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Classical Solutions of Nonlinear σ-Models

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Abstract

Nonlinear U(N) and O(N) σ-models are studied without imposing a constraint on the modulus of the field vector. Exact solutions in four-dimensional Minkowski space are presented, which have the form of plane resp. spherical waves. The singularities of the solutions as well as those of the Lagrangian density and the energy-momentum tensor are discussed. All results hold under the assumption, that space-time and internal symmetry of freedom are not mixed.

1. Introduction

This paper continues a previous investigation [Mitter, Widder 1979, referred to as [1]] on classical solutions of nonlinear field theories with quartic self-coupling in the Lagrangian. The analysis of the previous paper [1], which dealt with one complex scalar field, shall be extended here to a multiplet of N complex fields, whereby the theory has the internal symmetry group U(N). The special case of N real fields with O(N) symmetry is always contained. The quantum counterpart for N = 4 has been investigated in the past as a model for chiral symmetry [see e.g. Lee 1972], whereby the four real fields are identified with the pion and a σ-Meson, which has provided the name for the model. More recently the interest has shifted to classical solutions, which could eventually be of interest in connection with the confinement problem. In particular solutions of the instanton- resp. meron-type have been established and investigated for σ-models, whereby the real field multiplet is required to form an unit vector in O(N)-space [De Alfaro, Fubini, Furlán 1978]. In another interesting class of models the field multiplet is required to form a complex unit vector [Eichenherr 1978, D'Adda et. al. 1978]. We shall not impose such a constraint, but shall start, as in [1], from simple symmetry requirements in coordinate space, which allow for relatively large classes of exact solutions.

In some cases solutions with a constant value of the U(N) - or

O(N)- modulus of the field vector will be contained in these classes. For the coordinates we shall consider the four-dimensional Minkowski space. We shall pay particular attention to the Lagrangian and the energy-momentum tensor as computed with these solutions.

2. Field equations and physical quantities

The field is described by a set of N complex functions

$$\varphi_k(x) \qquad k = 1, \cdots N$$

with arguments x^μ in Minkowski space. The Lagrangian density is

$$(1) \qquad L = \frac{1}{2}\left[\partial^\mu \varphi_k^* \, \partial_\mu \varphi_k + \frac{\lambda}{2}(\varphi_k^* \varphi_k)^2\right]$$

where repeated latin indices have to be summed from 1 to N. The canonical formalism provides the field equation

$$(2) \qquad \Box \varphi_k - \lambda \varphi_k (\varphi_l^* \varphi_l) = 0$$

As a consequence of the invariance of L under U(N) rotations of the field we obtain continuity equations for the N^2 vectors

$$(3) \qquad M^\mu_{kl} = i\left(\varphi_k^* \partial^\mu \varphi_l - \varphi_l \partial^\mu \varphi_k^*\right)$$

The canonical energy-momentum tensor is

$$T^{\mu\nu} = \tfrac{1}{2}\left[\partial^\mu\varphi_k^* \partial^\nu\varphi_k + \partial^\nu\varphi_k^* \partial^\mu\varphi_k\right] - g^{\mu\nu}L \tag{4a}$$

We shall also consider the improved tensor [(Callan et al., 1970]

$$\theta^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{6}\left(\partial^\mu\partial^\nu - g^{\mu\nu}\Box\right)\varphi_k^*\varphi_k \tag{4b}$$

which gives the same global generators of translations and Lorentz transformations. Both $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ fulfill continuity equations.

We shall write the field in the form

$$\varphi_k(x) = r(x)\, e_k(x) \tag{5}$$

where e_k transforms as an unit vector under $U(N)$ rotations

$$r = r^*, \quad e_k^* e_k = 1, \quad e_k^*\partial^\mu e_k + e_k\partial^\mu e_k^* = 0$$

$$e_k^*\Box e_k + e_k\Box e_k^* = -2\,(\partial^\mu e_k^*)(\partial_\mu e_k) \tag{6}$$

Then M^μ takes the form

$$M^\mu_{kl} = M^{\mu\,*}_{lk} = i r^2\left(e_k^*\partial^\mu e_l - e_l\partial^\mu e_k^*\right) \tag{7}$$

The field equation becomes

$$e_k(\Box r-\lambda r^3)+2(\partial^\mu r)(\partial_\mu e_k)+r\Box e_k=0$$

By simple manipulation we obtain two equations: one of them is again the continuity equation for M

$$\partial_\mu M^\mu_{kl}=0 \tag{8}$$

whereas the other one can be written in the form

$$\Box r-\lambda r^3-\frac{M^2}{r^3}=0 \tag{9}$$

where we have used the abbreviation

$$M^2 = r^4(\partial_\mu e_k^*)(\partial^\mu e_k)=\frac{1}{2}M^*_{\mu,kl}M^\mu_{kl}-\frac{1}{4}M^*_{\mu,kk}M^\mu_{ll} \tag{10}$$

The Lagrangian density becomes

$$L=\frac{1}{2}\left[(\partial_\mu r)(\partial^\mu r)+\frac{\lambda}{2}r^4+\frac{M^2}{r^2}\right] \tag{11}$$

and the energy-momentum tensors read

$$T^{\mu\nu}=(\partial^\mu r)(\partial^\nu r)-\frac{1}{2}g^{\mu\nu}\left[(\partial_\lambda r)(\partial^\lambda r)+\frac{M^2}{r^2}+\frac{\lambda}{2}r^4\right]+ \tag{12a}$$
$$+\frac{r^2}{2}\left[(\partial^\mu e_k^*)(\partial^\nu e_k)+(\partial^\nu e_k^*)(\partial^\mu e_k)\right]$$

(12b)
$$\theta^{\mu\nu} = \frac{2}{3}(\partial^\mu r)(\partial^\nu r) - \frac{1}{3}\, r\partial^\mu\partial^\nu r - \frac{1}{6}\, g^{\mu\nu}\Big[(\partial_\lambda r)(\partial^\lambda r) + r\Box r - \frac{3}{2}\lambda r^4\Big] + \frac{r^2}{2}\Big[(\partial^\mu e_k^*)(\partial^\nu e_k) + (\partial^\nu e_k^*)(\partial^\mu e_k)\Big]$$

From the last formula one may easily check, that the trace of θ vanishes, as it must be. In addition it is evident, that the two tensors are identical for all solutions with constant r.

In order to find solutions of the continuity equation (8) we shall start from an ansatz for M of the form

(13) $$M^\mu_{kl} = i q^\mu L_{kl}$$

with constant $L^*_{kl} = -L_{lk}$ and

(14) $$\partial_\mu q^\mu = 0$$

This is the simplest possibility and means, that there is no mixing of space-time- and internal symmetry degrees of freedom. For M^2 we have

(15) $$M^2 = q_\mu q^\mu \Lambda^2$$

with the constant

$$(16) \quad \Lambda^2 = \frac{1}{2} L^*_{kl} L_{kl} - \frac{1}{4} L^*_{kk} L_{ll} \geq 0$$

If we start from an appropriate ansatz for q^μ fulfilling equ. (14), we can solve the problem in two steps: first we have to solve equ. (9) with (15) for r and then we have to determine e_k from equ. (7) and (13), viz.

$$(17) \quad e^*_k \partial^\mu e_l - e_l \partial^\mu e^*_k = \frac{q^\mu}{r^2} L_{kl}$$

It is even possible to compute L, $T^{\mu\nu}$ and $\theta^{\mu\nu}$ without knowing e_k. In order to demonstrate this we have to observe, that one may show by algebraic mainpulations of equ. (17) the relations

$$L_{kk} = e_k L_{kl} e^*_l \ , \quad L_{kl} L^*_{kl} + L_{kk} L^*_{ll} = -2 e_k L_{kl} L_{ln} e^*_n$$

With these we obtain form equ. (17)

$$(18) \quad \frac{r^2}{2} \left[(\partial^\mu e^*_k)(\partial^\nu e_k) + (\partial^\nu e^*_k)(\partial^\mu e_k) \right] = \frac{q^\mu q^\nu}{r^2} \Lambda^2$$

and all terms in L, $T^{\mu\nu}$ and $\theta^{\mu\nu}$ depend only on r and its derivatives.

If we have N real fields (i.e. the O(N) σ-model)the corresponding formulae are obtained by omitting the factor i in equs. (3), (7) and (13) and omitting the asterisk everywhere. Since the diagonal elements L_{kk} vanish in this case, we have N(N-1)/2 instead of N^2 conserved quantities (3).

3. Plane waves

Here we shall start from the simplest possible choice: we assume, that q^μ is proportional to a constant vector p^μ (which is not light-like; the case $p^2 = 0$ is considered in Section 6). In order to obtain plane wave solutions, we shall furthermore assume, that r depends on x only via p·x:

(19) $$r(x) = R(\tau) \quad , \quad \tau = \sqrt{|g|}\, p_\mu x^\mu \, , \quad g = \lambda / p^2$$

From relations (17), (18) we observe, that also e_k should depend only on τ

(20) $$e_k = e_k(\tau)$$

Denoting the derivative with respect to τ by a dot and choosing

(21) $$q^\mu = \sqrt{|g|}\, p^\mu$$

we obtain from equ.(17)

(22) $$R^2 \left(e_k^* \dot{e}_\ell - e_\ell \dot{e}_k^* \right) = L_{k\ell}$$

which we shall use in section 5 to determine e_k. The equation for R is obtained from (9) and reads

$$\ddot{R} - \varepsilon R^3 - \Lambda^2/R^3 = 0 \tag{23}$$

where ε is the sign of g. All solutions of this equation have been given in [1], section 3 (see formulae (I)-III)) in terms of Jacobian elliptic functions. The only difference is the relation [1,21] between the constants appearing in the solutions and the initial values, which has to be replaced by

$$D^2 = 4\Lambda^2 \,, \quad C = \frac{\varepsilon}{2} R_0^4 - \dot{R}_0^2 - \Lambda^2/R_0^2 \tag{24}$$

The physical densities take the form

$$L = \frac{\lambda}{2}\left(R^4 - \varepsilon C\right) \tag{25}$$

$$T^{\mu\nu} = \frac{\lambda}{2p^2}\left[\left(p^\mu p^\nu - p^2 g^{\mu\nu}\right)R^4 - \varepsilon C\left(2p^\mu p^\nu - g^{\mu\nu}p^2\right)\right] \tag{26a}$$

$$\Theta^{\mu\nu} = -\left|\frac{\lambda}{p^2}\right| \frac{C}{6}\left(4p^\mu p^\nu - p^2 g^{\mu\nu}\right) \tag{26b}$$

The results (25) and (26a) agree with the expressions given in [1,19]. Thus the positivity properties of T^{00} are the same as in [1]. In contrast Θ^{00} is positive for $C < 0$, irrespective of the sign of λ or p^2. The improved tensor $\Theta^{\mu\nu}$ does not depend on the explicit form of the solution R!

Now we shall briefly discuss the various solutions for R. The degenerate solution [1,32]

$$R = \sqrt{2}\, R_0 \, (R_0 \tau + \sqrt{2})^{-1}$$

is obtained only for

$$\varepsilon = +1 \ , \qquad C = \Lambda^2 = 0 \ .$$

T^{00} turns out positive, but the improved tensor vanishes. The most reasonable solutions (if one wants to give a physical interpretation to plane waves at all) seem to be those of type III: since for them R^2 is bounded, L is finite, Θ^{00} is positive and the integrated quantities diverge only for an infinite volume. For the usual (negative) sign of λ these solutions correspond to timelike p^μ. The solution with constant R is contained in this set, cf. [1,27]. For the opposite sign of p^2 or λ we have solutions of type I or II, which assume infinite values at infinitely many points. The improved tensor is obviously not affected by these singularities. In spite of the fact, that they are present in the first term of the Lagrangian (25), they are not interesting for the action: if we change L by a divergence

$$L \rightarrow L' = L - \frac{1}{6} \Box (\varphi_\kappa^* \varphi_\kappa)$$

we obtain

$$L' = -\lambda \varepsilon C/6$$

independent of R.

4. Spherical waves

For spherical waves the only relevant direction should be x. Then equ.(14) can only be satisfied, if we take

$$q^\mu \sim x^\mu / x^4$$

The scalar r should depend only on x^2 in order to obtain a spherical wave. If we use the variables [1,35]

$$(27) \qquad s = |\lambda|\,|x_\mu x^\mu|\,, \quad \tau = \ln\sqrt{s}\,, \quad \varepsilon = \mathrm{sgn}\,\lambda\ \mathrm{sgn}\,x_\mu x^\mu$$

and write

$$(28) \qquad r(x) = \frac{1}{\sqrt{s}}\,R(\tau), \qquad q^\mu = \varepsilon\lambda\,\frac{x^\mu}{s^2}$$

we see from equs. (17), (18)

$$(29) \qquad e_k = e_k(\tau)$$

and obtain again equ.(22) for the determination of e_k. The field equation for R becomes

$$(30) \qquad \ddot{R} - R - \varepsilon R^3 - \Lambda^2/R^3 = 0 \quad .$$

This equation has been solved in [1], section 4 and the solutions are again of the types (I)-(III) of [1], section 3. Instead of [1,40] we have now to use

(31) $$D^2 = 4\Lambda^2, \quad C = \frac{\varepsilon}{2} R_o^4 + R_o^2 - \dot{R}_o^2 - \Lambda^2/R_o^2 \quad .$$

The physical densities read

(32) $$L = \frac{1}{2\lambda x^4}\left[R^4 + \varepsilon(2R^2 - 2R\dot{R} - C)\right]$$

(33a) $$T^{\mu\nu} = \frac{1}{2\lambda x^6}\left[(x^\mu x^\nu - x^2 g^{\mu\nu})R^4 + \varepsilon(2x^\mu x^\nu - x^2 g^{\mu\nu})(2R^2 - 2R\dot{R} - C)\right]$$

(33b) $$\Theta^{\mu\nu} = -\frac{C}{6|\lambda|\,|x_\rho x^\rho|^3}(4x^\mu x^\nu - x^2 g^{\mu\nu})$$

Formulae (32) and (33a) agree with [1,39]. As for plane waves we observe, that Θ^{00} is positive for any $C < 0$ irrespective of the sign of λ or x^2. The total energy diverges logarithmically (as for meron solutions) due to the x^{-4} behaviour of the energy-density. Since the field equation (2) is invariant under translations and conformal transformations (cf. [1,9], [1,10]), any transformed solution is a solution as well. This fact can be used to shift the singularity. If we apply a translation by a constant vector $-c^\mu$ followed by a conformal transformation [1,9] with $b^\mu = -c^\mu/2c^2$, we obtain solutions of the form

(34) $$\varphi'_k(x) = \left|\frac{4c^2}{\lambda x_+^2 x_-^2}\right|^{1/2} R(\tau')\, e_k(\tau')\, \mathrm{sgn}\, c^2 x_+^2$$

where

(35) $$X_{\pm} = X \pm C \,, \quad \tau' = \ln\left|4\lambda c^2 \frac{X_-^2}{X_+^2}\right|^{1/2}$$

The Langrangian density and the improved tensor read

(36) $$L(\varphi') = \frac{2c^2}{\lambda X_+^4 X_-^4}\left[(R^4-\varepsilon C)a_-^2 + R^2(a_+^2+a_-^2) - 2R\dot{R}\, a_+ a_-\right]$$

(37) $$\theta^{\mu\nu}(\varphi') = -\frac{2c^2 C}{3\lambda X_+^4 X_-^4}\left(4a_-^\mu a_-^\nu - g^{\mu\nu} a_-^2\right)$$

Here the argument of R is τ' and we have

(38) $$a_{\pm}^{\mu} = X_-^\mu \left(\frac{X_+^2}{X_-^2}\right)^{1/2} \pm X_+^\mu \left(\frac{X_-^2}{X_+^2}\right)^{1/2}$$

Since

$$a_+^2 = 4x^2 \,, \quad a_-^2 = 4c^2 \,, \quad a_+ a_- = 4cx$$

it is obvious, that the singularities are now located at $\pm c^\mu$.

Finally we shall discuss the various possible solutions for $R(\tau)$. It is evident, that there are no solutions with r = const. fulfilling our basic ansatz (28). This is evident already from equ. (9), since constant r implies constant M^2, which cannot be fulfilled with q^μ from equ. (28).

The degenerate solution [1,55]

$$r = \frac{a}{1-\frac{a^2}{8}\lambda x^2}$$

with constant a is obtained only for $\Lambda^2 = C = 0$ and corresponds to a vanishing improved tensor. For the remaining set of solutions we have to observe that we cannot restrict the discussion to solutions with bounded R here. If we start with given values for the parameters Λ^2, λ and C, the two possible values of the sign ε correspond to spacelike resp. timelike regions of x^2 and we have to consider the solutions in both domains. Thus, if we take e.g. $C < 0$ (so that θ^{00} is positive) and choose the values of C and Λ^2 appropriately (cf.[1,44]), R is of type (III) and therefore bounded for $\varepsilon = -1$ (i.e. spacelike x^2 for the usual negative sign of λ), but the corresponding solution in the other sector $\varepsilon = +1$ (i.e. timelike x^2) is of type (II) or (I) and diverges for infinitely many values of τ . The simplest example for this fact (which is hard to discover in euclidean x-space) is the solution with constant R, which is contained in type (III) both for $\Lambda^2 \neq 0$ (cf.[1,47]) and $\Lambda^2 = 0$ (cf.[1,56]) and corresponds to $\varepsilon = -1$. For $\varepsilon = +1$ the corresponding solution is [1,46] which contains the cotangent and displays the infinities as mentioned above. The precise nature of the singularity at the light cone might differ from the one obtained by our formulae by distributions concentrated at $x^2 = 0$, since we have not paid attention to these terms when differentiating s.

As for plane waves the other singularities of the solutions of type (I) or (II) do not affect the action. Changing again L by a divergence

$$L \rightarrow L' = L - \frac{1}{6}\,\partial^{\mu}\left[\partial_{\mu}(\varphi_k^{*}\varphi_k) - q_{\mu} G\right]$$

we arrive at

$$L' = -\,\varepsilon C/(6\lambda x^{4})$$

if we take

$$G = s\,\varphi_k^{*}\varphi_k - \int^{s}\varphi_k^{*}\varphi_k\,ds = R^{2} - 2\int^{\tau} R^{2}\,d\tau$$

The last term could be expressed in terms of elliptic integrals with R^2 in the argument, so that G can be given as an explicit function of φ_k, if necessary.

5. Determination of e_k for plane and spherical waves

Both for plane and spherical waves we have to determine e_k from

$$R^{2}\,(e_k^{*}\dot{e}_l - e_l\dot{e}_k^{*}) = L_{kl}$$

$$(39) \qquad e_k^{*}\dot{e}_k = -\dot{e}_k^{*}e_k = L_{kk}/2R^{2}$$

$$\dot{e}_k^{*}\dot{e}_k = \Lambda^{2}/R^{4}$$

By elementary steps we obtain a linear system of first order equations, which reads in matrix notation

(40) $R^2\dot{e} = K.e$

with

$K = -K^{\dagger} = L^T - \frac{1}{2}\,SpL$

In the complex case the solution is obtained by standard methods. The components of e are linear combinations of exponentials

$$\exp\, ik_1 \int_0^{\tau} \frac{dx}{R^2(x)}$$

where ik_1 are the eingenvalues of K and the coefficients are determined up to some phase factors from

$e^{\dagger}.e = 1,\; R^2 e^{\dagger}\dot{e} = SpK,\; R^4\dot{e}^{\dagger}\dot{e} = \Lambda^2 = \frac{1}{2}\,Sp\,K^{+}K + \frac{N}{2}\,SpK\;SpK^{\dagger}$

In this fashion we obtain e.g. for N=1 the result of [I,18]

(41) $e_1 = e^{i\phi} \qquad \phi - \phi_o = 1 \int_o^{\tau} \frac{dx}{R^2(x)}, 1 = -iL_{11}/2$

For N=2 the matrix K is traceless and the eigenvalues turn out to be

$k_1 = -k_2 = \Lambda$

We shall neither write down the coefficients of the exponentials in this case nor consider higher values of N, since the results are not of particular interest. Instead we shall consider the real case (O(N) model). Then the matrix L is real and antisymmetric and it is better to solve the equations

(39) directly by representing the components of the real unit vector e in terms of appropriate angular variables. These are easy to understand, if we construct a mechanical analog by interpreting R as radial coordinate of a moving point in N-dimensional space and τ as the time. Then C is a multiple of the energy and L_{kl} are related to the angular momenta (The "real-field" solutions of ref. [1] correspond to zero angular momenta). We shall consider the lowest few values of N.

For O(2) we have

$$e_1 = \cos \phi,\ e_2 = \sin \phi$$

and obtain for ϕ the same result as in the U(1) case. For O(3) we have three constants, which form an axial vector under rotations

$$(42)\quad \vec{L} := (L_{23}, L_{31}, L_{12}),\ \Lambda^2 = \vec{L}^2$$

With the unit vector

$$(43)\quad \vec{n} = (e_1, e_2, e_3) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi, \cos\theta)$$

the equation for n reads

$$(44)\quad \vec{n} \times \vec{n} = \frac{1}{R^2}\vec{L}$$

By a rotation we can always obtain
L = (0,0,1)

so that the orbital plane of the mechanical analog is the 12-plane. Then we have

(45) $\dot{\theta} = 0,\ \theta = \frac{\pi}{2},\quad \dot{\phi} = \frac{1}{R^2}$

with the same solution as for N = 2.
For N = 4 the six constants can be combined into two 3-vectors

(46) $\vec{L} = (L_{23}, L_{31}, L_{12}),\quad \vec{F} = (L_{14}, L_{24}, L_{34}),\quad \Lambda^2 = \vec{L}^2 + \vec{F}^2$

(like in the Kepler problem, where $\vec{F}$ is the Lenz vector).
Writing

(47) $(e_1, e_2, e_3) = \vec{n}\sin\chi,\ e_4 = \cos\chi$

with the same unit vector as for N=3, we have

(48) $\frac{1}{R^2}\vec{L} = (\vec{n} \times \dot{\vec{n}})\sin^2\chi,\quad \frac{1}{R^2}\vec{F} = -\vec{n}\dot{\chi} - \dot{\vec{n}}\sin\chi\cos\chi$

and we infer, that

(49) $\vec{L}.\vec{F} = 0$

By rotation we can always arrange for the choice

$L = (0, 0, l),\qquad F = (f, 0, 0).$

Then we obtain again

(50) $\dot{\theta} = 0,\ \theta = \pi/2$

The remaining equations for $\phi, \dot{\chi}$ can be readily solved. After some elementary steps we obtain

$$\tan\phi = \frac{1}{a}\left[al\int_0^t \frac{dx}{R^2(x)} + \arc\tan\,(a\tan\phi_0)\right] \tag{51}$$

$$\tan\chi = \frac{1}{f\sin\chi},\qquad a^2 = 1 + (f/l)^2$$

For higher values of N one may proceed in similar fashion.

6. Lightlike plane waves

The field equation (2) does not allow for plane wave solutions (19), (20) with lightlike p^μ. We shall now show, that there are plane wave solutions with lightlike propagation character, if we allow for propagation in opposite directions. Let p^μ be a fixed, lightlike vector ($p^2 = 0$). We introduce a tetrad

$$(52) \qquad p^\mu = p^0 n^\mu = \frac{p_0}{\sqrt{2}}(1,\vec{m}), \quad \hat{n}^\mu = \frac{1}{\sqrt{2}}(1,-\vec{m}), \quad e_i^\mu = (0,\vec{e}_i) \quad i=1,2$$

where $\vec{e}_i, \vec{m}$ are three orthogonal unit vectors in 3-space. For the tetrad vectors we have then

$$(53) \qquad n^2 = \hat{n}^2 = n\cdot e_i = \hat{n}\cdot e_i = 0 \;, \quad n\cdot\hat{n} = 1 \;, \quad e_i\cdot e_j = -\delta_{ij}$$

Any vector a^μ can then be represented by its lightlike components

$$(54) \qquad a^\mu = n^\mu a_u + \hat{n}^\mu a_v + \sum_{i=1}^{2} e_i^\mu a_i$$

We shall reserve the special notation

$$(55) \qquad x_v = n\cdot x = u \;, \quad x_u = \hat{n}\cdot x = v$$

for the coordinate vector.

For a plane wave of the type mentioned above the only essential directions should be n and $\hat{n}$. Therefore we shall require

$$(56)\qquad q^{\mu} = n^{\mu} q_u + \hat{n}^{\mu} q_v = q^{\mu}(u,v)$$

and

$$(57)\qquad r = r(u,v) \quad , \quad e_k = e_k(u,v)$$

The field equations (9) resp. (14) amount to

$$(58)\qquad \begin{aligned} & 2\,\partial_u \partial_v r - \lambda r^3 - \frac{2\Lambda^2 q_u q_v}{r^3} = 0 \\ & \partial_v q_u + \partial_u q_v = 0 \end{aligned}$$

We shall be interested only in a separable solution, for which

$$(59)\qquad r(u,v) = R_1(u)\, R_2(v)$$

The field equation for r is separable, if the last term is a multiple of the second term. Therefore we put

$$(60)\qquad q_u q_v = -k\,(R_1 R_2)^6$$

with a constant k. The solution becomes

$$(61) \qquad R_1 = [2K(a+u)]^{-1/2}, \quad R_2 = [2K'(b+v)]^{-1/2}$$

where a,b are related to the initial values, K is the separation constant and

$$(62) \qquad 2KK' = \lambda - 2k\Lambda^2$$

The continuity equation for q^μ is solved by

$$(63) \qquad (q_u, q_v) = \frac{\varkappa}{4(\lambda - 2k\Lambda^2)} \big((u+a)(v+b)\big)^{-2} \big(v+b, -(u+a)\big)$$

where

$$(64) \qquad \varkappa = \left(\frac{2k}{\lambda - 2k\Lambda^2}\right)^{1/2} = \left(\frac{k}{KK'}\right)^{1/2}$$

The constants have to be chosen in such a way, that q_u, q_v and r are real. This leads to the restrictions

$$|\lambda| > 2|k|\Lambda^2$$

$$(65) \qquad \operatorname{sgn}\lambda = \operatorname{sgn}k = \operatorname{sgn}(u+a)(v+b)$$

The last condition shows, that we have a solution only in two opposite quadrants of the (u,v) plane, which have only the point $u + a = v + b = 0$ in common. The solution r is singular at this point and at the boundaries of the quadrants.

The physical densities read

$$L = (3\lambda - 8k\Lambda^2)\, F(u,v) \tag{66}$$

$$T^{\mu\nu} = -\lambda F(u,v)\Big[n^\mu \hat{n}^\nu + \hat{n}^\mu n^\nu - \Big(3 - \frac{8k\Lambda^2}{\lambda}\Big)\sum_{i=1}^{2} e_i^\mu e_i^\nu - 2 n^\mu n^\nu \frac{v+b}{u+a} - 2\hat{n}^\mu \hat{n}^\nu \frac{u+a}{v+b}\Big] \tag{67}$$

$$\theta^{\mu\nu} = \frac{1}{3}(\lambda - 8k\Lambda^2)\, F(u,v)\Big[n^\mu \hat{n}^\nu + \hat{n}^\mu n^\nu + \sum_{i=1}^{2} e_i^\mu e_i^\nu - 2 n^\mu n^\nu \frac{v+b}{u+a} - 2\hat{n}^\mu \hat{n}^\nu \frac{u+a}{v+b}\Big] \tag{68}$$

where

$$F(u,v) = \big[4(\lambda - 2k\Lambda^2)(u+a)(v+b)\big]^{-2} = \frac{1}{4}\big[R_1(u) R_2(v)\big]^4 \tag{69}$$

The positivity properties can be read off directly. For $\lambda < 0$ both θ^{00} and the generator density [Rohrlich 1971]

$$\theta_{uv} = \hat{n}_\mu n_\nu\, \theta^{\mu\nu} \tag{70}$$

of u-displacements can be made positive by an appropriate choice of k. For $\lambda > 0$ this remains true for Θ_{uv}, but not always for Θ^{00}.

The unit vectors e_k can be determined in a similar fashion as in section 5. The only difference is, that we obtain now two partial differential equations instead of one. In the complex case we have now instead of equ. (40)

$$(71)\quad r^2 \partial_u e = q_u K.e, \qquad r^2 \partial_v e = q_v K.e$$

with the same matrix K as before (correspondingly for the real case). For the U(1) or O(2) model the solution is again of the form (41) with

$$(72)\quad \phi - \phi = \frac{\varkappa l}{2} \ln \left(\frac{u+a}{v+b}\right)$$

For higher N we obtain linear combinations of exponentials with arguments of similar structure. As a result the components of e exhibit infinitely rapid oscillations at the singular lines $u+a=0$ resp. $v+b = 0$.

The solutions obtained here can be slightly generalized, if one assumes, that the last term in the field equation (58) for r is a linear combination of the first and the second term. Then one has two constants instead of k. The basic fact, that there is a solution only in the two opposite quadrants of the u, v-plane is, however, not changed.

Finally it has to be observed, that there are also solutions with constant r and lightlike propagation character. If r is constant, we must have

$$(73)\quad q_u q_v = - \frac{\lambda r^6}{2\Lambda^2}$$

The simplest solutions of the second equation (58) are obtained for constant q_u and q_v, i.e.

$$(74)\quad q_u = r^3 \left[- \frac{\alpha\lambda}{2\Lambda^2}\right]^{1/2}, \qquad q_v = r^3 \left[- \frac{\lambda}{2\alpha\Lambda^2}\right]^{1/2}$$

with constant α. The sign of α has to be chosen opposite of that of λ to render q_u, q_v real. The physical quantities read

$$
\begin{aligned}
&L = -\lambda r^4/4 \\
&\Theta^{\mu\nu} = T^{\mu\nu} = -\frac{\lambda}{4}\, r^4 [n^\mu \hat{n}^\nu + \hat{n}^\mu n^\nu + 2\sum_{i=1}^{2} e_i^\mu e_i^\nu - 2\alpha n^\mu n^\nu - \frac{2}{\alpha}\hat{n}^\mu \hat{n}^\nu]
\end{aligned} \tag{75}
$$

The equations (71) for e_k can be solved as indicated above. For the U(1) - or O(2) model the phase reads

$$\phi = r\,\left(-\frac{\lambda\alpha}{2}\right)^{1/2} [u + a + (v+b)/\alpha] \tag{76}$$

Another solution for q_u, q_v is obtained by replacing α by $-(v+b)/u+a$ in the expressions (74) and (75), whereby the sign of this ratio has to be chosen equal to that of λ. It turns out, however, that the equations for e have no solutions in this case (except for vanishing eigenvalues k_1).

All solutions obtained in this section can be understood also as solutions of the model 1+1 dimensions. In this case one takes the unit vector $\vec{m}$ parallel to the z-direction and interprets $u = (x^0 - z)\sqrt{2}$ and $v = (x^0 + z)\sqrt{2}$ as transformed coordinates.

7. Conclusion

Exact solutions φ_k of the field equations of the (unconstrained) U(N)-and O(N)-σ-model have been obtained, which correspond to plane and spherical waves. For plane waves the constant vector p^μ orthogonal to the wave fronts may be timelike, spacelike or lightlike. In the first two cases the solutions (which depend only on one variable p.x) may contain singularities. The Lagrangian and the energy-momentum-tensor can be made finite by subtracting appropriate derivative terms (which do not affect global quantities). The integrated densities diverge for an infinite volume. For light-like waves the solutions depend on two variables corresponding to propagation in opposite directions. The solutions and the densities may contain singularities. These solutions may be understood also as solutions to the model in 1+1 dimensions. For spherical waves the solutions (except in some degenerate case) have meron-like singularities at the light cone and infinitely many additional singularities, located either inside or outside of the light cone, depending on the sign of the coupling constant. The latter singularities, which are perhaps not expected from analysis in Euclidean x-space, can be made to disappear in the Lagrangian and the energy-momentum tensor by adding derivative terms, so that these densities contain only the meron-like singularity. For the energy-momentum tensor this amounts to using the improved tensor found in another context [Callan et al., 1970] both for plane and spherical waves. The tensor has the structure postulated from general requirements in this context [Butera et al. 1979].

All solutions found in this paper are based on a fundamental ansatz (13), which expresses the postulate, that internal symmetry and space-time structures are not mixed. If this is assumed, the internal symmetry group affects the physical densities only via a constant Λ^2. Solutions with constant U(N)- or O(N)-modulus of the field are then only possible for plane, but not for spherical waves.

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