $U(\Lambda^m)$ is continuous for all m at r=0 across t_i if $\lambda_i \pi = 2 n_i \pi$ with integer n_i. This means that β_i^m is defined /i.e. is gauge invariant/ mod $m\bar{m}$.

In conclusion to find SU(N+1) meron solutions with topological charge density $Q(\vec{r},t) = \sum_i \left(\sum_m \beta_i^m\right)\delta^{(3)}(\vec{r})\delta(t-t_i)$ is reduced to solving (19) with boundary conditions

$$\psi^m(0,t) = (-1)^{\frac{2}{m\bar{m}}\sum_{i=1}^{k}\beta_i^m}\sqrt{m\bar{m}} \qquad t_{k-1} < t < t_k,\ m=1,\dots N \tag{21}$$

where β_i^m are integer multiples of $\frac{1}{2}m\bar{m}$.

Let us now discuss the emerging structure of topological charges of SU(N+1) merons supposing for the moment that the appropriate solutio: s to (19,21) exist. We say that a meron configuration is elementary and is entirely in the k-th U(1) sector if $\beta_i^m = 0$ for $m \neq k$ /we recall that m-s index the different U(1) sectors (4-6)/. For all values of m the smallest /non-zero/ value of β_i^m is $\frac{1}{2}m\bar{m}$. As m ranges from 1 to N there are N types of possible elementary meron charges and of them $\frac{N+1}{2}$ are different for odd N and $\frac{N}{2}$ are different for even N /because $(\overline{m}) = m$ /. Thus for N=1 /SU(2)/ or N=2 /SU(3)/ we obtain the results of refs[3,4,7]: the elementary meron charges are $\frac{1}{2}$ and 1 respectively, in the latter case this value is valid for the elementary merons in both U(1) sectors. However, as we go beyond SU(3) i.e. as N exceeds 2 an unexpected thing happens: there are elementary merons in the different U(1) sectors with different charges! /e.g. for N=3 and N=4 the possible values are $\frac{3}{2}$ /m=1,3/ and 2 /m=2/ and 2 /m=1,4/ and 3 /m=2,3/ respectively/. It is clear that for even N all β_i^m will be integer while for odd N the β_i^m -s can be both integer /for even m/ and halfinteger /for odd m/. In a similar way as $\alpha_i = \sum_m \beta_i^m$ for non-elementary merons the total topological charge at t_i is integer for N even and can be both integer and halfinteger for N odd / >1/.

It is also obvious that for all N the minimal elementary meron charge at t_i is obtained for m=1 /m=N/ and is given by $\beta_i^1 = \frac{N}{2}$. This value is exactly the half of the minimal topological charge for non-reducible SU(N+1) instantons found in ref[8] within the ansatz (2,3). This fact gives support us to retain the name "meron" - that originated from fractional charges - even when $\frac{N}{2}$ is integer.

Finally, we would like to mention that these new values for the topological charge for merons are not artifacts of normalization. Rather they are the consequence of the form of the maximal representation of the O(3) generators allowed by the SU(N+1) generators.

IV. Existence and Numerical Solutions for Non-reducible Elementary SU(N+1) Meron Pairs

The Ψ^m functions contain a hidden dependence on the index m via the boundary conditions (21). We find it convenient to make this hidden dependence explicite by introducing $\Psi^m = \sqrt{m\bar{m}}\,\chi^m(r,t)$ as for χ^m the field equations (19) take the form:

$$r^2\Delta\chi^m - \chi^m\left(m\bar{m}\,\chi^{m\,2} - \frac{1}{2}(m-1)\overline{(m-1)}\,\chi^{m-1\,2} - \frac{1}{2}(m+1)\overline{(m+1)}\,\chi^{m+1\,2} - 1\right) = 0 \quad (22)$$
$$m = 1,\dots N$$

while boundary conditions (21) change to

$$\chi^m(0,t) = (-1)^{\frac{2}{m\bar{m}}\sum_{i=1}^{k}\beta_i^m} \qquad t_{k-1} < t < t_k \qquad m = 1,\dots N \quad (23)$$

i.e. χ^m are ± 1 on the r=0 line independently of m. Note that - as a consequence of the elementary identities $\bar{m}-1 = \overline{m+1}$, $\bar{m}+1 = \overline{m-1}$ - the system (22) has a nice symmetry under the change $\chi^m \leftrightarrow \chi^{\bar{m}}$ $m = 1,\dots N$. If β_i^m are such that the boundary conditions (23) respect this symmetry we can reduce the number of equations in (22).

If $\beta_i^m = \frac{1}{2}m\bar{m}$ /mod $m\bar{m}$/ for all values of m and i then the substitution $\chi^m = \chi$ reduces (22,23) to a single equation - to the equation for merons in SU(2) -

$$r^2\Delta\chi - \chi(\chi^2 - 1) = 0 \quad (24)$$

with appropriate boundary conditions for multiple merons. A proof of existence of solutions to (24) with these boundary conditions was given in ref [11]. As for $\chi^m = \chi$ m=1,..N $M_\pm \sim T_\pm$ everywhere in R_+^2 these solutions describe the trivial embedding of the $SU(2)$ /multiple/ merons into the maximal $SO(3)$ subgroup of $SU(N+1)$.

In establishing the existence of the general solutions to (24) a central role was played by the closed form two-meron solution. As we expect that in the general case of (22,23) the two meron solutions can play an equally important role we turn now to the discussion of existence of elementary two meron solutions to (22,23), though no analytic /closed form/ solution to (22,23) is known for any choice of β_i^m different from the previous one.

In what follows we consider only such elementary two meron configurations when one of the merons is sitting at the origin and the other is shifted to infinity. /The case when the two merons are at a finite distance from each other can be obtained from this configuration by a conformal transformation./ For the χ^m functions describing such an elementary two meron configuration in the <u>k-th</u> <u>U(1) sector</u> the (23) boundary conditions read as

$$\chi^m(0,t)=1 \quad m\neq k \ \text{all}\ t, \quad \chi^k(0,t)=1 \quad t<0, \quad \chi^k(0,t)=-1 \quad t>0 \tag{25}$$

because for such an elementary meron at t=0 $\beta^m = 0$ $m \neq k$, $\beta^k = \frac{1}{2} t\bar{t}$

Further we assume that for these two merons χ^m depend on r and t only via the combination $\Theta = \arg(-t+ir)$: $\chi^m(r,t) = \chi^m(\Theta)$ This assumption changes (22) into a system of ordinary differential equations / $\chi_m' = \frac{d\chi_m}{d\Theta}$ /

$$\sin^2\theta\, \chi_m'' - \chi_m \Big(m\bar{m}\, \chi_m^2 - \tfrac{1}{2}(m-1)\overline{(m-1)}\, \chi_{m-1}^2 - \qquad (26)$$

$$- \tfrac{1}{2}(m+1)\overline{(m+1)}\, \chi_{m+1}^2 - 1 \Big) = 0 \qquad m = 1, \dots N$$

while (25) become

$$\lim_{\theta\to 0} \chi^m = \lim_{\theta\to\pi} \chi^m = 1 \quad m \neq k\, ; \quad \lim_{\theta\to 0} \chi^k = \lim_{\theta\to\pi} (-\chi^k) = 1 \qquad (27)$$

We also express the action density (8) in terms of $\chi_m(\theta)$ /recall that $f_{01}^m = 0$ at $r > 0$ as a consequence of (14)/:

$$L = \frac{2\pi}{R^2} \sum_{m=1}^{N+1} \Big\{ \frac{1}{\sin^2\theta} \Big(\tfrac{1}{2} m\bar{m}\, \chi_m^2 - \tfrac{1}{2}(m-1)\overline{(m-1)}\, \chi_{m-1}^2 - \qquad (28)$$

$$- \big(\tfrac{1}{2}N+1-m\big)\Big)^2 + m\bar{m}\, (\chi_m')^2 \Big\}$$

where $R^2 = r^2 + t^2$. Note that (26) are the variational equations to (28) and that (27) are necessary to ensure the finitness of L. In fact $\chi_m(\theta) \equiv \pm 1$ constitute the minima of the / θ dependent/ "potential" in (28), for this case $L \equiv 0$; for any other $\chi_m(\theta)$ $L > 0$. Therefore it is not obvious that a minimum of $\int_0^\pi L\, d\theta$ -corresponding to a solution to (26,27)-exists, and we must look for an alternative proof. On the other hand from (28) it is clear that - for finite L - these SU(N+1) two merons have the same kind of logarithmic interaction - i.e. are just as singular - as the original SU(2) ones because the total action of the two meron is obtained by integrating L over R_+^2: $A = \int_0^\infty R\, dR \int_0^\pi d\theta\, L$. Therefore the integral of $\frac{R^2}{2\pi} L$ over θ just measures the coupling constant of this interaction.

The alternative proof that we shall now sketch is the adaptation to our problem of the proof given in ref[7] for the

N=2 case. The proof also establishes that $\chi_m'(0)=\chi_m'(\pi)=0 \quad m=1,\dots N$

First, we note that it is sufficient to work in the $(0,\frac{\pi}{2}]$ interval with boundary conditions $\chi^k(\frac{\pi}{2})=0,\ \chi_m'(\frac{\pi}{2})=0 \quad m\neq k$ because setting $\chi_k(\frac{\pi}{2}+\theta)=-\chi_k(\frac{\pi}{2}-\theta);\ \chi_m(\frac{\pi}{2}+\theta)=\chi_m(\frac{\pi}{2}-\theta)\ m\neq k$ will extend the solution to a continuous function on $(0,\pi)$.

The general strategy is to give upper and lower bounds for $\chi_m(\theta)$, i.e. to give 2N functions $\ell_m(\theta)<u_m(\theta)$ that satisfy both the boundary conditions at zero and at $\pi/2$:

$$\begin{aligned} &u_m(0)=\ell_m(0)=1 \qquad && u_m'(0)=\ell_m'(0) \qquad && m=1,\dots N \\ &u_k(\tfrac{\pi}{2})=\ell_k(\tfrac{\pi}{2})=0 \qquad && u_m'(\tfrac{\pi}{2})=\ell_m'(\tfrac{\pi}{2})=0 \qquad && m\neq k \end{aligned} \tag{29}$$

and the following differential inequalities:

$$\sin^2\theta\,\ell_m''-\ell_m\left(m\bar{m}\,\ell_m^2-\tfrac{1}{2}(m-1)(\overline{m-1})\,\ell_{m-1}^2-\tfrac{1}{2}(m+1)(\overline{m+1})\,\ell_{m+1}^2-1\right)>0 \tag{30}$$

$$\sin^2\theta\,u_m''-u_m\left(m\bar{m}\,u_m^2-\tfrac{1}{2}(m-1)(\overline{m-1})\,u_{m-1}^2-\tfrac{1}{2}(m+1)(\overline{m+1})\,u_{m+1}^2-1\right)<0 \tag{31}$$

and to prove that the solution satisfies $\ell_m(\theta)<\chi_m(\theta)<u_m(\theta)$

This proof goes as follows: With the aid of $\ell_m(\theta)$ and $u_m(\theta)$ we define a domain M in $(0,\frac{\pi}{2}]\times R^N$: $M=\{\theta,\chi_m \mid \ell_m(\theta)<\chi_m<u_m(\theta),\ 0<\theta<\frac{\pi}{2}\}$ and denote by S the boundary of M. As the equations (26) are sufficiently well behaved for any θ in $(\delta,\frac{\pi}{2}]$ $\delta>0$ there is a unique solution to (26) in $(\delta,\frac{\pi}{2})$ with the initial conditions $\chi_k(\frac{\pi}{2})=0,\ \chi_m(\frac{\pi}{2})=a_m\ m\neq k;\ \chi_k'(\frac{\pi}{2})=a_k,\ \chi_m'(\frac{\pi}{2})=0\ m\neq k$

The solution is a curve in $(0,\frac{\pi}{2})\times R^N$:

$$\gamma_{a_m}(\theta)=(\theta,\chi_m(\theta)\quad m=1,\dots N)$$

The $\gamma_{a_m}(\theta)$ curve starts in M if the a_m /m=1,N/ initial values are between the boundaries set by (29), i.e. if $(a_1 \dots a_N) \in I$ where $I = I_1 \times \dots I_N$ with $I_k = (u_k'(\frac{\pi}{2}), \ell_k'(\frac{\pi}{2}))$, $I_m = (\ell_m(\frac{\pi}{2}), u_m(\frac{\pi}{2}))$ For each $(a_1 \dots a_N) \in I$ there is a unique point T_{a_m} where γ_{a_m} first intersects S. In the case $\gamma_{a_m}(\theta) \in M$ for all $\theta \in (0, \frac{\pi}{2}]$ we define $T_{a_m} = \gamma_{a_m}(0) = (0, 1 \dots 1)$. As long as $\gamma_{a_m}(\theta) \in M$ we can divide (26) by $\sin^2\theta$ to obtain a uniform bound on $(\chi_m)''$ of order unity. Since $u_m'(0) = \ell_m'(0) = 0$ such a bound implies that $\lim_{\theta \to 0} \chi_m' = 0$. Therefore $\gamma_{a_m}(\theta)$ is a solution to our problem if $T_{a_m} = (0, 1 \dots 1)$

One can conclude that there must be an $(a_1 \dots a_N) \in I$ such that $\gamma_{a_m}(0) = (0, 1 \dots 1)$ by proving that the map T: $I \to S$ is continuous and that there is a curve C in I such that T(C) surrounds the point in question in S. The proofs of these statements for N=2 were given in ref [7], and by trivial modifications they apply in the general case too. Therefore we conclude that we can prove the existence of the solution to our problem if we can give the $\ell_m(\theta), u_m(\theta)$ functions with the necessary properties (29-31).

In what follows we give some ℓ_m and u_m functions that - according to the previous discussion - ensure the existence of all kinds of elementary two merons for N=3,4 /SU(4), SU(5)/. Although at present we have no upper and lower bounds for the general case we still expect that even in the cases not considered here they can be established.

For all of our two merons we take for $\ell_m(\theta)$ the following

$$\ell_k(\theta) = \cos\theta \qquad \ell_m(\theta) = 1 - \tfrac{1}{2}\sin^2\theta \qquad m \neq k$$

They obviously satisfy (29) and a straightforward substitution shows that they satisfy (30) for all m provided $k\tilde{k}<10$ /The restriction comes from the (k+1)-th and (k-1)-th inequalities, the others are satisfied for all values of k/. Note that the lower bound for the function going through 0 at $\pi/2$ is nothing but the closed form two meron solution for SU(2).

The determination of upper bounds is more tedious and less systematic. Finally, for N=3 /SU(4)/ and k=1 / $\beta^1=\frac{3}{2}$, $\beta^2=\beta^3=0$ / we obtained

$$u_1=1-\left(\frac{2\theta}{\pi}\right)^3 \qquad u_2=1-0.1\left\{\left(\frac{2\theta}{\pi}\right)^3-\frac{3}{5}\left(\frac{2\theta}{\pi}\right)^5\right\} \qquad u_3=1$$

With these functions it is easy to check the validity of the third inequality in (31), but we had to use a pocket calculator to establish the validity of the first and the second ones. /The k=3 case is obtained from this one by interchanging all quantities with indices 1 and 3/. For N=3 /SU(4)/ and k=2 / $\beta^1=\beta^3=0$, $\beta^2=2$ / when the boundary conditions allow the $\chi_1=\chi_3$ substitution and we have only two equations in (26) we found the following upper bounds:

$$u_1=1 \qquad u_2=1-\left(\frac{2\theta}{\pi}\right)^4$$

For N=4 /SU(5)/ and k=1 / $\beta^1=2$, $\beta^2=\beta^3=\beta^4=0$ / we found

$$u_1=1-\left(\frac{2\theta}{\pi}\right)^4,\ u_2=1-\frac{1}{4}\left\{\left(\frac{2\theta}{\pi}\right)^4-\frac{2}{3}\left(\frac{2\theta}{\pi}\right)^6\right\},\ u_3=1-\frac{1}{32}\left\{\left(\frac{2\theta}{\pi}\right)^4-\frac{2}{3}\left(\frac{2\theta}{\pi}\right)^6\right\},\ u_4=1$$

while for N=4 and k=2 ($\beta^1=\beta^3=\beta^4=0$, $\beta^2=3$)

$$u_1=1-0.42\left\{\left(\frac{2\theta}{\pi}\right)^5-\frac{5}{7}\left(\frac{2\theta}{\pi}\right)^7\right\},\ u_2=1-\left(\frac{2\theta}{\pi}\right)^5,\ u_3=1-0.38\left\{\left(\frac{2\theta}{\pi}\right)^5-\frac{5}{7}\left(\frac{2\theta}{\pi}\right)^7\right\}$$

$$u_4=1$$

The SU(5) two merons in the third or fourth U(1) sectors are obtained from the k=2 and k=1 cases respectively by interchanging all quantities with indices 1,4 and 2,3.

Having established the existence of all possible types of elementary two merons in SU(4) and SU(5) we display on figs 1-4. the numerical solutions of the equations describing them. From the figures we see that - as a consequence of the rather similar upper and lower bounds - the function going through zero at $\pi/2$ is almost the same in all cases. It is also clear that the farther is a function in (26) from this particular one the better it approaches the constant - identically one - function. These numerical solutions also enable us to compare the coupling constant of the interaction of the different elementary two merons in SU(4) and SU(5):

	SU(4)		SU(5)	
topological charge at t = 0	$\frac{3}{2}$	2	2	3
$\frac{R^2}{2\pi}\int_{\pi/2}^{\pi} L(\theta)\,d\theta$	4.75	6.54	6.99	11.05

These values of the coupling essentially determine /up to a factor of $4\pi^2$/ the action for our two merons in Minkowski space obtained by bringing back the other meron at finite distance via a conformal transformation and continuing in t analytically to ix_0 as in ref[3].

Finally, we note that for the existence of non-elementary merons at present we have no rigorous proof, however, the existence of two meron solutions - though not in a closed form - gives support to the expectation that even in the general case a proof could be given along the lines of the argument of ref[11]. In this case a search for numerical solutions can be carried out in a way similar to the determination of multiple merons in SU(2)[12].

V. Conclusion and Outlook

Let us now turn to the discussion of some further properties of the elementary two merons in SU(N+1). First, we note that for the merons with all of their χ_m -s /m=1,N/ different /e.g. for all N the meron with smallest charge /one can establish whether they represent a genuine non-reducible SU(N+1) configuration by looking at a component of the field strength /e.g. B_3/ which is is diagonal on the z axis. The fact that all the diagonal elements are different and non-zero /for all z/ implies that there exist no other linear relation between them /expect the tracelessness condition/ and guarantees that our solutions are not contained in any subgroup of SU(N+1).

As we mentioned earlier for all N the smallest elementary meron charge is just the half of the topological charge of the smallest non-reducible instanton[8]. For N=1 /SU(2)/ the relation between the two meron and one instanton solutions is even more intriguing: there exist a family of elliptic solutions - indexed by a continuous parameter k - interpolating between them [13] /the k=0 solution yields the two meron, the k=1 the one instanton/. This family may describe the dissociation of an instanton into two merons [14]. Naturally emerges the question: does the family of the same kind of solutions exist for the general non-reducible SU(N+1) instantons and two merons? We know of no definite answer to this question but we note a striking difference between the N=1 and the N>1 elementary two merons that may answer the question in the negative: the N=1 two meron has a higher symmetry, it is O(4) symmetric while for N>1 it is not. /Remember that we consider only such two merons when one of them is shifted to infinity/. To see we note that using the recent results for constructing gauge fields with given symmetry properties [9] one can show that the most general O(4) symmetric ansatz for an SU(N+1) gauge theory has the form:

$$W_\mu = \phi(R)\, \Sigma_{\mu\nu} x^\nu \qquad \mu,\nu = 1,\dots 4 \tag{32}$$

where $R = (r^2 + t^2)^{1/2} = (x_\mu x_\mu)^{1/2}$ and $\Sigma_{\mu\nu}$ are some representations of the O(4) generators constructed from matrices of the Lie algebra of SU(N+1): $\Sigma_{ij} = [T_i, T_j]$ $\Sigma_{4i} = T_i$ where T_i $(i = 1,\dots 3)$ form an SO(3)/SU(2)/ subgroup of SU(N+1). (32) implies in our ansatz that $\chi_m = \chi$ for all m. Therefore the field equations reduce to (24) that really allows the O(4) symmetric two meron solution $\chi = \cos\theta$. /The O(4) symmetry can be made manifest by a gauge transformation/.

However, as we mentioned earlier in this case we have $\beta^m = \frac{1}{2} m\bar{m}$ $m = 1, \ldots N$ therefore only for N=1 is this an elementary two meron.

Let us finally mention the possible physical application of the SU(N+1) /SU(4) and SU(5)/ merons in the mechanisms[1,2] suggested so far to explain the confinement of quarks. According to the first[1] in a non-abelian gauge theory of quarks and gluons the contribution of instantons to the functional integral determines the vacuum structure of the theory while the contribution of merons gives rise to the confinement of quarks via the dissociation of instantons into merons. The critical value of the coupling constant where this dissociation - and the transition to confinement phase - begins depends crucially on the value of the coupling in the two meron interaction. If we apply this picture to an SU(N+1) theory then the existence of non-reducible two merons with different couplings may signal the existence of several such phase transition points. /Provided the assumptions of the model are valid even after the first phase transition./

In the alternative mechanism - that assumes no meron pair dissociation[2] - the meron pairs at high density behave like a dense fluid thus changing the exponent of Wilson loop from perimeter to area behaviour[2]. Perhaps the best way to clarify the role of non-reducible SU(N+1) merons in this scheme would be to obtain O(2) symmetric ones. For O(2) symmetric merons in SU(2) it was shown that they have 1/2 topological charge concentrated in points in a <u>plane</u>[15]. It needs further clarification whether the generalization of this for $SU(N+1)$ holds or not.

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Figure Captions

Fig. 1. The functions describing SU(4) two meron with $\beta^1 = \frac{3}{2}, \beta^2 = \beta^3 = 0$

Fig. 2. The functions describing SU(4) two meron with $\beta^1 = \beta^3 = 0, \beta^2 = 2$

Fig. 3. The functions describing SU(5) two meron with $\beta^1 = 2, \beta^2 = \beta^3 = \beta^4 = 0$

Fig. 4. The functions describing SU(5) two meron with $\beta^1 = \beta^3 = \beta^4 = 0, \beta^2 = 3$

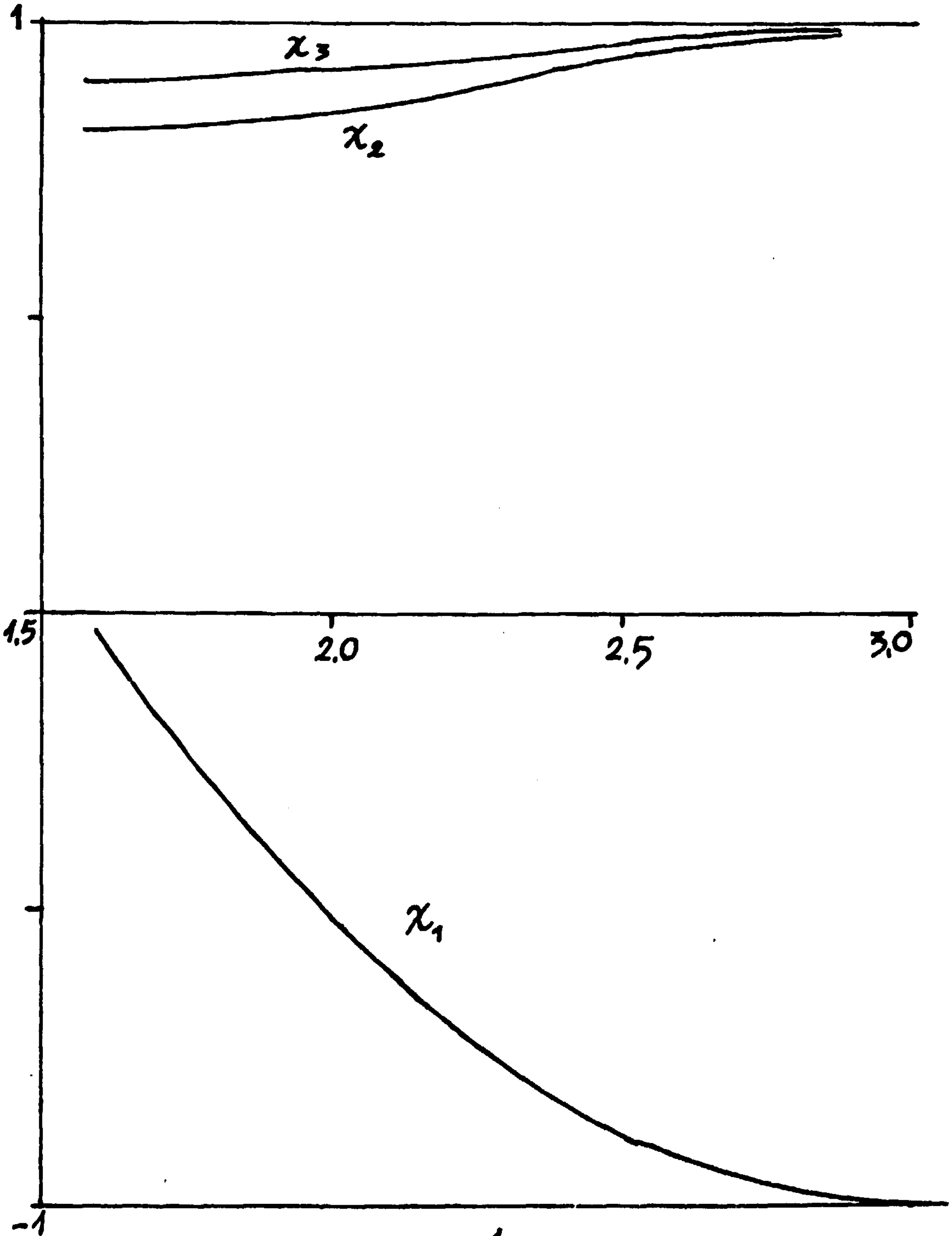

Fig. 1.

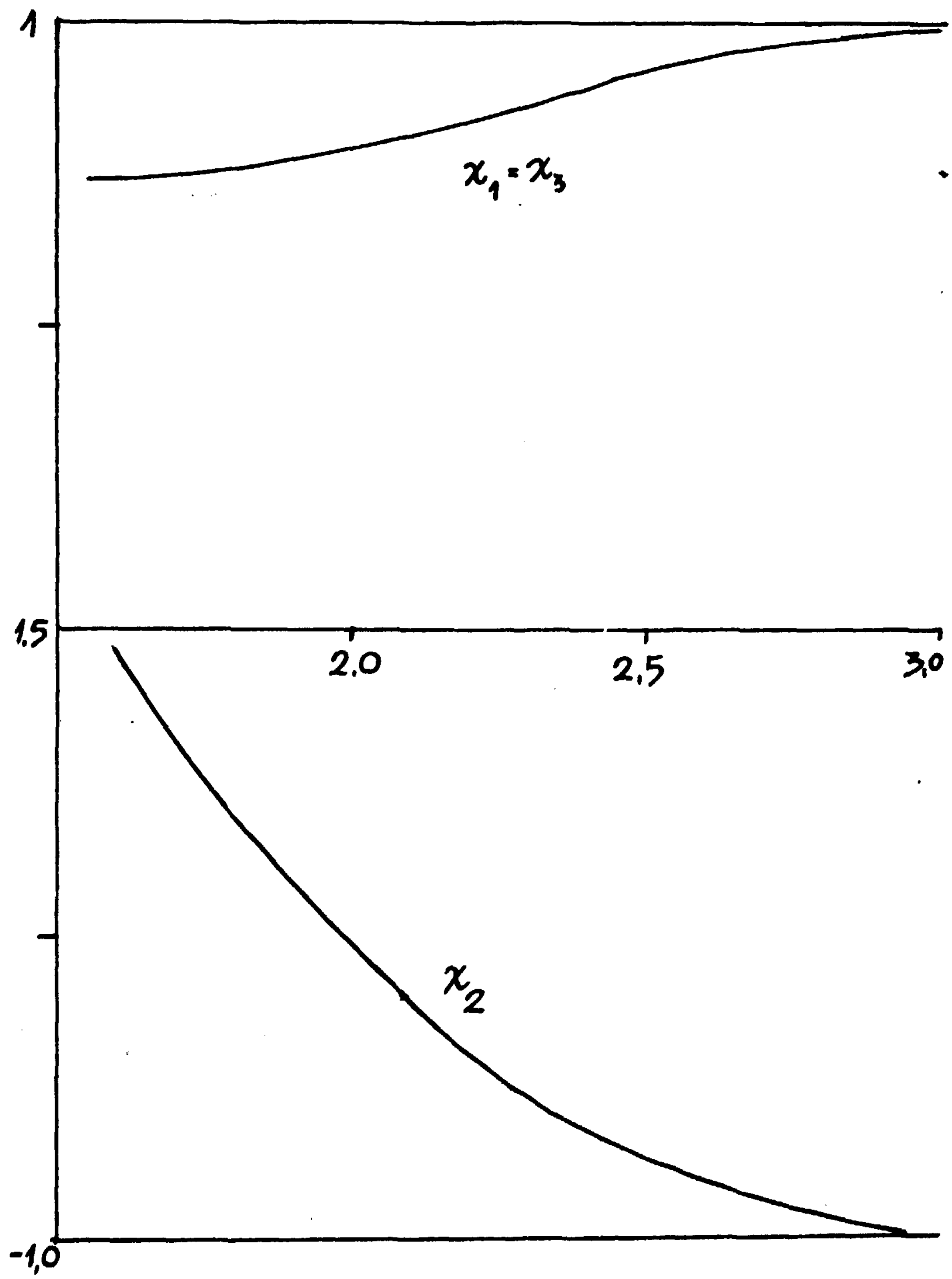

Fig. 2

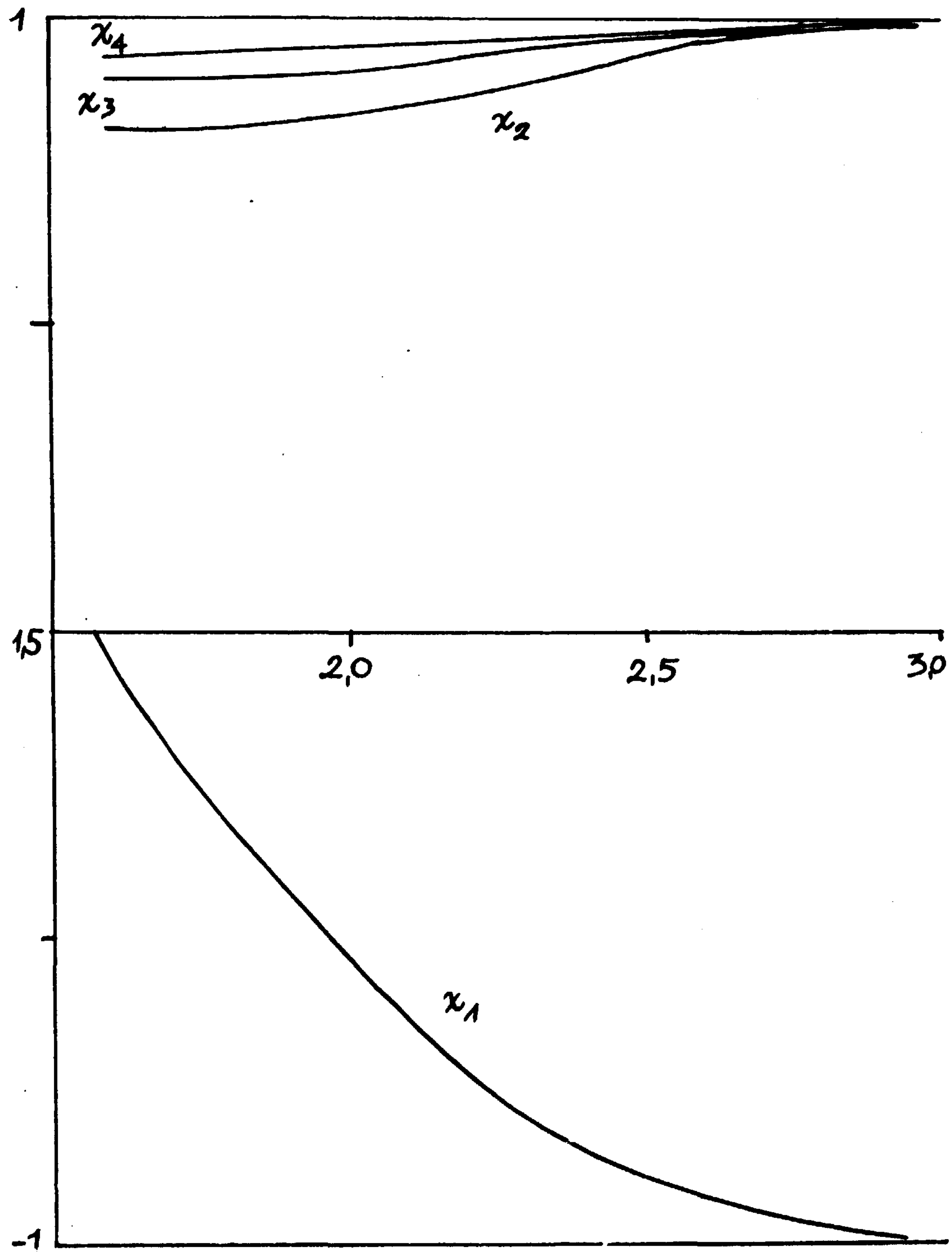

Fig. 3

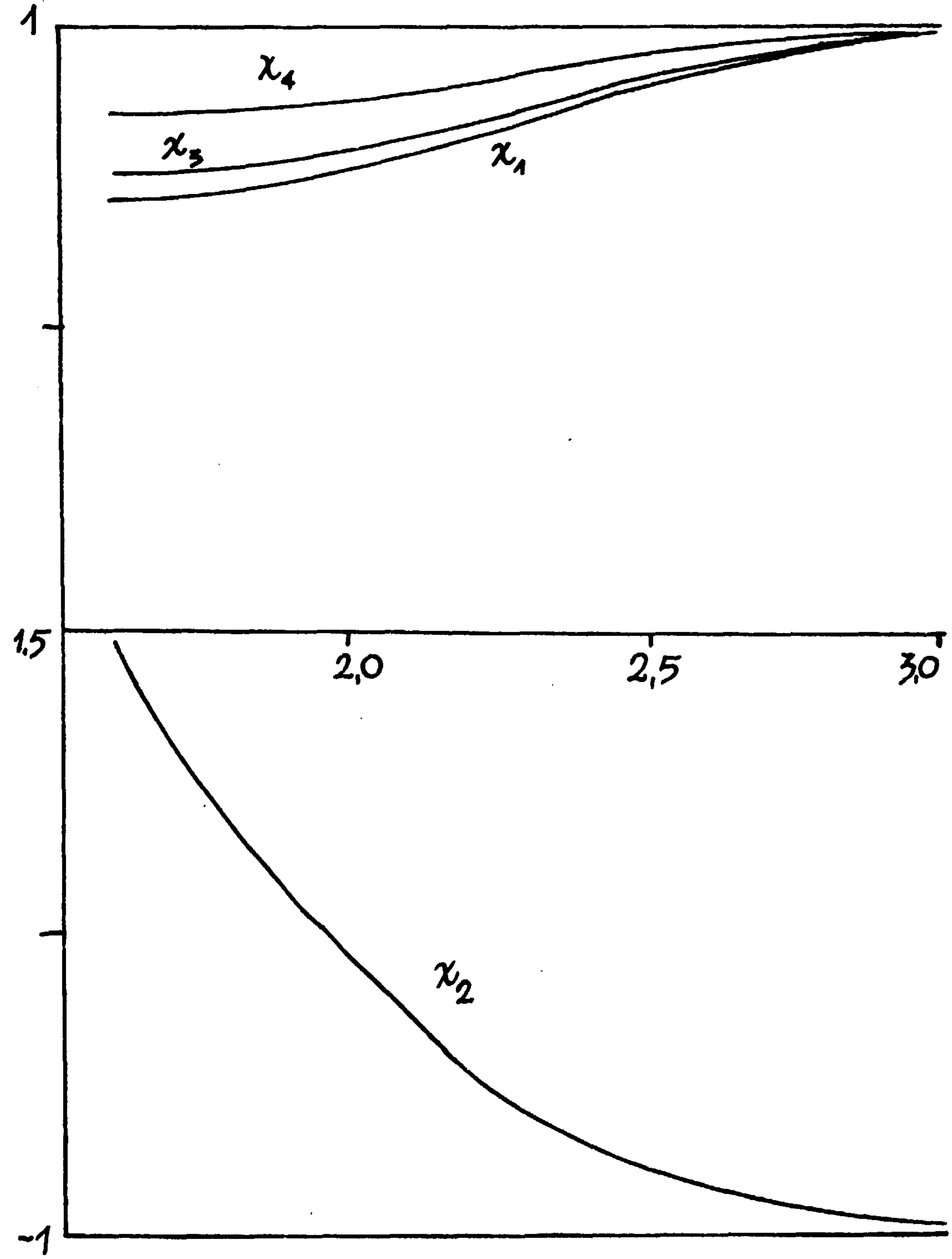

Fig. 4

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