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A GRAPHICAL APPROACH OF THE VECTOR ANALYSIS

E. ELBAZ and J. MEYER

Institut de Physique Nucléaire (and IN2P3)
Université Claude Bernard Lyon-1
43, Bd du 11 Novembre 1918 - 69622 Villeurbanne Cedex, France

The Graphical Spin Algebra has been shown to be applicable in a cartesian coordinate system without major modification. Then the (G. S. A.) allows a new and very easy approach of the usual vector analysis. Some examples of application are given.

1. INTRODUCTION

The Graphical Spin Algebra (G. S. A.) [1] is now well-known as a useful tool to handle the Racah algebra of the SU_2 group. Different extension have been given for the SU_3 group or even for all compact groups [2-5].

The G. S. A. lies on a one-to-one diagrammatic representation of the elements of the group and on some fundamental rules of transformation based on the rotational invariance orthogonality and completeness relations. For the purpose of this paper only some basic aspects of the G. S. A. have to be known [1]. On the other hand[6] we have shown that one could define a graphical representation of the vectors or vector operators in a spherical coordinate system. When a graphical representation of the metric tensor [7] $E^j(r \mid s)$ is given one can use the G. S. A. without alteration in a spherical coordinate system as usual or even in a cartesian system.

The choice of a proper convention to link cartesian and spherical systems was then important and it appeared that the use of the Biedenharn-Rose convention allowed and identical graphical representation of the scalar and dot products of two vector operators. Moreover since the " 3nj " coefficients are scalars, independent of the coordinate system, one could use the G. S. A. without specifying the reference frame and defining it only at convenience.

Such a result was sufficiently intringuing to reconsider the graphical representation of the vector operators. It appeared effectively that the usual graphical rules of the G. S. A. and the knowledge of two special cartesian Clebsch-Gordan coefficients gave immediately all the usual results known as the vector analysis.

2. DIAGRAMMATIC REPRESENTATION OF VECTOR OPERATORS

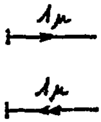
2.1 Standardization of vector operators :

Using the Biedenharn-Rose convention [8] for the standardization of vector operators, the linear relation between the cartesian $(A_x A_y A_z)$ components of the $\vec{A}$ vector operator and its $A_{1\mu}$ standard components can be written

$$\begin{pmatrix} A_{11} \\ A_{10} \\ A_{1-1} \end{pmatrix} = \begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \qquad (2.1)$$

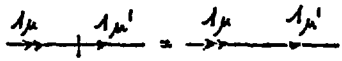
2.2 Spherical (standard) coordinates :

Using the G.S.A. diagrams [6], we define a spherical, or standard basis

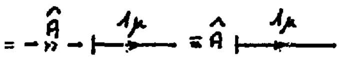
$| 1\mu >$ = , $< 1\mu |$ =  (2.2)

which verifies completeness and orthonormalization relations

$\sum_\mu | 1\mu > < 1\mu | =$  $= 1$ (2.3)

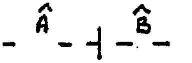
$< 1\mu | 1\mu' > =$  $= \delta_{\mu\mu'}$ (2.4)

In such a basis a vector operator $\vec{A}$ wil be defined by its components

$A_{1\mu} = < \hat{A} | 1\mu> =$  (2.5)

$A^+_{1\mu} = < 1\mu | \hat{A} > =$  (2.6)

The scalar product is introduced through

$< \hat{A} | \hat{B} > =$  $= \sum_\mu < \hat{A} | 1\mu > < 1\mu | \hat{B} > =$  (2.7)

$$= \sum_\mu A_\mu B^+_\mu = \sum_\mu (-)^{1-\mu} A_\mu B_{-\mu}$$

and the standardization gives with the Biedenharn-Rose convention

$$< \hat{A} | \hat{B} > = \vec{A} . \vec{B} \tag{2.8}$$

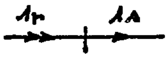
2.3 Cartesian coordinates :

One introduces the cartesian vector basis e_r and for the sake of simplicity we set

$$| e_r > = | 1r > =$$  $\equiv$  (2.9)

with $r = 1, 2, 3 = x, y, z$. They form a complete orthonormal basis

$$\sum_r | 1r > < 1r | =$$  $= 1$

$$< 1r | 1s > =$$  $= \delta_{rs}$ (2.10)

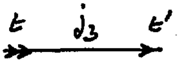
At this stage, one must point out that the above relations are particular cases of the description of tensors in cartesian coordinates.

If $A^{(j_1)}(m_1)$ defines a tensor of rank m_1 (3^{m_1} components) and of order j_1 ($(2j_1+1)$ independent components), a Cartesian Clebsch-Gordan coefficient defines the decomposition of two irreducible spaces H^{j_1} and H^{j_2} into a sum of subspace H^{j_3} 7),

$$(A^{j_1}(m_1) \otimes B^{j_2}(m_2))^{j_3 p}_{m_3} = \sum_{m_1 m_2} A^{j_1}(m_1) \; B^{j_2}(m_2) < j_1 m_1 j_2 m_2 | j_3 m_3 >_p \tag{2.11}$$

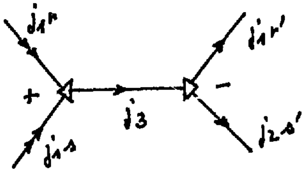
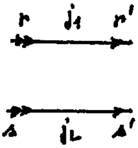
where $3^{m_1+m_2} = 3^{m_3}$ is the dimension of the product space, and p is the multiplicity.

One then defines the metric tensor 7)



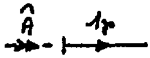
$$= E^{j_3}(t | t') \tag{2.12}$$

and the closure relation becomes

$\Sigma_{j_3 p}$  =  $= E^{j_1}(r|r')\ E^{j_2}(s|s')$ (2.13)

Since $E^1(r|s) = \delta_{rs}$ we find the (2.10) relation.

The cartesian components of vector operators are then

$A_r = <\hat{A}|1r> = <\hat{A}|e_r> =$  =  (2.14)

with $r = x, y, z = 1, 2, 3$.

The scalar product is now

$<\hat{A}|\hat{B}> = \sum_i <\hat{A}|1i><1i|\hat{B}> =$  $= \sum_i A_i B_i^+$ (2.15)

when dealing with hermitian operators $B_i^+ = B_i$ and

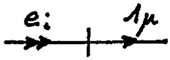
$$<\hat{A}|\hat{B}> = \vec{A}.\vec{B} \qquad (2.16)$$

2.4 <u>Cartesian spherical transformation coefficients</u> :

We can express a spherical component $A_{1\mu}$ of the $\vec{A}$ vector operator in a cartesian coordinate system

$$A_\mu = <\hat{A}|1\mu> = \sum_i <\hat{A}|e_i><e_i|1\mu> = \sum_i A_i <e_i|1\mu> \qquad (2.17)$$

Graphically it follows that

$<e_i|1\mu> =$  (2.18)

It defines the matrix element of the unitary U-transformation matrix

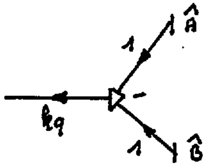
$$A_\mu = U^i_\mu \, A_i = \sum_i < e_i \mid \mu > A_i \tag{2.19}$$

with

$$U^i_\mu = \begin{pmatrix} <e_1 \mid 1> & <e_2 \mid 1> & <e_3 \mid 1> \\ <e_1 \mid 0> & <e_2 \mid 0> & <e_3 \mid 0> \\ <e_1 \mid -1> & <e_2 \mid -1> & <e_3 \mid -1> \end{pmatrix} = \begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{2.20}$$

3. TENSOR PRODUCT IN SPHERICAL COORDINATES

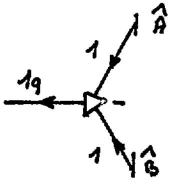
The A_μ considered as ITO of rank 1 allows the determination of the tensorial product $(A_\mu \times B_\nu)_{kq}$ and graphically it immediately follows that

$(A_\mu \times B_\nu)_{kq} =$  (3.1)

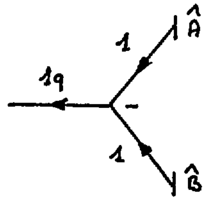
The tensorial product of zero rank is related to the scalar product since

$(A_\mu \times B_\nu)_{oo} = \frac{1}{\sqrt{3}}$  $= \frac{1}{\sqrt{3}} (\vec{A} . \vec{B})$ (3.2)

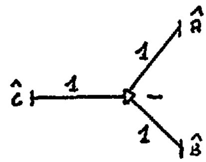
and the rank one to the cross-product

$(A_\mu \times B_\nu)_{1q} = -\frac{1}{\sqrt{2}} (\vec{A} \wedge \vec{B})_{1q} =$  (3.3)

or equivalently

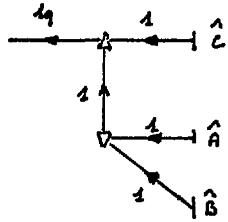
$(A \wedge B)_{1q} = \sqrt{6}$  (3.4)

One can then obtain the graphical representation in spherical coordinates of more complex products like

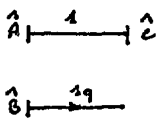
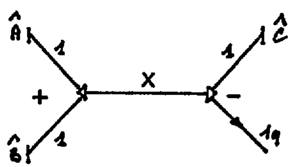
$(\vec{A} \wedge \vec{B}) . \vec{C} = (\vec{B} \wedge \vec{C}) . \vec{A} = (\vec{C} \wedge \vec{A}) . \vec{B} = \sqrt{2}$ 

(3.5)

and

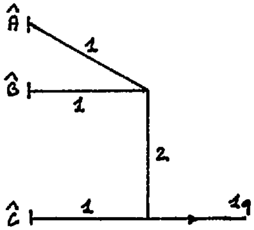
$((\vec{A} \wedge \vec{B}) \wedge \vec{C})_{1q} = 2$  (3.6)

or even construct other tensors of rank one. We have for instance

$(\vec{A} . \vec{C})\, \vec{B}_{1q} =$  $= \sum_X$  (3.7)

We then express the sum over X = 0, 1, 2 to get with (3.6)

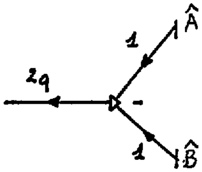
$T_{1q}(\vec{A}, \vec{B}, \vec{C}) =$



$$= \left[(\vec{A}.\vec{C})\, B_{1q} - \frac{1}{3}(\vec{A}.\vec{B})\, C_{1q} - \frac{1}{2}\left((\vec{A} \wedge \vec{B}) \wedge \vec{C}\right)_{1q} \right] \qquad (3.8)$$

One can also determine the $k = 2$ tensor components of the tensorial product ot two vector operators

$(A_\mu \times B_\nu)_{2q} =$



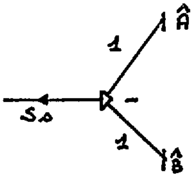
(3.9)

and its spherical components which can be found in the literature [9].

4. SCALAR AND CROSS PRODUCTS IN CARTESIAN COORDINATES

Let us construct the tensor product of two vector operators in cartesian coordinates

$(A_{r_1} \otimes B_{r_2})_{Ss} =$



$$= \sum_{r_1 r_2} \langle 1r_1\ 1r_2 | Ss \rangle A_{r_1} B_{r_2} \qquad (4.1)$$

In order to obtain the scalar and the cross-product we have to evaluate two particular cartesian Clebsch-Gordan coefficients $\langle 1r_1\ 1r_2 | 00 \rangle$ and $\langle 1r_1\ 1r_2 | 1s \rangle$.

It can be easily found that

$$\langle 1r_1\ 1r_2 | 00\rangle = \sum_{\mu_1\mu_2} \langle 1r_1 | 1\mu_1\rangle \langle 1r_2 | 1\mu_2\rangle \langle 1\mu_1\ 1\mu_2 | 00\rangle \tag{4.2}$$

$$= \sum_{\mu_1} \langle 1r_1 | 1\mu_1\rangle \langle 1r_2 | 1-\mu_1\rangle \frac{1}{\sqrt{3}} (-)^{1-\mu_1} = \frac{1}{\sqrt{3}} \delta_{r_1 r_2}$$

since

$$\langle 1r_2 | 1-\mu_1\rangle (-)^{1-\mu_1} = \langle 1\mu_1 | 1r_2\rangle \tag{4.3}$$

It follows that

$$(A_{r_1} \otimes B_{r_2})_{oo} = \frac{1}{\sqrt{3}} \vec{A}.\vec{B} \tag{4.4}$$

and a comparison with (2.8) shows that the Biedenharn's convention leads to the same value of the tensor product of zero order in spherical and cartesian coordinates.

Let us now evaluate the cartesian Clebsch-Gordan coefficient

$$\langle 1r_1\ 1r_2 | 1s\rangle = \sum_{\mu_1\mu_2\mu_3} \langle 1r_1 | 1\mu_1\rangle \langle 1r_2 | 1\mu_2\rangle \langle 1\mu_3 | 1s\rangle \, . \, \langle 1\mu_1\ 1\mu_2 | 1\mu_3\rangle \tag{4.5}$$

The use of the $U_{r_1}^{\mu_1}$ matrix elements gives without difficulty the value

$$\langle 1r_1\ 1r_2 | 1s\rangle = -\frac{1}{\sqrt{2}} \epsilon_{r_1 r_2 s} \tag{4.6}$$

$\varepsilon_{r_1 r_2 s}$ is the Levi-Civittà antisymmetric tensor .

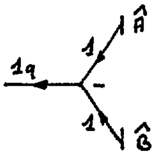
$$\varepsilon_{r_1 r_2 s} = \begin{cases} 1 \text{ if } r_1 r_2 s \text{ is an even permutation of } 1,\ 2,\ 3 \text{ indices} \\ -1 \text{ if } r_1 r_2 s \text{ is an odd permutation} \\ 0 \text{ elsewhere} \end{cases}$$

In cartesian coordinates one thus finds that

$$(A_{r_1} \times B_{r_2})_s = -\frac{1}{\sqrt{2}} (\vec{A} \wedge \vec{B})_s \tag{4.7}$$

It is exactly the result obtained in spherical coordinates. We have then

$= \vec{A} . \vec{B}$

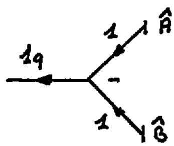


$= \frac{1}{\sqrt{6}} (\vec{A} \wedge \vec{B})_q$ (4.8)

in any reference frame. When working in spherical coordinates $q = \mu = 1, 0, -1$ and in cartesian coordinates $q = s = 1, 2, 3 = x, y, z$.

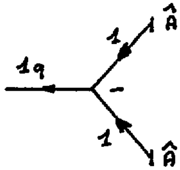
5. VECTOR ANALYSIS

Let us first recall some obvious but useful results. The cross-product of two vector operators reads now

$(\vec{A} \wedge \vec{B})_q = \sqrt{6}$


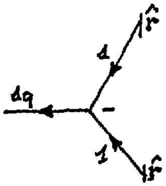
(5.1)

If the components of $\vec{A}$ operator commute, one can change the lecture order of the diagram without affecting the result ; one knows however that such a change multiplies the result by $(-)^{1+1+1}$. It then follows

$[\vec{A}, \vec{A}] = 0 \iff$


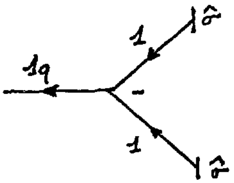
$= 0$ (5.2)

We obtain for instance

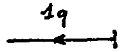


$= 0$ equivalent to $\vec{r} \wedge \vec{r} = 0$ (5.3)

while with the Pauli matrices



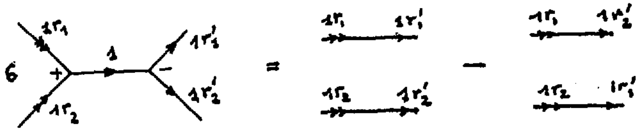
$= \frac{2i}{\sqrt{6}}$



$\hat{\sigma}$ (5.4)

equivalent to the commutation : $\vec{\sigma} \wedge \vec{\sigma} = 2i\, \vec{\sigma}$ (5.5)

An other interesting result is obtain with the cartesian coordinates coefficients



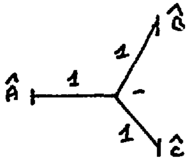
(5.6)

since 7)

$$\sum_s \epsilon_{r_1 r_2 s}\, \epsilon_{r_1' r_2' s} = \delta_{r_1 r_1'}\, \delta_{r_2 r_2'} - \delta_{r_1 r_2'}\, \delta_{r_2 r_1'} \qquad (5.7)$$

We shall denote this rule as the " crossing rule ".

5.1 The triple scalar product :

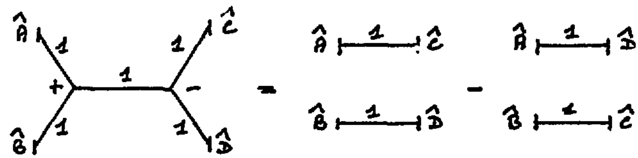
$$\vec{A}.(\vec{B}\wedge\vec{C}) = \sqrt{6}$$  (5.8)

Due to the symmetry property of the " 3jn " coefficient one can start the lecture from any vector operator and thus obtain

$$\vec{A}.(\vec{B}\wedge\vec{C}) = \vec{B}.(\vec{C}\wedge\vec{A}) = \vec{C}.(\vec{A}\wedge\vec{B}) \qquad (5.9)$$

5.2 Scalar products of two dot products :

The use of (5.6) gives an interesting expression of the scalar products of two dot products

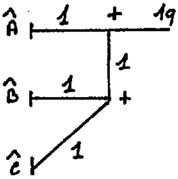
$$(\vec{A}\wedge\vec{B}).(\vec{C}\wedge\vec{D}) = 6$$  (5.10)

$$(\vec{A}\wedge\vec{B}).(\vec{C}\wedge\vec{D}) = (\vec{A}.\vec{C})(\vec{B}.\vec{D}) - (\vec{A}.\vec{D})(\vec{B}.\vec{C}) \qquad (5.11)$$

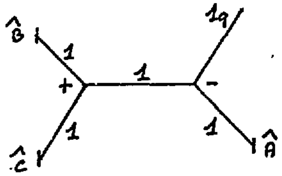
When dealing with vector operators which do not necessarily commute, one must take care of the order of the operator in the left and right hand sides. When the above are only vectors the order is unimportant.

5.3 The double cross-product :

$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q = 6$$



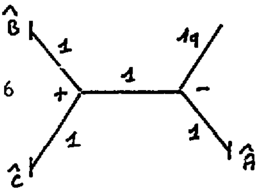
$$= 6$$



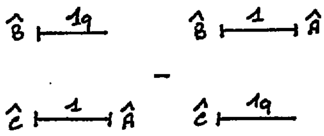
(5.12)

One can consider that one works in cartesian coordinates and uses (5.6) to immediately obtain the well-known relation

$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q = 6$$



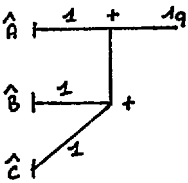
$$=$$



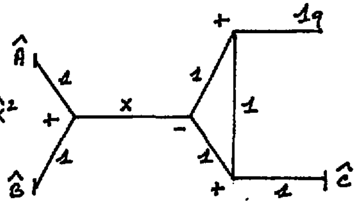
$$\vec{A} \wedge (\vec{B} \wedge \vec{C}) = \vec{B}(\vec{A} . \vec{C}) - (\vec{A} . \vec{B}) \vec{C} \qquad (5.13)$$

One can use now the graphical representation of the double cross-product and the usual rules of the G.S.A. to get the analytical expression of a particular tensor

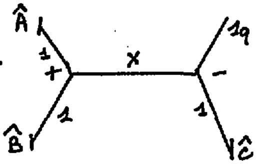
$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q = 6$$

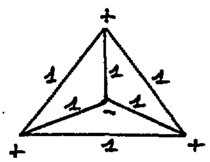


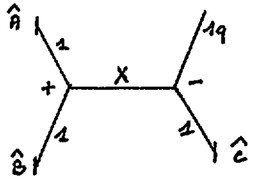
$$= 6 \sum_X \hat{X}^2$$



$$= 6 \sum_X \hat{X}^2$$





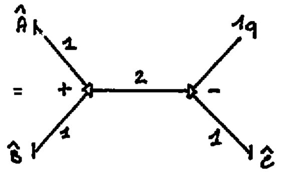
$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q = 6 \sum_X \hat{X}^2 (-)^X \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & X \end{Bmatrix}$$ 

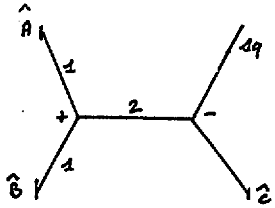
(5. 14)

One can develop that expression since $X = 0, 1, 2$ and the corresponding "6j" coefficient take the values $-\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{6}$. One then obtains

$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q = -\frac{2}{3} (\vec{A} . \vec{B}) C_q - \frac{1}{2} (\vec{C} \wedge (\vec{A} \wedge \vec{B}))_q + T_{1q} (\vec{A}, \vec{B}, \vec{C}) \qquad (5.15)$$

with

$T_{1q} (\vec{A}, \vec{B}, \vec{C}) =$ 

$= 5$ 

(5. 16)

and it follows that

$$T_{1q}(\vec{A}, \vec{B}, \vec{C}) = (\vec{A} \wedge (\vec{B} \wedge \vec{C}))_q + \frac{2}{3} (\vec{A} . \vec{B}) C_q + \frac{1}{2} (\vec{C} \wedge (\vec{A} \wedge \vec{B}))_q \qquad (5.17)$$

or with (5. 13)

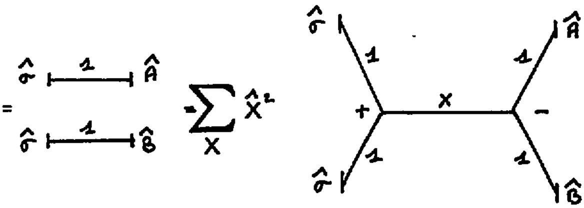
$$T_{1q} = \frac{1}{2} (\vec{A} . \vec{C}) B_q - \frac{1}{3} (\vec{A} . \vec{B}) C_q + \frac{1}{2} (\vec{B} . \vec{C}) A_q \qquad (5.18)$$

6. SOME EXAMPLES OF APPLICATION

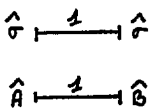
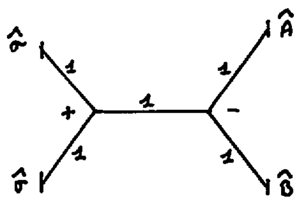
One can use (2. 15) to show that for the σ Pauli matrices considered as vector operators

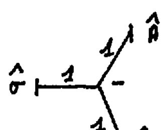
$$T_{2q}(\vec{\sigma}, \vec{\sigma}) = 0 \qquad (6.1)$$

One can then easily obtain the following

$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) =$ 

We use the X = 0, 1, 2 and (5. 5) , (6. 1) to get

$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \frac{1}{3}$  $+ 3$ 

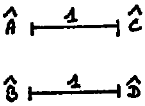
$= \frac{1}{3}\sigma^2 \vec{A}.\vec{B} + 3i\sqrt{\frac{2}{3}}$ 

and since $\sigma^2 = 3$

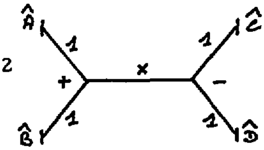
$$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = (\vec{A}.\vec{B}) + i\vec{\sigma}.(\vec{A} \wedge \vec{B}) \qquad (6.2)$$

One can obtain a more general expression when starting with the product of two scalar products of vector operators

$$(\vec{A}\,.\,\vec{C})\;(\vec{B}\,.\,\vec{D}) \;=$$



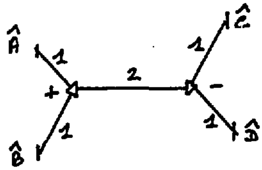
$$= \sum_X \hat{X}^2$$



(6. 3)

We set

$$T_2(\vec{A}, \vec{B})\;.\;T_2(\vec{C}, \vec{D}) \;=$$



(6. 4)

and one easily obtains the well-known form

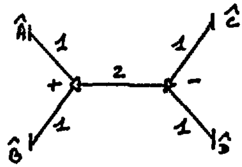
$$T_2(\vec{A}, \vec{B})\;.\;T_2(\vec{C}, \vec{D}) \;=\; (\vec{A}\,.\,\vec{C})\;(\vec{B}\,.\,\vec{D}) - \frac{1}{3}\,(\vec{A}\,.\,\vec{B})\;(\vec{C}\,.\,\vec{D}) - \frac{1}{2}\,(\vec{A}\wedge\vec{B})\,.\,(\vec{C}\wedge\vec{D}) \qquad (6.\,5)$$

If $\vec{C} = \vec{D} = \vec{\sigma}$, one refinds (6. 2) . If all the vector operators are different, one can express the scalar product of the two dot products with (5. 6) getting

$$T_2(\vec{A}, \vec{B})\;.\;T_2(\vec{C}, \vec{D}) \;=\; \frac{1}{2}\,(\vec{A}\,.\,\vec{C})\;(\vec{B}\,.\,\vec{D}) + \frac{1}{2}\,(\vec{A}\,.\,\vec{D})\;(\vec{B}\,.\,\vec{C}) - \frac{1}{3}\,(\vec{A}\,.\,\vec{B})\;(\vec{C}\,.\,\vec{D}) \qquad (6.\,6)$$

A comparison with (5.16) and (5.18) shows that

$$T_2(\vec{A},\vec{B})\,.\,T_2(\vec{C},\vec{D}) = T_1(\vec{A},\vec{B},\vec{C})\,.\,\vec{D} =$$



(6.7)

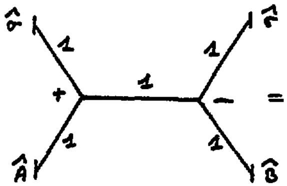
or equivalently

$$T_2(\vec{A},\vec{B})\,.\,T_2(\vec{C},\vec{D}) = T_1(\vec{C},\vec{D},\vec{B})\,.\,\vec{A} = T_1(\vec{C},\vec{D},\vec{A})\,.\,\vec{B}$$
$$= T_1(\vec{A},\vec{B},\vec{D})\,.\,\vec{C} \qquad (6.8)$$

since the above diagram can be cut by isolating any component.

We note that when $\vec{C} = \vec{D} = \vec{\sigma}$ one can reach the dot product $(\vec{\sigma}\wedge\vec{A}).(\vec{\sigma}\wedge\vec{B})$

$$(\vec{\sigma}\wedge\vec{A}).(\vec{\sigma}\wedge\vec{B}) = 6$$

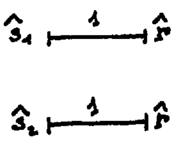


$$= 3\vec{A}.\vec{B} - (\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B})$$

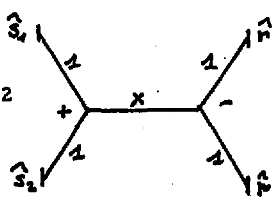
(6.9)

Let us finish by an example in which both cartesian and spherical aspects of the G.S.A. have to be used

$$(\vec{S}_1.\vec{r})\,(\vec{S}_2.\vec{r}) =$$

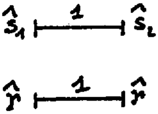
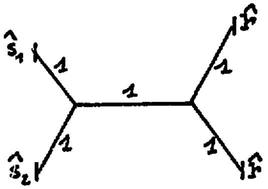


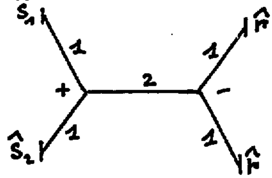
$$= \sum_X \hat{X}^2$$



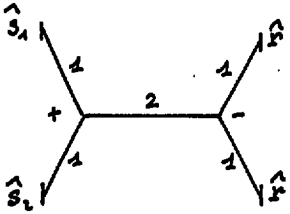
(6.10)

and since $X = 0, 1, 2$

$$(\vec{S}_1 \cdot \vec{r})(\vec{S}_2 \cdot \vec{r}) = \frac{1}{3}$$  $+ 3$ 

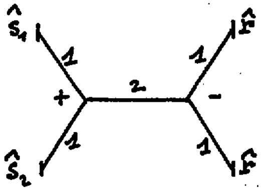
$+ 5$  (6.11)

Since $\vec{r} \wedge \vec{r} = 0$ the second diagram vanishes and we are left with

$$(\vec{S}_1 \cdot \vec{r})(\vec{S}_2 \cdot \vec{r}) - \frac{1}{3}(\vec{S}_1 \cdot \vec{S}_2)\, r^2 = 5$$ 

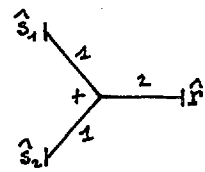
$$= T_2(\vec{S}_1 \cdot \vec{S}_2) \cdot T_2(\vec{r}, \vec{r}) \qquad (6.12)$$

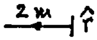
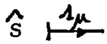
We divide the two sides by the length r^2 of the $\vec{r}$ vector and set

$$S_{12} = \frac{(\vec{S}_1 \cdot \vec{r})(\vec{S}_2 \cdot \vec{r})}{r^2} - \frac{1}{3}\vec{S}_1 \cdot \vec{S}_2 = 5$$  (6.13)

Since the only directions of the $\vec{r}$ vector are now involved in the diagram, one can normalize it by $\sqrt{\frac{4\pi}{3}}$ in order to have  $\hat{r} \equiv Y_{1m}(\hat{r})$ and use the usual technique of the G.S.A. on the two spherical harmonics thus left

$$S_{12} = \frac{(\vec{S}_1 \cdot \vec{r})(\vec{S}_2 \cdot \vec{r})}{r^2} - \frac{1}{3}\, \vec{S}_1 \cdot \vec{S}_2 = \sqrt{\frac{8\pi}{3}} \tag{6.14}$$



where  $= Y_{2m}(\hat{r})$ the usual spherical harmonic in the $\hat{r}$ direction, and  $= S_{1\mu}$ is the standard form of the spin vector operator.

7. CONCLUSION

W have shown in this paper the following results. First if we use the Biedenharn-Rose convention for the transformation of the cartesian basis into a standard (spherical) basis, the G. S. A. is applicable without modification in cartesian coordinates. Moreover, the graphical representations of the scalar and dot products and of the scalar " 3nj " coefficients in the two coordinates are identical. One can thus work without specifying a priori the coordinate system. The second result is that the G. S. A. can give a new useful approach of the vector analysis in its more usual aspect. In that case one can deal with the only few graphical representations and rules (2. 15) , (2. 16) and (2. 17), (4. 8) , (5. 6) . One note that when dealing with these rules only, one can avoid the $\sqrt{6}$ numerical coefficient in the dot product, but the use of the other rules of the G. S. A. makes this coefficient indispensable.

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