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INCOHERENT NOISE AND SELF-CONSISTENCY IN STOCHASTICALLY UNSTABLE PLASMAS*

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It is argued that Dupree's procedure for computing self-sustaining clump spectra is tautological.

It is well-known that stochastically unstable Hamiltonian systems exhibit orbit exponentiation. That is, when the appropriate resonances overlap the system develops positive Kolmogorov entropy, which implies that, on the average,[1] the separation between two orbits initially close grows exponentially rapidly. Dupree has called this phenomenon "phase space granulation."[2] Particularly important for applications is the problem where the forces are self-consistent with the dynamical flow in phase space. For example, it has been proposed[3] that the anomalous electron heat transport observed in tokamak devices may be explained by the development of stochastic magnetic fields[1,4] produced in a self-consistent manner[5] by turbulent micro-instabilities with electromagnetic polarizations.

Dupree has studied the steady-state spectrum of stochastically unstable Vlasov plasma,[2] taking into account certain aspects of the exponential orbit divergence. In particular, he argued heuristically that the stochastic evolution of phase space blobs of limited extent (which he calls "clumps") gives rise to a certain so-called "incoherent" electric field $\delta\tilde{E}$, which is then shielded by the dielectric ε to give the total (fluctuating) field δE. He argued that such a state will be self-consistent or "self-sustaining" if a certain regeneration factor,

$$R_f\{\mathcal{E}\} \equiv \langle\delta\tilde{E}^2\{\mathcal{E}\}\rangle/(|\varepsilon|^2\mathcal{E}) \ , \qquad (1)$$

equals unity. Here $\mathcal{E} \equiv \langle\delta E^2\rangle$ and $\{\cdots\}$ indicates functional dependence.

The procedure to assess whether a self-sustaining state is possible is two-fold. First, one expresses the incoherent fluctuations in terms of $\mathcal{E}$ and thus finds the functional form $R_f\{\mathcal{E}\}$. Second, one examines the structure and size of the functional form and attempts to determine whether the critical state $R_f = 1$ can be achieved for any value of $\mathcal{E}$. Dupree's estimates show that $R_f\{\mathcal{E}\}$ involves the plasma susceptibilities and phase space gradients of the distribution. However, his results do not contain the fluctuation intensity explicitly and are thus of order unity for $\varepsilon = O(1)$. This feature has led to considerable uncertainty as to whether a self-sustaining state in fact exists, since variations of order unity of numerical factors — which are not accurately determined by the approximate theory — give large relative variations of $R_f\{\mathcal{E}\}$ around $R_f = 1$. In fact, it is very peculiar that the dimensionless parameter which determines the existence of a certain turbulent state does not explicitly involve the turbulence level. This implies that the theory has no continuous connection to weak turbulence theory, for which it is readily verified that $R_f = O(\mathcal{E})$, and suggests that the self-sustaining state may exist at all turbulence levels, which is improbable.

The structure of Dupree's formulas suggest that his theory is a (severe) simplification of the direct-interaction approximation.[6] Krommes has previously discussed[7] the relations of this and other renormalizations to Dupree's concept of incoherent noise. Recently, DuBois[8] extended the direct-interaction version of Krommes' formulas to give an explicit prediction for the form of the incoherent source. Unlike that of Dupree, DuBois' formula does explicitly involve the turbulence level and is continuously connected to weak turbulence theory. The purpose of this paper is to discuss a possible explanation for this discrepancy which, when ε is taken to be $O(1)$, amounts to a whole order in $\mathcal{E}$.

We propose that Dupree's heuristic procedure of screening the incoherent noise is, though intuitively correct, in fact a tautology as implemented. That is, if Dupree were to retain certain terms which he inconsistently ignores and were then to recompute the regeneration factor following the logical steps he outlines, he would find $R_f\{\mathcal{E}\} \equiv 1$, corresponding to the identity $1 \equiv \mathcal{E}/\mathcal{E}$. Though the appearance of such an identity serves as a useful self-consistency check on the manipulations, it clearly cannot determine either the fluctuation level or spectral details at saturation. These would appear to follow only by explicit solution of equations of the direct-interaction type.

Let us recall certain general aspects of the renormalized theory of the electrostatic Valsov-Poisson system

$$\partial_t f + v\cdot\nabla f + E\cdot\partial f = 0 \quad , \tag{2a}$$

$$\nabla\cdot E = 4\pi \sum (ne) \int dv\, f \quad . \tag{2b}$$

We shall write the solution of Eq. (2b) as $E = \mathcal{E}f$, which defines the operator $\mathcal{E}$. If non-Gaussian initial conditions are ignored, the statistical solution of Eqs. (2) can be conveniently developed[9,10] in terms of the mean distribution $\langle f\rangle$, the two-point correlation function $C(1,1') \equiv \langle\delta f(1)\delta f(1')\rangle$, where $\delta f \equiv f - \langle f\rangle$, and the mean infinitesimal response function $R(1;1') \equiv \langle\delta f(1)/\hat{\eta}(1')\rangle|_{\hat{\eta}=0}$, where $\hat{\eta}$ is a non-random source term added to the right-hand side of Eq. (2a). *The response function includes the self-consistent* effects of dielectric shielding. Instead of R, we may introduce the bare particle "propagator" g (which obeys a certain Dyson equation[10,11] which we need not write here). One has in operator notation[11]

$$R = g - g\partial\bar{f}\varepsilon^{-1}\mathcal{E}g \quad , \tag{3}$$

where $\bar{f} \equiv \langle f \rangle + \delta\bar{f}$, $\delta\bar{f}$ being a certain ponderomotive type of nonlinear term defined in Ref. 11, and where the dielectric is defined by[8,10-12]

$$\varepsilon \equiv 1 - Eg\partial\bar{f} \quad . \qquad (4)$$

An important identity is

$$ER = \varepsilon^{-1}Eg \quad , \qquad (5)$$

which describes shielding of test particles moving in the turbulent medium.

It can now be shown[8-12] that the fluctuations are determined from the balance equation

$$C = R\tilde{F}R^t, \qquad (6)$$

where $R^t(1;1') \equiv R(1';1)$ and $\tilde{F}$ is a linear functional of a certain four-point function[7,9,10] $K(1,2;1',2')$ which describes the propagation of two-point fluctuations or pairs of test particles and obeys the Bethe-Salpeter equation.[7,10] In the direct-interaction approximation,

$$K(1,2;1',2') = \tfrac{1}{2}[C(1,1')C(2,2') + (1'\leftrightarrow 2')] = O(\mathcal{E}^2) \quad . \qquad (7)$$

A balance equation for the field spectrum can be constructed from Eq. (6) by applying the E operator from both the left and the right and by using Eq. (5):

$$\mathcal{E} = [E(g\tilde{F}g^t)E^t]/|\varepsilon|^2 \qquad (8)$$

We interpret this[7,10] by saying that there arises a certain "incoherent noise" $\delta\tilde{f}$ with correlation

$$\tilde{C} \equiv \langle\delta\tilde{f}\delta\tilde{f}\rangle \equiv g\tilde{F}g^{t} \quad . \tag{9}$$

Associated with $\delta\tilde{f}$ is an incoherent field $\delta\tilde{E} \equiv E\delta\tilde{f}$ with correlation

$$\tilde{\mathcal{E}} \equiv \langle\delta\tilde{E}\delta\tilde{E}\rangle \equiv E\tilde{C}E^{t} \quad . \tag{10}$$

The turbulent plasma then shields the incoherent field, giving the total field fluctuation as $\delta E = \delta\tilde{E}/\varepsilon$ — which is to be more precisely interpreted as

$$\langle\delta EA\rangle = \varepsilon^{-1}\langle\delta\tilde{E}A\rangle \tag{11}$$

for arbitrary A. Setting $A = \delta E$ leads to $\mathcal{E} = \tilde{\mathcal{E}}/|\varepsilon|^2$, which is just Eq. (8). Equation (6) may be interpreted similarly [7,10] by defining a "coherent response" $\delta f^{(c)}$ according to $\delta f^{(c)} \equiv -g\partial\bar{f}\delta E$ or, more precisely,

$$\langle\delta f^{(c)}A\rangle = -g\partial\bar{f}\langle\delta EA\rangle \quad . \tag{12}$$

Using (3), we can then write

$$C = \tilde{C} + \langle\delta f^{(c)}\delta\tilde{f}\rangle + \langle\delta\tilde{f}\delta f^{(c)}\rangle + \langle\delta f^{(c)}\delta f^{(c)}\rangle \quad , \tag{13}$$

where the last term, for example, is equal to

$$<\delta f^{(c)}\delta f^{(c)}> \equiv (g\partial\bar{f}\varepsilon^{-1}Eg)\ddot{F}(g\partial\bar{f}\varepsilon^{-1}Eg)^t$$
$$= (g\partial\bar{f})\,\mathcal{E}\,(g\partial\bar{f})^t \quad . \tag{14}$$

In arriving at the last line of Eq.(14), we used Eq. (8); note that Eq. (14) agrees with Eq. (12) . Dupree neglects the cross terms $<\delta f^{(c)}\delta\tilde{f}>$, arguing that $\delta\tilde{f}$ is a very random function. However, upon noting Eqs. (12) and (11), we find

$$<\delta f^{(c)}\delta\tilde{f}> \approx - g\partial\bar{f}\varepsilon^{-1}E<\delta\tilde{f}\delta\tilde{f}> \ , \tag{15}$$

so that $<\delta f^{(c)}\delta\tilde{f}>$ does not vanish if $\tilde{C}$ does not. That a rigid correlation exists between $\delta\tilde{f}$ and $\delta f^{(c)}$ vitiates the terminology "coherent" and "incoherent"; however, we retain it for comparison purposes.

We may now describe our interpretation of Dupree's procedure for computing $\tilde{C}$. As we understand it, Dupree works not with the formal solution (υ), but rather with the differential equation $R^{-1}C = \tilde{F}R^t$, which can be written for the two-time function C(t,t') as

$$g^{-1}C - \tilde{F}g^t = -\delta\bar{f}<\delta\varepsilon\delta f> - \tilde{F}(g\partial\bar{f}\varepsilon^{-1}Eg)^t \quad . \tag{16}$$

(The corresponding equation for C(t,t) follows by appropriate symmetrization, which we occasionally indicate by the subscript "s".) Dupree argues that at long wavelengths the term in $\tilde{F}g^t$ is of the form $\partial_1 D_{1,1'}\partial_{1'}C(1,1')$, where $D_{1,1'}$ is a cross-diffusion coefficient defined by Eq. (42) of Ref.3 . At equal times, then, Dupree approximates the operator on C on the left-hand side of Eq. (16) as a bivariate Fokker-Planck operator g_2^{-1} . If one notes Eqs. (13), (12), (4), and (11) so that, for example,

$$<\delta E\delta f> = -\mathcal{E}(g\delta\overline{f})^{t} + <\delta E\delta\tilde{f}> \quad , \tag{17}$$

one finds

$$C(t,t) = g_2\{g^{-1}[<\delta f^{(c)}\delta f^{(c)}> + (\delta f^{(c)}\delta\tilde{f}> + <\delta\tilde{f}\delta f^{(c)}>]\}_s \ . \tag{18}$$

Dupree neglects the last two terms.

Of course, (18) describes the total fluctuations, including shielding. Following Dupree, we obtain the incoherent fluctuations by subtracting off everything else according to Eq. (13):

$$\tilde{C}(t,t) = g_2\{g^{-1}[<\delta f^{(c)}\delta f^{(c)}> + <\delta f^{(c)}\delta\tilde{f}> + <\delta\tilde{f}\delta f^{(c)}>]\}_s - \{[<\delta f^{(c)}\delta\tilde{f}> + <\delta\tilde{f}\delta f^{(c)}>]_s + <\delta f^{(c)}\delta f^{(c)}>\} \quad . \tag{19}$$

Dupree would retain only the terms in $<\delta f^{(c)}\delta f^{(c)}>$.

We can now construct the incoherent spectrum by applying the $\mathcal{E}$ operator to Eq. (19). Dupree has shown[2] that g_2 propagates pairs of particles along stochastically unstable orbits and includes the effect of exponential orbit divergence. There is thus in Eq. (19) a class of particles, satisfying 1∿1', for which g_2 becomes effectively the "clump lifetime" τ_{cl} — the time for the initially closely-separated particles to diverge a typical wavelength. Since τ_{cl} is logarithmically larger than the inverse K-entropy, it can be argued that the first term dominates in Eq. (19). Thus, according to Dupree, the incoherent clump noise becomes

$$\langle\delta\tilde{E}\delta\tilde{E}\rangle_{cl} \simeq EEg_2g^{-1}\langle\delta f^{(c)}\delta f^{(c)}\rangle$$
$$\simeq EE\tau_{cl}D_{1,1'}(\partial\tilde{f})^2 \quad . \qquad (20)$$

Dupree then asserts that Eq. (20) should be shielded:

$$\mathcal{E} = \langle\delta\tilde{E}\delta\tilde{E}\rangle_{cl}/|\varepsilon|^2 \quad . \qquad (21)$$

We believe that this shielding recipe is incorrect. As we have already remarked in the paragraph following Eq. (18), the right-hand side of Eq. (20) is in fact Dupree's approximation ($\langle\delta\tilde{f}\delta f^{(c)}\rangle = 0$) to the shielded spectrum $\mathcal{E}$. Equation (20) thus reads $\tilde{\mathcal{E}} \simeq \mathcal{E}$ and Eq. (21) becomes $\mathcal{E} \simeq \mathcal{E}/|\varepsilon|^2$, correct only for $\varepsilon \simeq 1$. More generally, we compute from Eq. (19)

$$\tilde{\mathcal{E}} = \mathcal{E} - [-(\varepsilon-1)\mathcal{E}\varepsilon^t - \varepsilon\mathcal{E}(\varepsilon-1)^t + (\varepsilon-1)\mathcal{E}(\varepsilon-1)^t] \qquad (22a)$$
$$= \varepsilon\mathcal{E}\varepsilon^t \quad , \qquad (22b)$$

where we used Eqs. (18) and (4). This agrees with Eq. (11), as it must. In Eqs. (22), the $\mathcal{E}$ may be either the total spectrum (including all nonlinear processes) or the contribution to the spectrum due to a particular nonlinear process such as clumps; the important fact is that the same factor of $\mathcal{E}$ appears in each term of Eq. (22a). We observe that the correct shielding law emerges from cancellations between the first term of Eq. (22a) (which gives rise to Dupree's clump source) and the last term (the coherent response, which Dupree includes in principle but ultimately neglects), and between the coherent response and the mixed terms stemming from $\langle \delta f^{(c)} \delta \tilde{f} \rangle$ (which Dupree neglects). In fact, though any of the terms of Eq. (19) may be small in a particular region of phase space, integration changes their order so that they all compete at the more macroscopic level of the electric fields. Furthermore, if we attempt to follow Dupree and shield Eq. (22), we find

$$\mathcal{E} \equiv \tilde{\mathcal{E}} / |\varepsilon|^2 = \mathcal{E} \ ,$$

which is a manifest tautology. The size and structure of $\mathcal{E}$ cannot be determined in this way. One must revert to the explicit formula for $\tilde{F}\{K\}$.

In the direct-interaction approximation, it is clear from DuBois' work[8] that $\tilde{F}$ is one order smaller in $\mathcal{E}$ than Dupree's prediction. In fact, DuBois' result has dimensionally the same form as Eq. (20), with the important difference that $(\partial \bar{f})^2$ is replaced by $\langle [\partial \delta f^{(c)}]^2 \rangle$! Since $\delta f^{(c)}$ is driven by $\partial \bar{f}$, it can still be said that gradients of the mean distribution drive phase space granulation. However, the detailed dynamics differ considerably from Dupree's proposal.

Beyond direct interaction, the formula for $\tilde{F}\{K\}$ is modified according to the solution of the Bethe-Salpeter equation.[7,10] Indeed, K is itself evolved by g_2, dielectrically shielded, and also driven by higher-order noise terms. However, we have been unable to show, and we do not believe, that a higher-order renormalization can lead to Dupree's theory. In accordance with general principles, such renormalizations affect detailed dynamics but do not alter the gross order of the terms they describe.

Finally, we emphasize that the considerations in this paper are restricted to systems well into the stochastic regime, and whose autocorrelation time is short compared to the inverse K-entropy. When either of these criteria is not satisfied, partial trapping results and the nature and dynamics of "clumps" are strongly modified.[2] The standard renormalized kinetic equations are not well-adapted to such regimes, which thus represent an interesting area for future work.

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