568

179
3-18-80

FEBRUARY 1980

PPPL-1568
UC-20g

DR. 887

RENORMALIZATION AND PLASMA PHYSICS

BY

MASTER

J. A. KROMMES

PLASMA PHYSICS LABORATORY

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

This work was supported by the U. S. Department of Energy Contract No. EY-76-C-02-3073. Reproduction, translation, publication, use and disposal, in whole or in part, by or for the United States Government is permitted.

Renormalization and Plasma Physics

John A. Krommes

Plasma Physics Laboratory, Princeton University,
Princeton, New Jersey 08544

A review is given of modern theories of statistical dynamics as applied to problems in plasma physics. The derivation of consistent renormalized kinetic equations is discussed, first heuristically, later in terms of powerful functional techniques. The equations are illustrated with models of various degrees of idealization, including the exactly soluble stochastic oscillator, a prototype for several important applications. The direct-interaction approximation is described in detail. Applications discussed include test particle diffusion and the justification of quasilinear theory, convective cells, $\vec{E}\times\vec{B}$ turbulence, the renormalized dielectric function, phase space granulation, and stochastic magnetic fields.

(Running title: "Renormalization")

TABLE of CONTENTS

PREFACE 5

1: RENORMALIZATION and PLASMA PHYSICS—An INTRODUCTION . . . 9

2: The STOCHASTIC OSCILLATOR 15

2.1: Introduction 15

2.2: Exact Solution of the Stochastic Oscillator 17

2.3: Approximate Closure Theories for the Stochastic Oscillator 21

3: TEST PARTICLE DIFFUSION in a DISCRETE SPECTRUM 26

3.1: Introduction 26

3.2: Stochastic Instability and Test Particle Diffusion. 27

3.3: Quasilinear Theory and Renormalization 33

4: HYDRODYNAMICS I 37

4.1: Introduction 37

4.2: Guiding Center Plasma 39

4.3: Convective Cells 42

5: The DIRECT-INTERACTION APPROXIMATION 46

5.1: Introduction 46

5.2: Kraichnan's Original Arguments 47

5.3: Generalized Langevin Representation 53

5.4: Random Coupling Models 54

5.5: Series Reversion 55

6: SYSTEMATOLOGY of RENORMALIZATION 59

6.1: Introduction 59

6.2: The Renormalized Equations of Martin, Siggia, and Rose 67

6.3: The Operator $\hat{\psi}$ and Some Technical Details of the Functional Approach 77

6.4: The Stochastic Oscillator Revisited 83

6.5: The Bethe-Salpeter Equation and Higher Order Renormalizations 87

7: HYDRODYNAMICS II 91

7.1: Introduction 91

7.2: Exact Consequences of the Hasegawa-Mima and Related Equations 93

7.3: Closure Approximations and Random Galilean Invariance 101

8: DIELECTRIC RESPONSE in VLASOV PLASMA 107

8.1: Introduction 107

8.2: The Renormalized Dielectric 112

8.3: Dielectric and Propagator in the Direct-Interaction Approximation, and Reduction to Weak Turbulence Theory 116

8.4: Dielectric Function of Guiding Center Plasma . . . 119

8.5: Aspects of the Drift Wave Dielectric 121

9: SELF-CONSISTENT FLUCTUATIONS and the BALANCE EQUATION . . 124

9.1: Incoherent Noise and the Balance Equation 124

9.2: Phase Space Granulation 129

10: MISCELLANEOUS 135

10.1: Renormalization and Stochastic Magnetic Fields . . 135

10.2: Projection Operator Techniques 138

10.3: Canonical Transformations 141

10.4: Additional References 142

11: AFTERWORD 143

ACKNOWLEDGMENTS 143

APPENDIXES 144

A: Cumulants 144

B: Fourier Transform Conventions 148

REFERENCES 150

TABLE I 163

FIGURES 164

LIST OF SYMBOLS 167

A slightly abridged version of this article will appear as Chapter 5.4 of the Handbook of Plasma Physics, edited by R. N. Sudan and A. A. Galeev, to be published by North-Holland Publishing Company. Chapter references in the article are to other articles in this collection.

- PREFACE -

In preparing the following review of renormalization as applied to plasma physics, I was particularly struck by two rather embarrassing facts. First, although the general theory of renormalization appears to be in reasonable shape (in spite of several important and difficult outstanding problems), successful (i.e., believable) practical plasma physics applications of the formalism are remarkably few, if not non-existent. Of course, calculations involving some aspects of renormalization and aimed at practical problems have proliferated since the early days of quasilinear theory, and each of these calculations has added to our understanding of stochasticity and turbulence in plasmas. Unfortunately, since the systematology of most of these early calculations has been obscure or questionable, there has arisen in the community at large a considerable aura of doubt around any statistical calculation which attempts to go beyond regular perturbation theory. In some cases, skepticism has been elevated to dogma: "Renormalized theories of plasma turbulence have nothing to offer the practical person concerned with scaling laws and transport in real devices."

Before attempting to ameliorate this impression, let me mention the second embarrassment (not unrelated to the first). Careful study of the fundamental plasma physics references on renormalization and turbulence reveals a quite pronounced historical lack of contact with the ideas and techniques of other fields, particularly fluid dynamics. (I do not wish to imply that the plasma physicists alone are to be blamed for this shortcoming.) There is, of course, some justification: insofar as the plasma is perceived as a system of weakly interacting waves and particles whereas the ordinary Navier-Stokes fluid can be thought of as a system of strongly interacting, critically damped eddies, the mathematical and physical analyses of the two systems would appear to be quite distinct. However, explicit renormalization in plasma physics is required mostly to describe non-wave-like phenomena: low frequency hydrodynamic excitations (convective cells), phase space granulation (clumps), etc.; for these effects, the physical distinction between, and mathematical analysis of, plasma and fluid blur considerably. Nevertheless, cross-fertilization between fields has been less than adequate. Perhaps the single most important approximation in nonlinear classical physics, the direct-interaction approximation (DIA), was first written for fluids in 1958, for general quadratically nonlinear systems in 1961, and specifically for Vlasov plasmas in 1967. Nevertheless, it was apparently almost completely ignored by most plasma theorists until as recently as 1976, when this author as well as others appreciated the compelling unification and generalization of many diverse theories which the DIA made possible. Presently, it would seem that the DIA may finally be receiving the long-overdue attention it deserves. Other standard techniques of fluid dynamics, critical phenomena, and other specialties of modern physics are also being examined for possible applications in plasma physics. It is to be hoped that this trend toward unification continues. Any other course would be scientifically deplorable and, in the long run, likely detrimental to the healthy development of research on controlled fusion, astrophysics, and other disciplines in which the fundamentals of plasma physics are important.

Let me return to the problem of credibility. In my opinion, although much of the early skepticism was justified, our ability to make sensible approximations has increased dramatically in recent years along with our understanding of the general structure and meaning of renormalized theories. The direct-interaction approximation, or several of its relatives, for example, furnishes a reasonable starting point for many practical applications. True, the DIA and similar closures are quite complicated and further approximations may be required. However, we have available at least the proper foundations. Whether or not simple, heuristic, and/or dimensional arguments ultimately turn out to give the proper scaling in a particular application, it will be (I assert) our new-found understanding of the systematology which will lead us to believe those scalings. In any case, only systematically derived equations enable one to compute precise numbers. Considering the very considerable amount of effort which has gone toward precise determination of linear stability thresholds, this point cannot be ignored.

Let me include here a few words about the article itself. Because the length constraint is severe, I have not attempted to develop any of the topics completely. History is given short shrift, both as manifested in technique and in attempts at practical computation. Nevertheless, historical developments have been essential to our present understanding; I ask the understanding and forgiveness of the many authors I have slighted.

The article contains little new material; a slightly larger portion may still be controversial. The article is aimed at the non-specialist. By design, the level of difficulty fluctuates substantially (though not, I believe, excessively).

The emphasis of the material is distinctly on fundamentals and philosophy rather than on applications. Thus, I specifically do not attempt to develop the relevance of the techniques for problems of controlled thermonuclear research--an area which, from the point of view of turbulence, remains poorly and inadequately explored. Furthermore, in spite of their importance, I have essentially or entirely omitted several broad specialties, including approximation techniques for strongly coupled plasma, Langmuir turbulence, saturation of parametric instabilities, most aspects of equilibrium hydrodynamics, and particle discreteness. However, references to the literature on these topics are included, and these should be accessible once the techniques discussed in the present article are mastered.

Finally, I wish to thank all those who have contributed either to my understanding of the physics or to the preparation of the manuscript. For the latter, I am particularly grateful to Rick Jensen, Bob Kleva, Mike Kotschenreuther, Harry Mynick, Philippe Similon, and Gary Smith, who diligently read an early draft of the manuscript and offered many valuable suggestions. One person deserves special note. Carl Oberman--former advisor, now coworker and friend--has been instrumental in awakening my interests in the field and in underscoring the need for, and importance of, firm foundations. He has been a constant source of inspiration, patience, and encouragement, for which I will always be grateful.

John A. Krommes

Princeton, New Jersey

August, 1979

1: RENORMALIZATION and PLASMA PHYSICS--An INTRODUCTION

Consider the problem of measuring the properties of a classical plasma. In the ideal experiment, one might insert point probes which do not disturb the plasma at all. This would allow a point-to-point mapping of some quantity Q--e.g., the local electromagnetic fields, particle densities, or temperature--as a function of time. Of course, because of the intrinsic fluctuations present in all systems, the probe trace will likely appear quite irregular, with little or no discernible coherent structure. Let now the experiment be repeated many times. If the experiment is determined by a small number of macroscopic parameters (e.g., filling pressure, wall temperature, Ohmic heating current, etc.), as is typical, and if the probe is measuring microscopic information, then the probe traces of each realization will be quite different in detail. Nevertheless, averages of the traces over the ensemble of realizations will yield smooth information, reproducible if a second ensemble of macroscopically identical experiments is performed. The mean quantity $\langle Q\rangle$ may vanish, and the two-time correlation $C(t,t') \equiv \langle\delta Q(t)\delta Q(t')\rangle$ (where $\delta Q \equiv Q-\langle Q\rangle$) may become a function of just the time difference $\tau \equiv t-t'$ (statistical stationarity). Very often, $C(\tau)$ will decay to zero. The structure and decay rate of $C(\tau)$ will be linked to linear and, in particular, intrinsically nonlinear properties of the plasma and thus will often be useful as a diagnostic. More generally, let us call ensemble averages like C "observables". The nomenclature has quantum-mechanical origins; here, it is intended to suggest the averaging whereby the generally unwanted microstructure of the system is eliminated and a few reproducible ("observable") numbers like decay rates or transport coefficients emerge. We can then say that renormalization is the science of correctly computing observables. As we will see, naive perturbation theory usually fails in this regard by predicting unphysical secularities. Renormalization as a technique thus transcends perturbation theory.

In the succeeding chapters, we shall encounter renormalization in a number of complicated but important applications: weakly turbulent Vlasov plasma, strongly turbulent Navier-Stokes-like fluids, stochastic magnetic fields, etc. Here, it is useful to consider a few simple examples in which the physical and mathematical problems facing a renormalized theory can be made explicit with a minimum of complexity. Consider first a weakly damped oscillator

$$\ddot{x} + 2\nu\,\mathrm{sgn}\,t\;\dot{x} + \omega_0^{\,2} x = 0 \;, \quad x(0) = 1,\; \dot{x}(0) = 0 \;, \tag{1}$$

where $\nu/\omega_0 \ll 1$. [A related, though more complicated, example is discussed by Martin (1973, footnote 22)]. Physically, the displacement x might represent the electric field amplitude of some plasma wave. To solve Eq. (1), it is convenient to introduce the one-sided quantities

$$\begin{aligned} x_+(t) &= H(t)\,x(t) \;, \\ x_-(t) &= H(-t)\,x(t) \end{aligned} \tag{2}$$

(where H(t) is the Heaviside function), so that

$$x(t) = x_+(t) + x_-(t) \;. \tag{3}$$

The equation for x_+ is

$$\ddot{x}_+ + 2\nu\dot{x}_+ + \omega_0^{\,2} x_+ = \delta'(t) + 2\nu\delta(t) \;, \tag{4}$$

and the solution, which may be obtained conveniently by Fourier transformation, is

$$x_+(t) = H(t)e^{-\nu t}[\cos(\Omega_0 t) + (\frac{\nu}{\Omega_0}) \sin(\Omega_0 t)] , \qquad (5)$$

$$\Omega_0 \equiv (\omega_0{}^2 - \nu^2)^{\frac{1}{2}} . \qquad (6)$$

From Eqs. (2) and (1),

$$x_-(t) = x_+(-t) , \qquad (7)$$

so we find

$$x(t) = e^{-\nu|t|}[\cos(\Omega_0 t) + (\frac{\nu}{\Omega_0}) \sin(\Omega_0 |t|)] , \qquad (8)$$

$$x_\omega = \frac{4\nu\omega_0{}^2}{(\omega^2-\omega_0{}^2) + 4\nu^2\omega^2}$$

$$= (\frac{\nu}{\Omega_0}) \left[\frac{(2\Omega_0-\omega)}{(\omega-\Omega_0)^2+\nu^2} + \frac{(2\Omega_0+\omega)}{(\omega+\Omega_0)^2+\nu^2}\right]$$

$$\simeq \pi[\delta(\omega-\Omega_0) + \delta(\omega+\Omega_0)] \qquad (9)$$

Thus, the "spectrum" consists of two lines of equal weight π and width ν, displaced from the undamped oscillator frequency by an amount

$$\delta\omega_0/\omega_0 \equiv (\Omega_0-\omega_0)/\omega_0 \sim -\nu^2/\omega_0{}^2 . \qquad (10)$$

(In normal usage, the spectrum is defined in terms of a statistical average of the oscillator intensity. This technical distinction is irrelevant in this introductory example.) The width and line shift are not readily computable from regular perturbation theory based on small ν. Indeed, the first-order perturbation theory

$$\ddot{x}_1 + \omega_0{}^2 \dot{x}_1 = 2\nu \,\mathrm{sgn}\, t\, \dot{x}_0 = 2\nu\omega_0 \sin(\omega_0|t|) \qquad (11)$$

exhibits a resonance at the unperturbed frequency ω_0 so that x_1

is secular; through first order,

$$x = (1-\nu|t|)\cos(\omega_0 t) \; . \tag{12}$$

In general, the result through n orders in perturbation theory will be just the first n+1 terms in the Taylor expansion of the solution (8) around $\nu = 0$; this does not exhibit the line shift at all and hints at the true width only through the scale on which secularities develop. We say that the nonsecular result (8) is renormalized (by the damping ν). If we persisted in attempting to generate the answer through regular perturbation theory, we would have to sum an infinite series of individually secular terms to arrive at the renormalized, nonsecular result. In physical problems, the friction coefficient ν often arises from the statistical effects of nonlinear interactions of the oscillator with a bath of many other oscillators (normal modes of a plasma, for example). A major aim of this article is to show how to deal with these nonlinear and, especially, random interactions and to arrive at a physically meaningful, renormalized result analogous to (8) without the need for such cumbersome series summations.

Consider now a more realistic experiment in which the probes do disturb the plasma. For example, sheaths may form and, we may say, physically renormalize the size, shape, and characteristics of the probe. To interpret the probe traces in this case, one would have to understand the linear and nonlinear processes leading to the renormalization, then "invert" the result. As a simple example of such a process, consider static Debye shielding. In linear theory, the potential distribution around a static test charge is

$$\phi(r) = Q\exp(-r/\lambda_D)/r \; , \tag{13}$$

which can be written in terms of a "renormalized charge" $\overline{Q}(r)$:

$$\phi(r) = \overline{Q}(r)/r \ , \quad \overline{Q}(r) \equiv Q\exp(-r/\lambda_D) \ . \qquad (14,15)$$

Experimentally, one would have available $\phi(r)$. One could determine the strength Q of the bare test charge only if he knew both its position and the proper law of Debye shielding. Furthermore, though the form of Debye shielding is well-known linearly, it is modified when nonlinear effects are included. In this case, it is more convenient to work in Fourier space. The total plasma potential to first order in the strength of the test charge is

$$\phi_k = 4\pi\overline{Q}_k/k^2 \ , \quad \overline{Q}_k \equiv Q/\varepsilon(\vec{k},0) \ , \qquad (16)$$

where $\varepsilon(\vec{k},\omega)$ is the plasma dielectric. The renormalization law for test charges then emerges from a (renormalized) theory of the low-frequency dielectric. We will learn how to define ε in the presence of turbulence, and how to deal usefully with it. We may also note that, at finite frequency, ε provides a major part of the nonlinear description of the normal modes of the plasma, and is thus related to our previous example.

Another use of renormalization occurs in the theory of low frequency, long wavelength "hydrodynamic" fluctuations of plasma, for which the fluid equations afford an adequate description. In the presence of instabilities, fluctuations can readily grow to a level at which the energetics are controlled by nonlinear processes. The description of the observable, mean rate of energy transfer between various scales of the turbulence then requires an understanding of how a particular fluid element is strained and distorted by all of the other fluid elements. Because the distortion is nonlinear, statistical, and on a time scale generally short compared to the scale for linear viscous or resistive dissipation, renormalization will be essential.

The turbulent distortion of fluid elements has an analog in the chaotic motion of phase space elements described by the full Vlasov equation. Here, the fundamental process is the stochastic instability, responsible for random wandering of single orbits and the exponentially rapid separation of pairs of orbits. Renormalization is essential to describe these intrinsically nonlinear and statistical phenomena.

The remainder of the article is organized as follows. In Sec. 2 we describe an exactly soluble model problem, the stochastic oscillator, for which many of the problems and techniques of renormalization in a statistical theory can be illustrated in considerable detail. In Secs. 3 and 4 we discuss in simple terms the use of renormalization in several applications of physical interest: test particle diffusion in Sec. 3, several hydrodynamics examples in Sec. 4. In Sec. 5 we derive the direct-interaction approximation (DIA)--in some ways, the most fundamental of all renormalizations--by appealing to the original historical arguments. The DIA provides a central focus throughout the remainder of the article. In Sec. 6--by far the most difficult technically--we discuss aspects of the systematic techniques which have been advanced to construct and justify various renormalizations, including the DIA. In Sec. 7, we discuss some problems of plasma hydrodynamics in more detail, including the physical and quantitative description of inertial-range cascades. In Sec. 8 we show how to construct the renormalized dielectric function, then sketch the salient features of several applications. Sec. 9 is devoted to the details of the equations which describe self-consistent fluctuations in, and saturated states of, Vlasov turbulence. There, we also discuss the concept of phase space granulation. In Sec. 10 we briefly mention various miscellaneous applications and techniques, including the problem of stochastic magnetic fields. We offer some concluding comments in Sec. 11. In App. A we review the theory of cumulants, which is used repeatedly throughout the article. In App. B we collect our Fourier transform conventions and some associated formulas.

2: The STOCHASTIC OSCILLATOR

2.1: Introduction

Our goal in this chapter is to discuss in reasonably complete detail a prototype equation which exhibits many of the features and difficulties of any statistical problem requiring renormalization, but which is exactly soluble. The exact solution serves as a reference to which approximate theories can be compared. The model is a variant of one first discussed by Kraichnan (1961) and Kubo (1963, and refs. therein):

$$\frac{d\psi(t)}{dt} + i\omega(t)\,\psi(t) = 0\ . \tag{17}$$

Here ω is a stationary, centered, Gaussian stochastic process functionally independent of ψ . We also prescribe a distribution of initial conditions, independent of the distribution of ω, such that ψ is stationary with zero mean and with Gaussian equal-time moments--in particular, $\langle|\psi(t)|^2\rangle = u_0^2$. (Here, as in the remainder of the article, the angular brackets denote the average over whatever statistical distribution has been specified. When no distribution is prescribed explicitly, the average can be assumed to be over an appropriate ensemble of initial conditions.)

The goal is to find a statistical solution of Eq. (17). In general, this implies determining all the many-time moments of ψ. However, we shall be concerned in particular with the two-point correlation function $C(t,t') \equiv \langle\delta\psi(t)\,\delta\psi^*(t')\rangle = C(t-t')$ and with a certain Green's function $R(t;t')$, to be defined. It is important to note that, because the random coefficient ω enters Eq. (17) multiplicatively, the statistics of ψ are not, in general, Gaussian, but rather depend nonlinearly on the statistics of ω. We say that Eq. (17) is stochastically nonlinear, thus distinguishing it from problems of dynamic nonlinearity in which the random coefficient depends functionally on ψ itself. Practical examples of dynamically nonlinear equations are those of Navier-Stokes (cf. Secs. 4 and 7) and Vlasov (cf. Secs. 8 and 9), while a practical problem involving stochastic nonlinearity is the description of particle transport in stochastic magnetic fields caused by fixed external perturbations (cf. Sec. 10.1).

In Sec. 2.2 we solve Eq. (17) exactly. However, in practical problems, the function ω is replaced, in general, by a complicated integro-differential operator and a general solution is usually not available in tractable form. In lieu of an exact solution, the obvious first approach to finding $\langle\psi\rangle$ would be to average Eq. (17) [over the independent distributions of ω and of $\psi(0)$]:

$$\frac{d}{dt}\langle\psi\rangle + i\langle\omega\psi\rangle = 0 \ . \qquad (18)$$

That $\langle\psi\rangle$ happens to be identically zero here does not vitiate the following arguments; similar procedures apply to the correlation function.) We may decompose both ω and ψ into mean fluctuating components,

$$\begin{aligned} \omega &= \langle\omega\rangle + \delta\omega \ , \\ \psi &= \langle\psi\rangle + \delta\psi \ , \end{aligned} \tag{19}$$

so that

$$\frac{d}{dt}\langle\psi\rangle + i\langle\omega\rangle\langle\psi\rangle + i\langle\delta\omega\delta\psi\rangle = 0 \ . \tag{20}$$

By assumption, $\langle\omega\rangle = 0$. However, Eq. (20) is not a closed equation for $\langle\psi\rangle$; rather, the unknown two-point mixed correlation $\langle\delta\omega\delta\psi\rangle$ has appeared. In fact, Eq. (20) is the lowest member of a hierarchy of equations, analogous to the familiar BBGKY hierarchy in many-body physics (Oberman, Chap. 2.3) in which n-point correlations are determined by (n+1)-point correlations. This is the well-known closure problem. To proceed, one seeks to approximately express some n-point function in terms of lower-order functions, thus finding a closed, coupled system for those functions. Most of renormalization theory is concerned with effecting this procedure efficiently and accurately.

In the following discussion, we shall have much use for the so-called cumulant functions (perhaps better known to workers in many-body kinetic theory as "cluster functions" or "irreducible parts"). These are defined and discussed in App. A. The reader unfamiliar with cumulants is encouraged to read App. A at this time.

2.2: Exact Solution of the Stochastic Oscillator

It will be sufficient to compute $C(\tau)$ for $\tau>0$; time reversibility implies that $C(-\tau) = C(\tau)$. To this end, define the one-sided function

$$C_+(\tau) \equiv H(\tau)\,C(\tau) \ . \tag{21}$$

Upon applying $\partial/\partial\tau$ to (21) and using Eq. (17), we find

$$\frac{d}{d\tau}C_+(\tau) + i\langle\delta\omega(\tau)\,\delta\psi(\tau)\,\delta\psi^*(0)\rangle = \delta(\tau)\,C(0)\;. \qquad (22)$$

The precise initial condition is inessential to the structure of Eq. (21). Let us remove it by defining the stochastic "infinitesimal response function" $\tilde{R}$ according to

$$\tilde{R}(t;t') \equiv \delta\psi(t)/\delta\hat{\eta}(t') \qquad (23)$$

[where $\hat{\eta}(t)$ is a non-random source term inserted on the right-hand side of Eq. (17) and $\delta/\delta\hat{\eta}$ denotes functional differentiation], and the mean response function R by

$$R(t;t') \equiv \langle\tilde{R}(t;t')\rangle\;. \qquad (24)$$

The quantity $\tilde{R}$ obeys

$$\frac{\partial\tilde{R}}{\partial\tau} + i\omega(\tau)\,\tilde{R}(\tau) = \delta(\tau)\;, \qquad (25)$$

and is thus a Green's function for Eq. (21), so that $\psi(t)$ is propagated from its value at $t = 0$ according to

$$\psi(t) = \tilde{R}(t;0)\psi(0)\;. \qquad (26)$$

Because of the assumed independence of ω from $\psi(0)$ we then find

$$\begin{aligned} C(\tau) &= \langle\tilde{R}(\tau;0)\rangle\langle\psi(0)\psi^*(0)\rangle \\ &= R(\tau)C(0) \qquad\qquad (\tau > 0)\,. \end{aligned} \qquad (27)$$

This also follows immediately from Eq. (22) upon noting that R is the Green's function for that equation.

The exact solution of Eq. (25) is (Kubo 1963)

$$\tilde{R}(\tau) = H(\tau)\exp\left[-\int_0^\tau d\tau' i\omega(\tau')\right]\;. \qquad (28)$$

Let us define the covariance matrix

$$F(t,t') \equiv \langle \delta\omega(t)\delta\omega(t')\rangle = F(t-t') \ . \tag{29}$$

Then cumulant expansion (App. A) or direct averaging over the Gaussian distribution assumed for ω leads to

$$\begin{aligned} R(\tau) &= H(\tau)\exp\left[-\tfrac{1}{2}\int_0^\tau d\tau'\int_0^\tau d\tau'' F(\tau'-\tau'')\right] \\ &= H(\tau)\exp\left[-\int_0^\tau d\bar{\tau}(\tau-\bar{\tau})F(\bar{\tau})\right] \ . \end{aligned} \tag{30}$$

For definiteness, let us take

$$F(\tau) = \beta^2 \exp(-\tau/\tau_{ac}) \ , \tag{31}$$

where $\beta \equiv \langle \delta\omega(0)^2\rangle^{\frac{1}{2}}$ and τ_{ac} (the "autocorrelation" time) are fixed numbers. The dynamical behavior of R is then parametrized by a single dimensionless parameter, the so-called Kubo number $K \equiv \beta\tau_{ac}$. If we normalize the time to τ_{ac},

$$\bar{\tau} \equiv \tau/\tau_{ac} \tag{32}$$

we have

$$R(\tau) = H(\tau)\exp[-K^2(\bar{\tau}-1+e^{-\bar{\tau}})] \ . \tag{33}$$

The short and long time limits of this expression are given in Table I. Alternatively, we may write

$$R(\tau) = H(\tau)\exp\left[-\int_0^\tau d\tau'\bar{\nu}(\bar{\tau}')\right] \ , \tag{34}$$

where

$$\bar{\nu}(\bar{\tau}) = K^2(1-e^{-\bar{\tau}}) \ . \tag{35}$$

Let us consider the behavior of R, hence C, for times longer than an autocorrelation time ($\bar{\tau} > 1$). In this limit $\bar{\nu}(\bar{\tau}) \to K^2$ = constant, and R decays exponentially :

$$R \sim \exp(-K^2\bar{\tau}) \ . \tag{36}$$

Doob's theorem (cf. Wang and Uhlenbeck 1945) then allows us to argue that ψ is approximately Gaussian in this limit. The mechanism is that the rapid fluctuations of the bath cause decorrelation on the scale τ_{ac}. Thus, if we coarse-grain the system by dividing the time axis into units $\Delta\tau$ satisfying $\tau_{ac} \ll \Delta\tau \ll \beta^{-1}$, the system at time τ will depend on the detailed physics only in the preceding interval $\Delta\tau$ and is Gaussian-Markov (cf. Wang and Uhlenbeck 1945) in the coarse-grained time. This implies that the nonlinear term can be modeled by a smooth friction $\nu(\infty) \equiv \nu$ $(\nu \equiv \bar{\nu}/\tau_{ac})$ and a white noise source $f(t)$:

$$\frac{d\psi}{dt} + \nu\psi = f(t) \qquad (t > 0). \qquad (37)$$

The intensity of f is fixed by requiring that the driven fluctuations $\langle|\psi(t)|^2\rangle$ agree with the assumed intensity $C(0)$:

$$\langle f(\bar{\tau})\, f(0)\rangle = 2u_o{}^2\nu\delta(\bar{\tau}) \; . \qquad (38)$$

In Secs. 5 and 10, we shall discuss approximate ways of finding Langevin-type representations of the form (38) without having explicit knowledge of the exact solution

Observe from Eq. (34) that one can write for <u>all</u> times

$$\frac{d}{d\tau} R(\tau) + \nu(\tau)\, R(\tau) = \delta(\tau) \; . \qquad (39)$$

Equations of the form (39), in which $\dot{R}(\tau)$ depends only on R at the same time τ, are often called Markovian. Given any nonlinear equation of the general form (17), formal techniques exist which allow one to find such Markovian representations (Misguich and Balescu 1975a). However, these forms are, in general, illusory. Eq. (39) is not coarse-grained and describes exact dynamics, including the effects of events in the past on the present; Eq. (37) is coarse-grained and contains no nonlocal effects.

For infinite autocorrelation time or Kubo number, $R \sim \exp(-\frac{1}{2}\beta^2 t^2)$; in this limit, the statistics of ψ are strongly non-Gaussian. This limit (Kraichnan 1961) affords a particularly difficult test of approximate theories, as we will see later in some detail. The point is that as $K \to \infty$ the nonlinear term of Eq. (17) dominates and standard perturbation theories fail badly. This limit is the prototype for high-Reynolds-number Navier-Stokes turbulence. In plasmas, we have occasion to study the limits of both long and short autocorrelation times. Systems with short τ_{ac} include those with wave-like fluctuations with broad spectrum (Sec. 3), whereas long τ_{ac} often occurs in the theory of hydrodynamic excitations (Secs. 4 and 7).

2.3: Approximate Closure Theories for the Stochastic Oscillator

In the limit of small K, one might attempt to solve Eq. (17) by regular perturbation theory. It is easily seen that one generates thereby the Taylor expansion of the true solution (Kraichnan 1961). Because any finite-order truncation of this expansion is secular, the procedure fails for physically interesting times unless one is able to sum the perturbation series to all orders--in general, a difficult task. The nature of the problem is illustrated by the solution in the Markovian limit $t > \tau_{ac}$. In perturbation theory, it is often convenient to proceed in Fourier space, so consider the Fourier transform of Eq. (36):

$$R_\omega \to [-i(\omega + i\nu)]^{-1} . \tag{40}$$

If one blindly treats ν as small and expands, he arrives at

$$R_\omega = R_\omega^{(o)} - R_\omega^{(o)} \nu R_\omega^{(o)} + \cdots , \tag{41}$$

where

$$R_\omega^{(o)} \equiv [-i(\omega + i0)]^{-1}. \tag{42}$$

Upon inverse Fourier transformation, Eq. (41) leads to

$$R(\tau) = \int \frac{d\omega}{2\pi} e^{-i\omega\tau} R_\omega = 1 - \nu\tau + \cdots , \tag{43}$$

the Taylor expansion of expression (36). Of course, the problem with this approach is that expression (40) is a resonance function,

$$R_\omega = \frac{i\omega + \nu}{\omega^2 + \nu^2} . \tag{44}$$

Frequencies in the heart of the resonance, $\omega < \nu$, are not large compared to ν. Since an uncertainty principle tells us that low frequencies determine the characteristic long time behavior, we would expect that long time behavior is badly represented by a finite-order truncation of the high frequency expansion (41)--in agreement with our explicit results.

A more useful approach is to make a statistical ansatz. Construct the integral form of the exact equation (17) and insert the result into Eq. (22):

$$\frac{d}{d\tau} R(\tau) + \int_0^\tau d\tau' \langle\omega(\tau)\, \omega(\tau')\, \tilde{R}(\tau')\rangle = \delta(\tau) . \tag{45}$$

The triple correlation has the cumulant expansion (App. A; Kraichnan 1961)

$$\begin{aligned} \langle\omega(\tau)\, \omega(\tau')\, \tilde{R}(\tau')\rangle &= \langle\omega(\tau)\rangle\langle\omega(\tau')\rangle\langle R(\tau')\rangle + \langle\delta\omega(\tau)\,\delta\omega(\tau')\rangle\langle\tilde{R}(\tau')\rangle \\ &\quad + \langle\delta\omega(\tau)\,\delta\tilde{R}(\tau')\rangle\langle\omega(\tau')\rangle + \langle\omega(\tau)\rangle\langle\delta\omega(\tau')\,\delta\tilde{R}(\tau')\rangle \\ &\quad + \langle\langle\omega(\tau)\,\omega(\tau')\,\tilde{R}(\tau')\rangle\rangle \\ &= F(\tau - \tau')R(\tau') + \langle\langle\omega\omega\tilde{R}\rangle\rangle . \end{aligned} \tag{46}$$

If it is assumed that the joint statistics of ω and ψ are approximately

Gaussian so that the triple cumulant can be ignored, we arrive at the so-called Bourret approximation (Bourret 1962, Van Kampen 1976)

$$\frac{d}{d\tau}R(\tau) + \int_0^\tau d\bar{\tau}\, F(\bar{\tau})\, R(\tau-\bar{\tau}) = \delta(\tau) \ . \qquad (47)$$

Employing the convolution theorem, we can solve Eq. (47) by Fourier transformation:

$$R(\tau) = \int \frac{d\omega}{2\pi} \exp(-i\omega\tau)\, [-i\omega + (F_+)_\omega]^{-1}, \qquad (48)$$

where, as usual, $F_+(\tau) \equiv H(\tau)\, F(\tau)$. For the special case (31), we find

$$R(\tau) = H(\tau) \exp(-\tfrac{1}{2}\tau/\tau_{ac}) \begin{cases} \cosh(|\omega_0|\tau) + \tfrac{1}{2}(|\omega_0|\tau_{ac})^{-1} \sinh(|\omega_0|\tau) & (K < \tfrac{1}{2}), \quad (49a) \\ \cos(\omega_0\tau) + \tfrac{1}{2}(\omega_0\tau_{ac})^{-1} \sin(\omega_0\tau) & (K > \tfrac{1}{2}), \quad (49b) \end{cases}$$

where

$$\omega_0 \equiv (\beta^2 - \tfrac{1}{4}\tau_{ac}^{-2})^{\frac{1}{2}} \ . \qquad (50)$$

For $K \ll \frac{1}{2}$, $|\omega_0| \simeq \frac{1}{2}\tau_{ac}^{-1}(1-2K^2)$ and one readily verifies that the long time limit of Eq. (49a) agrees with the correct result [Eq. (36) and Table I]. However, Eq. (49b) fails badly for $K \gg \frac{1}{2}$, where β^{-1} is the shortest scale. Here, for $\beta\tau \lesssim 1$ the correct behavior is $R \sim \exp(-\frac{1}{2}\beta^2\tau^2)$, which is incorrectly replaced in the prediction (49b) of the Gaussian hypothesis by an oscillation at frequency $\omega_0 \sim \beta$. In the limit $K \to \infty$, the approximate solution $R \sim \cos(\beta\tau)$ does not decay at all. Observe, however, that the cumulant expansion gives correctly the first two terms of the exact result, $R \sim 1 - \frac{1}{2}\beta^2\tau^2$.

To heuristically derive an improved equation for R in the limit of large K, let us introduce the zero-th order Green's function

$$R^{(o)}(\tau) = H(\tau) \tag{51}$$

[cf. Eq. (42)] and rewrite Eq. (45) in the form

$$\frac{d}{d\tau}R(\tau) + \int_0^\tau d\tau' \langle\omega(\tau)\, R^{(o)}(\tau-\tau')\, \omega(\tau')\, \tilde{R}(\tau')\rangle = \delta(\tau) \; . \tag{52}$$

The cumulant expansion (46) becomes

$$\langle\omega(\tau)\, R^{(o)}(\tau-\tau')\, \omega(\tau')\, R(\tau')\rangle = R^{(o)}(\tau-\tau')\, F(\tau-\tau')\, R(\tau') + \langle\langle\omega\omega\tilde{R}\rangle\rangle \; . \tag{53}$$

At large K we know that we cannot neglect the triple cumulant because the statistics are not Gaussian. On the other hand, the explicit appearance in (53) of an unperturbed propagator $R^{(o)}$ in a regime of strong nonlinearity is somewhat unsettling We can argue heuristically that one effect of the nonlinearities contained in $\langle\langle\omega\omega\tilde{R}\rangle\rangle$ will be to "renormalize" the response function according to $R^{(o)} \to R$, so that (53) can alternatively be written

$$\langle\omega(\tau)\, R^{(o)}(\tau-\tau')\, \omega(\tau')\, \tilde{R}(\tau')\rangle \equiv R(\tau-\tau')\, F(\tau-\tau')\, R(\tau') + C_3' \; , \tag{54}$$

thus defining some "residual" cumulant C_3' . If we ignore C_3' without further justification, we are led to the nonlinear equation

$$\frac{d}{d\tau}R(\tau) + \int_0^\tau d\bar{\tau}\, R(\bar{\tau})\, F(\bar{\tau})\, R(\tau-\bar{\tau}) = \delta(\tau) \; . \tag{55}$$

This important closure is called the <u>direct-interaction approximation</u> (DIA) [derived in the context of the stochastic oscillator by Kraichnan (1961)]. Considerably more profound and systematic derivations of the DIA can be given (cf. Secs. 5 and 6). We must stress here that for problems involving dynamic nonlinearity the DIA is more complicated, as we discuss in great detail in later sections.

Clearly, the DIA reduces to the Bourret approximation when the decay of $R(\tau)$ is slow compared to that of $F(\tau)$--i.e., for $K \ll 1$. In the opposite limit, however, they differ significantly. Kraichnan (1961) has considered the extreme case of $K = \infty$, in which case $F(\tau) = \beta^2$ and Eq. (55) can be solved exactly by Fourier transformation:

$$R(\tau) = H(\tau)\, J_1(2\beta\tau)/\beta\tau \quad . \tag{56}$$

Formula (56) is graphed in Fig. 3 of Kraichnan (1961) along with the exact solution. Unlike that of Bourret, the DIA decays on the proper scale β^{-1}, although with a spurious oscillation and with only algebraic rather than exponential envelope. However, the area under the curve,

$$A \equiv \int_0^\infty d\tau \, \exp(-\tfrac{1}{2}\beta^2\tau^2) = (\pi/2)^{\frac{1}{2}}\beta^{-1} \ , \tag{57}$$

is approximated remarkably well, to about 20%, by the DIA:

$$A_{DIA} \equiv \int_0^\infty d\tau \; J_1(2\beta\tau)/\beta\tau = \beta^{-1} \ . \tag{58}$$

This agreement is of considerable significance. In many of the applications, transport coefficients will be determined by time integrals over the response or correlation function and will be more sensitive to gross properties, like the area under the curve, than to detailed structure.

Kraichnan (1961) has shown how to derive the DIA for the stochastic oscillator by summing a certain subseries of perturbation theory. He also used the same technique to derive various higher order renormalizations. One of these was very successful and gave excellent agreement with the exact solution; the others were badly misbehaved, though they seemed superficially to be as well-motivated.

We shall return to these matters in Sec. 6, where we develop the systematology of renormalization. There, we will discuss Kraichnan's successful higher-order renormalization of the oscillator and will also attempt to provide some intuition as to why other approximations fail. First, however, we discuss in the next two sections various physical applications of heuristic renormalizations.

Additional references related to the material of this section include the papers by Leslie (1973), Frisch and Bourret (1970), and Cook (1978b).

3: TEST PARTICLE DIFFUSION in a DISCRETE SPECTRUM

3.1: Introduction

The elementary notions of test particle diffusion in a weakly turbulent plasma have been known, in the form of quasilinear theory (QLT) (Vedenov, Velikhov, and Sagdeev 1961, Drummond and Pines 1962), since almost the beginnings of theoretical plasma physics research. However, only in recent years have several of the fundamental assumptions, both physical and mathematical, been fully understood; in fact, there still remain certain outstanding questions. In this section we give an introduction to these matters (see also Galeev and Sagdeev, Chap. 4.1). The fundamental physical process underlying QLT is the stochastic instability; the fundamental mathematical justification for it involves renormalization.

For purposes of illustration we assume a stationary, homogeneous, one-dimensional spectrum of electrostatic waves, quantized in an unmagnetized plasma of length L, and described by the discrete fluctuation spectrum

$$<\delta E \delta E>_{k,\omega} = 2\pi <\delta E^2>_k \delta[\omega - \omega(k)] \quad . \tag{59}$$

It is then the fundamental hypothesis of QLT that resonant particles diffuse in velocity space according to the diffusion coefficient

$$D(v) = \pi(q/m)^2 \sum_k <\delta E^2>_k \delta[\omega(k) - k\ v] \tag{60}$$

or, in the continuum limit $L \to \infty$ (see App. B for conventions and notation),

$$D(v) = \pi(q/m)^2 \int \frac{dk}{2\pi} <\delta E^2>(k)\ \delta[\omega(k) - k\ v] \ . \tag{61}$$

Now the forms (60) and (61) are far from equivalent even in the limit of large box size. Formula (60) describes a very singular function of velocity, infinite at a wave-particle resonance and zero otherwise. Formula (61), however. is a smooth function of velocity. In fact, it is this formula which is correct even in the discrete spectrum, as we will explain. We will learn that a diffusive description is appropriate only when islands overlap in velocity space. In this case, the nonlinear mixing of stochastically unstable orbits gives rise to a broadening which renormalizes the propagators of linear theory and smooths the singularities manifest in (60) such that the integral expression (61) is the proper approximate description <u>even in a discrete spectrum</u>.

3. : Stochastic Instability and Test Particle Diffusion

The physics of stochastic instability is developed in detail elsewhere (White Chap. 3.5, Zaslavskii and Chirikov 1971, Chirikov 1969, Chirikov 1979 and refs. therein). Here, we gather the fundamental results we shall need. A test particle in a discrete spectrum is described by the Hamiltonian

$$H(x,p;t) = p^2/2m + q\sum_k \phi_k \exp i\,[kx - \omega(k)t] \ . \tag{62}$$

To examine the structure in phase space, we isolate the terms corresponding to a single wave-particle resonance:

$$H_k = p^2/2m + 2q\phi_k \cos[kx - \omega(k)t] \ . \tag{63}$$

Because H_k is not conserved, it is convenient to transform to the wave frame via the generating function (Goldstein 1950)

$$S(x,P) = (P + mv_\phi)(x - v_\phi t) \ , \tag{64}$$

where $v_\phi \equiv \omega/k$. If we redefine the zero of energy to include the kinetic energy $\frac{1}{2}mv_\phi^2$ of Galilean translation, then the transformed Hamiltonian K becomes

$$K(Q,P) = H_k + \partial S/\partial t + \tfrac{1}{2}mv_\phi^2$$

$$= P^2/2m + 2q\phi_k \cos(kQ) \ , \tag{65}$$

where

$$Q \equiv x - v_\phi t \ , \qquad P \equiv mV \ , \qquad V \equiv v - v_\phi \ . \tag{66}$$

Because K is conserved,

$$V = \pm(2/m)^{\frac{1}{2}}[K - 2q\phi_k \cos(kQ)]^{\frac{1}{2}} \ . \tag{67}$$

We see that the single resonance divides the phase space into trapping and passing regions, with the separatrix described by $K = 2q\phi_k$ or

$$V = 2(2q\phi_k/m)^{\frac{1}{2}} \sin(\tfrac{1}{2}kQ) \ . \tag{68}$$

The separatrix or island width ΔV is then

$$\Delta V = 4(2q\phi_k/m)^{\frac{1}{2}} \equiv 4v_{tr} \ , \tag{69}$$

where v_{tr} is the trapping velocity, related to the trapping frequency ω_{tr} at the center of the island by $v_{tr} = \omega_{tr}/k$.

We now return to the original Hamiltonian with many resonances. As is well-known (Zaslavskii and Filonenko 1968, Chirikov 1969, Chirikov 1979, Greene 1979), stochasticity ensues when, in order of magnitude, adjacent islands overlap--that is, when the stochasticity parameter

$$S \equiv \frac{\Delta V}{\delta V} \gtrsim 1 \ . \tag{70}$$

Here δV is the velocity spacing between adjacent resonances and is given by

$$\begin{aligned} \delta V &= \delta(\omega/k) \\ &= \delta k (v_g - v_\phi)/k, \end{aligned}$$

$$v_g \equiv \partial\omega(k)/\partial k; \qquad v_\phi \equiv \omega(k)/k \ . \tag{71}$$

In the stochastic limit, numerical evidence (Chirikov 1979 and refs. therein) verifies well that velocity space diffusion occurs. We define the velocity space diffusion coefficient by

$$D = \lim_{t\to\infty} \langle [v(t) - v_o]^2 \rangle / 2t \ , \tag{72}$$

where the average is to be taken over the wave phases and an ensemble of particles, each of initial velocity v_o but distributed uniformly in space. Then

$$D = \int_0^\infty d\tau \, C(\tau) \ , \tag{73}$$

where $C(\tau)$ is the Lagrangian (taken along the orbits) acceleration correlation

$$C(\tau) = (q/m)^2 \langle \delta E(x(\tau),\tau)\,\delta E(x(0),0) \rangle$$

$$= (q/m)^2 \sum_{k,k'} \langle \delta E_k(\tau)\,\delta E_{k'}(0) \exp i\,[kx(\tau) + k'x(0)] \rangle \ . \qquad (74)$$

In conventional QLT it is assumed that particles follow free-streaming orbits: $x(\tau) = x(0) + v\tau$. The averages over initial conditions and wave phases can then be performed separately, leading to

$$C(\tau) = (q/m)^2 \sum_k \langle \delta E^2 \rangle_k \exp i\,[kv - \omega(k)]\tau \ . \qquad (75)$$

Time integration of (75) gives rise to the singular result (60). The singularities have physical significance; they arise from quasi-recurrences in formula (75). In understanding this, it is convenient to view (75) as describing the interaction of a wave packet, moving with the group velocity v_g, with a particle of velocity v resonant with some typical phase velocity v_ϕ. If the width of the packet in k space is Δk, then the particle will lose correlation with the packet in an autocorrelation time

$$\tau_{ac} \equiv \frac{2\pi}{|v_g - v_\phi|\,\Delta k} \ . \qquad (76)$$

However, because of the periodic boundary conditions in the discrete spectrum, in a frame in which the group velocity vanishes the packet will reconstruct itself in a distance $L = 2\pi/\delta k = 2\pi N/\Delta k$, where N is the number of modes in the packet. Thus, if we ignore the possibility that the particle may hit the side of the box, the particle will suffer recurrent kicks each recurrence time

$$\tau_r \equiv N\tau_{ac} \qquad (77)$$

and the time integral of $C(\tau)$ will not exist for resonant particles.

To make this argument somewhat more quantitative, consider a dispersion law such as would be encountered in Langmuir turbulence,

$$\omega = \omega_0 + \alpha k^2 \ , \tag{78}$$

where ω_0 and α are constants. Let the particle be resonant with some typical wave of wavenumber k_0: $v = \omega_0/k_0 + \alpha k_0$. Then if $\delta k \equiv k - k_0$, the exponent of (75) becomes

$$\delta k \{ v - v_g(k_0) - \alpha\delta k \} \tau \ . \tag{79}$$

If $v - v_g = \omega_0/k_0 - \alpha k_0$ is sufficiently large, we can ignore the term $\alpha\delta k$ in Eq. (79). If we assume that the spectrum is reasonably flat over some range Δk, we have then to sum

$$\sum_{n=-\Delta n/2}^{\Delta n/2} \exp(in\delta k\Delta v\tau) = \frac{\sin(\pi\tau/\tau_{ac})}{\sin(\pi\tau/N\tau_{ac})} \ , \tag{80}$$

where $N \equiv \Delta n + 1$. For $\tau \ll N\tau_{ac}$ and $N \gg 1$, Eq. (80) can be approximated by

$$N \frac{\sin(\pi\tau/\tau_{ac})}{(\pi\tau/\tau_{ac})} \rightarrow N\delta(\tau/\tau_{ac}) \ . \tag{81}$$

This describes the initial decorrelation of particle from packet in an autocorrelation time. However, the function (80) recurs on the recurrence time (77), in agreement with our heuristic argument.

The recurrence phenomenon is related both to the discrete spectrum and the assumption of free orbits. As we approach the continuum limit $L \rightarrow \infty$, the recurrence time also approaches ∞ and we are led immediately to the form (61). However, in a discrete spectrum, the experimentally interesting case, recurrence can be prevented only by invoking nonlinear effects. In fact, we know from the theory of stochastic instability (Chirikov 1969) that the exponentially rapid divergence of adjacent orbits and consequent nonlinear mixing of orbits in phase space gives rise to irreversible, nonlinear decay of correlations on a time scale τ_K, the inverse of the Kolmogorov entropy. In Sec. 9 we estimate the rate of orbit divergence, finding thereby

$$\tau_K \sim (k^2 D)^{-1/3}, \tag{82}$$

where D is the quasilinear diffusion coefficient (61). More realistically, then, formula (75) must be modified to read

$$C(\tau) = (q/m)^2 \sum_k \langle \delta E^2 \rangle_k \exp\{i[kv - \omega(k)]\tau\}\, r(\tau/\tau_K) , \tag{83}$$

where $r(\bar{\tau})$ is a function, describing the nonlinear effects, which decays in exponential fashion and satisfies $r(0) = 1$. For purposes of estimation, we may take

$$r(\bar{\tau}) \approx e^{-\bar{\tau}} . \tag{84}$$

Let us compare the nonlinear decay time to the recurrence time. If we use the estimate

$$D \sim (q/m)^2 \langle \delta E^2 \rangle \tau_{ac} , \tag{85}$$

which follows from Eq. (61), and note Eqs. (76), (69), (70), and (71), it then follows simply that

$$\tau_r/\tau_K \sim S^{4/3} . \tag{86}$$

Since by assumption we are in the stochastic limit $S \gg 1$, the nonlinear mixing always prevents the recurrence and the diffusion coefficient is well-defined. For $S \ll 1$, the K-entropy vanishes and recurrence is not prevented. The singularities of (60) then signal the existence of trapping, not properly included in the quasilinear description.

Since the discrete nature of the spectrum manifests itself only on the recurrence scale, the existence of nonlinear mixing implies that the wavenumber summation can be changed to integration with impunity. Formally, such a procedure is justified if one increment in the sum over k produces a small fractional change in the summand. The increment in the linear theory exponent is $\delta(kv-\omega)\tau = \delta k \Delta v \tau$, which is to be compared with the nonlinear damping τ/τ_K:

$$\delta k \Delta v/(k^2 D)^{1/3} = \tau_K/\tau_r \ll 1 \ , \qquad (87)$$

so integration is justified. Since $C(\tau)$ will decay by phase mixing on the τ_{ac} scale, we can ignore the nonlinear term as long as $\tau_{ac} \ll \tau_K$. This gives an upper bound for the turbulence level or stochasticity parameter under which the quasilinear theory is valid:

$$\tau_K \gg \tau_{ac} \equiv \tau_r/N \rightarrow \tau_r/\tau_K < N \ . \qquad (88)$$

Thus, QLT is valid in the regime

$$1 < S < N^{3/4} \ .$$

Then,

$$C(\tau) \approx (q/m)^2 \int \frac{dk}{2\pi} < \delta E^2 > (k) \exp i[kv - \omega(k)]\tau \ . \qquad (89)$$

Time integration according to (83) then gives rise to (61).

3.3: Quasilinear Theory and Renormalization

By introducing the K-entropy as a description of mixing and decay of correlations, we have performed a renormalization. Indeed, if we perform the time integration of (83), we can write

$$D = (q/m)^2 \sum_k <\delta E^2>_k g^*_{k,\omega(k)} \quad , \tag{90}$$

where we have introduced a renormalized particle propagator g:

$$g_{k,\omega} \equiv [-i(\omega - kv + i\tau_K^{-1})]^{-1} \, . \tag{91}$$

One could equally well proceed from this form to justify the continuum representation (61). Now a first-principles derivation of the form (90) with (91) is exceedingly difficult. We are of the opinion that it has never been done satisfactorily; we describe a possible approach in Sec. 9 after we develop some more machinery. Here, we give for completeness two current arguments which, although non-rigorous, are physically suggestive.

Chirikov (1969) has estimated the K-entropy using his concept of renormalized resonances. He argues that when bare resonances overlap strongly, they superimpose in such a way that new, "macro-" or "renormalized" resonances form with width $(\Delta\omega)_\Sigma$ such that the renormalized resonances just touch. He then takes the trapping frequency Ω_Σ in a typical renormalized resonance as an estimate of the K-entropy.

The detailed law of superposition of bare resonances depends on the phase relations of the amplitudes of the bare resonances. We write

$$\Omega_\Sigma{}^n = \frac{(\Delta\omega)_\Sigma}{\Delta} \Omega_\phi{}^n \quad , \tag{92}$$

where Ω_ϕ is the trapping frequency in a typical bare resonance, Δ is the frequency spacing between bare resonances, and n is to be determined. The factor $(\Delta\omega)_\Sigma/\Delta$ is the number of bare resonances in a renormalized resonance. Because Ω_ϕ is proportional to the square root of the perturbation, n would equal 4 for randomly phased amplitudes. The island width $(\Delta\omega)_\Sigma$ obeys a law similar to Eq. (92):

$$(\Delta\omega)_\Sigma^{\,n} = \frac{(\Delta\omega)_\Sigma}{\Delta}(\Delta\omega)_\phi^{\,n} \quad . \tag{93}$$

Here $(\Delta\omega)_\phi$ is the island width of a bare resonance. Combining Eqs. (92) and (93), we find

$$\Omega_\Sigma = \left(\frac{(\Delta\omega)_\Sigma}{\Delta}\right)^{1/(n-1)} \Omega_\phi \quad . \tag{94}$$

For the problem of a test particle in a stochastic wave field, we can take to within numerical factors $\Omega_\phi \sim (\Delta\omega)_\phi \sim \omega_{tr}$. Also, $\Delta = \delta k(v_g - v_\phi) = (N\tau_{ac})^{-1}$. Noting that the diffusion coefficient scales as $D \sim (q/m)^2 N E_k^{\,2} \tau_{ac}$, that $\omega_{tr} \sim [(q/m)kE_k]^{\frac{1}{2}}$, and taking $n = 4$, we can substitute into Eq. (94) to find

$$\Omega_\Sigma \sim (k^2 D)^{1/3} \tag{95}$$

as Chirikov's estimate of the K-entropy for a randomly-phased spectrum. Other techniques which we shall develop later, including a direct estimate of the rate of divergence of adjacent orbits, agree with Eq. (95).

The approach to renormalization of the wave-particle resonance which is currently most popular in the literature is based on Dupree's work on orbit diffusion (Dupree 1966, Tetreault 1976). If one returns to Eq. (74) and makes the independence hypothesis that the particle motion is only weakly correlated with the Fourier amplitudes, one finds

$$C(\tau) = (q/m)^2 \sum_k \langle \delta E^2 \rangle_k \exp\{i[kv - \omega(k)]\tau\} \exp[ik\delta x(\tau)] \quad , \tag{96}$$

where $\delta x(\tau)$ is the deviation of the particle's position from its free-streaming value. Since for $\tau > \tau_{ac}$ the particle diffuses in velocity space, cumulant expansion assuming Gaussian statistics gives

$$\langle \exp[ik\delta x(\tau)]\rangle = \exp(-{}^1/_3\, k^2 D\tau^3)^{-1/3} \quad (\tau > \tau_{ac}). \qquad (97)$$

This argument introduces the diffusion time

$$\tau_d \equiv ({}^1/_3 k^2 D)^{-1/3}, \qquad (98)$$

which is manifestly of order τ_K. If we deal only with the time scale and ignore the strong cubic dependence of the decay on time, the remainder of the justification of the continuum representation proceeds as before.

The problem with these arguments lies with the independence hypothesis; Eq. (96) is not correct for all times. On physical grounds, the K-entropy τ_K^{-1}, definable in terms of the separation rate of pairs of orbits (Benettin et al. 1976), cannot correctly arise from a statistical theory of single particle orbits, which is what approximation (96) has introduced. The actual evaluation of the form (74) is quite nontrivial. Aspects of the formalism will be indicated in Secs. 6 and 9, although the solution will not be given. At present, it is believed that the approximate forms (96) and (97) are correct for times longer than τ_d, whereas the detailed time dependence of (74) for times shorter than τ_d is unknown. Inasmuch as only the nonlinear time scale is important, these details are of little practical significance for the present problem, especially when the autocorrelation time is short. However, any result which depends sensitively on the τ^3 behavior should be viewed with considerable skepticism. Also, we argue in Sec. 10.1 that an approximation analogous to Eq. (96) may be grossly inadequate for certain collisionality regimes in the problem of particle transport in stochastic magnetic fields.

A number of authors have argued that the Dupree-like renormalized theories of diffusion coefficients give negligible corrections to the simple quasilinear result (cf. Cook and Sanderson 1974). It must be noted that those authors use the integral representation in k space (in which case their comments are well-taken for $\tau_{ac} < \tau_K$); they are not addressing the important role which renormalization plays in the theory of the discrete spectrum.

Many references related to Dupree's approach can be found in Sec. 8.

4: HYDRODYNAMICS I

4.1: Introduction

In quasilinear theory, the effective Kubo number is $\tau_{ac}/\tau_K < 1$. Though non-vanishing K-entropy was essential for irreversibility, τ_K never entered into the final expression for the diffusion coefficient and the renormalization was more or less benign. In this chapter, by contrast, we introduce problems for which the nonlinearity is large, so that renormalization becomes essential. The examples are drawn from the hydrodynamics of plasma and are severe idealizations. They are, however, very useful because they demonstrate certain physical points and mathematical techniques with a minimum of labor.

Consider by way of introduction a strongly magnetized, shear-free plasma in which particles move cross-field with the $\vec{E}\times\vec{B}$ drift (Taylor and McNamara 1971). Let us estimate the perpendicular diffusion coefficient, assuming naively that it exists. We can use the random walk estimate

$$D \sim \Delta x^2/\Delta t \qquad (99)$$

if we can identify the fundamental steps Δx and Δt. To this end, suppose there arises some fluctuation in charge $\delta\rho$, uniform along the field, with characteristic perpendicular space scale Δx. Associated with $\delta\rho$ are two kinds of perpendicular velocities--a possible translational velocity of the fluctuation as a whole, not presently of interest, and an internal velocity Δv of deformation. The fluctuation will thus tend to tear itself apart--that is, send its energy to different space scales--in a time

$$\Delta t \sim \Delta x/\Delta v \ . \tag{100}$$

Now in this model particles move with the fluid, so the particle diffusion due to fluctuations of scale Δx is, from (99) and (100), of order

$$D \sim \Delta x \Delta v \ . \tag{101}$$

Since Δv is the velocity across the structure, it can be determined from the local field gradient

$$\Delta v \sim (c/B)\,\Delta x \partial \delta E/\partial x \sim (c/B)\,\delta\phi/\Delta x \ , \tag{102}$$

where $\delta\phi$ is the characteristic (rms) potential fluctuation across the structure. Amazingly, the unknown scale Δx cancels out in (101), leaving

$$\begin{aligned} D &\sim (c/B) < \delta\phi^2 >^{\frac{1}{2}} \\ &= \left(\frac{cT}{eB}\right) < \left(\frac{e\delta\phi}{T}\right)^2 >^{\frac{1}{2}}. \end{aligned} \tag{103}$$

This result obeys the Bohm scaling with B, as it must from considerations of dimensional analysis (Taylor and McNamara 1971). It is assumed that the fluctuation level $<\delta\phi^2>$ is known, either from experiment or from additional theoretical considerations.

There is a wealth of physics hidden in this simple estimate.

First of all, it is important to stress the two-dimensional assumption about the fluctuations. If fluctuations were set up with structure along the field, as would be the case if finite $k_{\parallel}$ instabilities were operative, then parallel streaming would serve as an additional rapid decorrelation mechanism and the estimate (100) of Δt would likely be incorrect. The simple estimates also afford no understanding of the mechanics of deformation, so we have no quantitative values for Δt or D itself. Furthermore, notice that the correlation time can be written $\Delta t \sim \Delta x^2/D$; that is, Δt is the time for a fluid element to "diffuse" across the scale Δx. Although standard, such a description is poor and misleading in the present context. In the usual Langevin picture, diffusion sets in only on a time much longer than the correlation time. Here, however, no such separation of scales exists. The rate and details of deformation of the fluid elements, which give rise to transport, are self-consistently determined by that same transport. The process is highly nonlinear. We can expect that the details of deformation and decorrelation will be described by nonlocal operators in space and time. Furthermore, we must allow for the possibility that the "D" which enters into Δt may differ from the actual fluid element diffusion coefficient.

4.2: Guiding Center Plasma

In later chapters we develop machinery capable of dealing systematically with the problems outlined above. Here, however, we proceed more simply and follow heuristic procedures developed by Taylor and McNamara (1971) and others (cf. Montgomery et al. 1972). We may write as usual

$$D = \int_0^\infty d\tau\, C(\tau) \ , \tag{104}$$

$$\begin{aligned} C(\tau) &\equiv (c/B)^2 \langle [\delta\vec{E}(\tau)\times\hat{n}]\cdot[\delta\vec{E}(0)\times\hat{n}]\rangle \\ &= (c/B)^2 \langle \delta\vec{E}_\perp(\tau)\cdot\delta\vec{E}_\perp(0)\rangle \\ &= (c/B)^2 \sum_{\vec{k},\vec{k}'} \langle \delta\vec{E}_{\perp\vec{k}}(\tau)\cdot\delta\vec{E}_{\perp\vec{k}'}(0) \exp i\,[\vec{k}\cdot\vec{x}(\tau) + \vec{k}'\cdot\vec{x}(0)]\rangle \ . \end{aligned} \tag{105}$$

A common approximation is the independence hypothesis (Weinstock 1976)

$$C(\tau) \simeq (c/B)^2 \sum_{\vec{k}} \langle \delta\vec{E}_\perp(\tau)\cdot\delta\vec{E}_\perp(0)\rangle_{\vec{k}} \langle \exp[i\vec{k}\cdot\delta\vec{x}(\tau)]\rangle \ . \tag{106}$$

This approximation is much harder to justify here than in the quasi-linear case, because of the tight coupling which exists between particle and fluid in the guiding center model (Dupree 1974). In fact, the most important deficiency of the present procedure is that it provides few clues to the nature or size of omitted terms. We will gain further insight into the nature of the approximation when we discuss systematic renormalization procedures in Sec. 6; however, the problem is extremely complicated and even now not satisfactorily resolved. Possibly the best a priori statement which can be made about the independence hypothesis is that it is not patently ridiculous.

We shall assume that we are given the static fluctuation spectrum $\langle \delta E^2\rangle_{\vec{k}}$. It is reasonable to assume that the turbulent fluid motion causes decay of field correlations, so we try the ansatz

$$\langle \delta\vec{E}_\perp(\tau)\cdot\delta\vec{E}_\perp(0)\rangle_{\vec{k}} \simeq \langle \delta E_\perp{}^2\rangle_{\vec{k}} \exp(-k^2\mu|\tau|) \ , \tag{107}$$

where the "turbulent viscosity" μ is to be determined. The cusped form (107) is incorrect near $\tau = 0$, but this should not be too significant. We also make the Gaussian hypothesis for the test particle:

$$\langle \exp i\,\vec{k}\cdot\delta\vec{x}(\tau)\rangle = \exp(-k^2 D|\tau|) \quad . \tag{108}$$

Then

$$D = \int_0^\infty d\tau\,(c/B)^2 \sum_{\vec{k}} \langle \delta E_\perp{}^2 \rangle_{\vec{k}} \exp[-k^2(\mu+D)\tau] \quad . \tag{109}$$

In lieu of better information, we shall take $\mu \approx D$.

Taylor and McNamara considered explicitly the case of thermal equilibrium, for which

$$\frac{\langle \delta E^2 \rangle (k)}{8\pi} = \frac{\frac{1}{2}T}{1+(k\lambda_D)^2} \quad . \tag{110}$$

If we perform the time integral in (109) and rearrange, we then arrive at an explicit result for D:

$$D = 4\pi(c/B)^2 T \int \frac{d\vec{k}}{(2\pi)^2} \, \frac{1}{2k^2[1+(k\lambda_D)^2]} \quad . \tag{111}$$

If we introduce the two-dimensional plasma parameter

$$\varepsilon_p \equiv (n\lambda_D{}^2)^{-1} = 4\pi e^2/T \tag{112}$$

(where n is the area density of particles in the plane perpendicular to B) and integrate (111) over azimuth, we find

$$D = \left(\frac{cT}{eB}\right)\left(\frac{\varepsilon_p}{4\pi}\right)^{\frac{1}{2}}\left(\int_0^\infty \frac{dk/k}{1+(k\lambda_D)^2}\right)^{\frac{1}{2}} \quad . \tag{113}$$

The integral in (113) is logarithmically divergent at long wavelengths. This indicates the breakdown of the Langevin assumption about scale separations; the transport is too nonlocal and decay of correlations too slow for a wavenumber-independent diffusion coefficient to exist. If we insert an artificial long wavelength cutoff $k_< \ll k_D$, then

$$D = \left(\frac{cT}{eB}\right)\left(\frac{\varepsilon_p}{4\pi}\right)^{\frac{1}{2}} \ln^{\frac{1}{2}}\left(\frac{[1+(k_<\lambda_D)^2]^{\frac{1}{2}}}{k_<\lambda_D}\right) . \tag{114}$$

Most authors (Taylor and McNamara 1971, Montgomery et al. 1972, Okuda and Dawson 1973) have rectified the divergence by quantizing in a box of side L (though somewhat inconsistently maintaining the integral representation) and taking $k_<$ to be the lowest allowed wavenumber: $k_< = 2\pi/L$. This, however, gives the illusory impression that there exists a constant diffusion coefficient with associated Markovian description of transport. This is not correct, as will be made clear in Sec. 7. A better procedure is to abandon the Markovian description entirely. We may argue that a given scale k^{-1} should be diffused only by shorter scales, and take $k_< = k$. The resulting wavenumber-dependent transport coefficient D(k) then determines a nonlocal transport law:

$$D\nabla^2\psi(\vec{x}) \rightarrow \nabla^2\left[\int_{-\infty}^{\infty} d\vec{\bar{x}} \int_0^{\infty} \frac{k\,dk}{2\pi} J_0(k|\vec{x}-\vec{\bar{x}}|)D(k)\right] \psi(\vec{\bar{x}}) . \tag{115}$$

We shall not attempt to make the form of D(k) more precise here (see, however, Sec. 7). In general, the transport law is nonlocal in time as well as in space and detailed closure approximations such as the DIA are required to satisfactorily determine the form of $D(k,\omega)$.

4.3: Convective Cells

The terminology "convective cell" has been employed in multiple usages by the plasma physics community to the point where its denotative value has essentially vanished. There are at least three connotations: (1) bulk, coherent fluid motions of plasma caused by ambipolar potentials or stray dc fields associated, for example, with internal multipole supports; (2) a specific low-frequency eigenmode of the two-dimensional fluid equations (see the following discussion); (3) any low-frequency, predominantly cross-field motion of magnetized plasma. Meaning (3) presently predominates in the literature, particularly in the interpretations of computer simulations; broad spectral activity around $\omega = 0$ would typically be cited as evidence for "convective cells" (Cheng and Okuda 1977, 1978). Unfortunately, the literature connected with this generalized definition is too vast to be properly discussed here. Instead, we shall briefly review some of the research on the specific convective cells of connotation (2).

The need for a renormalized theory of plasma transport was vividly emphasized by persuasive computer simulations of Dawson and Okuda (Dawson et al. 1971, Okuda and Dawson 1973). Those authors considered the diffusion of test particles in two-dimensional, shear-free magnetized plasma for various values of the magnetic field B. They clearly observed three regimes: (1) a classical regime for very small B, in which D scaled as $1/B^2$; (2) a plateau regime for larger B, in which D was independent of B; (3) a Bohm-like regime for very large B, in which D scaled as 1/B as in the guiding center models. Dawson and Okuda offered simple random walk arguments, similar to those given in Secs. 4.1 and 4.2, which satisfactorily explained the most important aspects of their observations, including the magnetic field scaling. However, those authors did not attempt to discuss the precise way in which classical kinetic theory broke down. Krommes and Oberman (1976b) discussed this question from the point of view of fluctuation theory (Oberman Chap 2.3) and succeeded in providing a kinetic equation whose solution would, at least in principle, provide a smooth transition between the classical regime (1) and the anomalous regimes (2) and (3). Although the techniques used by those authors were, in retrospect, rather primitive, at the time they represented a substantial advance in the theoretical technology available to plasma physicists.

In brief, the salient points of the argument were as follows. As is well-known, classical transport follows from a Chapman-Enskog solution of the Balescu-Lenard kinetic equation. (A review and many references can be found in Krommes 1975.) Generally, the wavenumber integral which appears is truncated to $k \gtrsim k_D$; in the opposite limit, the linearized Vlasov dielectric $\varepsilon^{(\ell)}$ which provides the shielding of the bare Coulomb force becomes large if the possibility of normal mode contributions in that regime is ignored. However, as the magnetic field is increased to $\omega_c > \omega_p$, simple estimates (Okuda and Dawson 1973) based on the fluctuation-dissipation theorem show that fluctuation energy shifts into low frequency, long wavelength modes. In particular the dielectric function which follows from analysis of the classical, linearized 2-D fluid equations predicts a shear mode, absent from linearized Vlasov theory, the dispersion relation of which is (Okuda and Dawson 1973, Krommes and Oberman 1976a,b)

$$\omega = -i\mu_c k_\perp{}^2/(1 + \omega_c{}^2/\omega_p{}^2) \ . \tag{116}$$

Here μ_c is the classical collisional shear viscosity specialized to two dimensions (Krommes 1975).

Krommes and Oberman noted that the appearance of the Vlasov dielectric resulted from the use of a perturbation theory which, in fact, breaks down in the hydrodynamic regime. That is, in the usual hierarchy approach (Oberman Chap. 2.3) the triple cumulant is neglected. The resulting solution for the pair correlation introduces the dielectric and, when inserted into the equation for the one-body distribution, gives rise to the Balescu-Lenard operator. However, for sufficiently small k and ω the triple cumulant must compete with the action of the streaming and Vlasov operators on the pair cumulant. Furthermore, a similar argument holds at every order in the hierarchy. By classifying the various terms which appeared in the hierarchal structure, Krommes and Oberman were able to argue that, for hydrodynamic fluctuations, the dominant effect of the higher cumulants was to renormalize the single particle propagator by a linearized, but also renormalized, Balescu-Lenard operator. The resulting self-consistent equation for the propagator--called the self-consistent field approximation by Krommes and Oberman--was, in fact, the DIA for the special case of thermal equilibrium which was being studied.

The renormalized kinetic equation correctly predicted classical transport in the regime of weak magnetic fields. In the strong-field regime, Krommes and Oberman used projection operator techniques (cf. Sec. 10.2) and eigenfunction expansions to derive a nonlinear equation for the viscosity, of the form

$$\mu(k,\omega) = \frac{1}{16} r^2 D_p^{\,2} \int d\vec{p}\, d\vec{q}\, \frac{M(\hat{k},\hat{q})\,\delta(\vec{k}-\vec{p}-\vec{q})}{-i\omega + r[\mu(p,\omega)p^2 + \mu(q,\omega)q^2]}\,, \qquad (117a)$$

where

$$r \equiv (1 + \omega_c^{\,2}/\omega_p^{\,2})^{-1}\,, \qquad (117b)$$

$$D_p \equiv (v_t^{\,2}/2\pi n)^{1/2}\,, \qquad (117c)$$

and $M(\hat{k},\hat{q})$ is a certain dimensionless function of its arguments.

Equations of the form (117) also follow from detailed closures like the DIA applied directly to the nonlinear fluid equations. The solution of Eq. (117) for μ exhibits a smooth transition between the plateau and guiding center regimes, is weakly nonlocal in both k and ω, and seems to agree well in magnitude with the experimental measurements.

Further problems in hydrodynamics are discussed in Sec. 7 after we develop more technical machinery in Secs. 5 and 6.

5: The DIRECT-INTERACTION APPROXIMATION

5.1: Introduction

In Sec. 2 we introduced the direct-interaction approximation by means of rather heuristic arguments. However, the foundations of the approximation are much more compelling than that introductory discussion may have suggested. Our deepest understanding of the systematology is described in the next chapter. Here, we briefly review some of the original arguments put forth by the fluid dynamicists. Each reveals a subtly different aspect of the DIA.

In the discussions which follow, we shall consider for explicitness and simplicity a model dynamical system which is a prototype for both the Navier-Stokes and the Vlasov equations, but which dispenses with a number of irrelevant details of those systems. The model (cf. Leith 1971) is a quadratically nonlinear coupled system of N real variables $u_\alpha(t)$ which evolve according to

$$\left(\frac{d}{dt} + \nu_\alpha\right)u_\alpha(t) = \frac{1}{2}\sum_{\substack{\alpha+\beta\\+\gamma=0}} M_{\alpha\beta\gamma}u_\beta(t)u_\gamma(t) + f_\alpha(t). \tag{118}$$

The indices can be thought of as generalized wavenumbers; they may also stand for vector indices, velocities, etc. The real constants ν_α represent viscous damping, modal instabilities, or, in general, linear dynamics. The functions $f_\alpha(t)$ allow for the possibility of external forcing, as in pipe flow; for conservative systems $(\nu = 0)$, all f's would vanish. We may assume that each mode-coupling coefficient $M_{\alpha\beta\gamma}$ is symmetric in its last two indices, and vanishes unless the sum of its indices vanishes. Additional assumptions which we impose to gain correspondence to real flows are that M vanishes if any two of its arguments are equal or if any one vanishes, and

$$M_{\alpha\beta\gamma} + M_{\gamma\alpha\beta} + M_{\beta\gamma\alpha} = 0. \tag{119}$$

Finally, we assume that the "energy",

$$\varepsilon \equiv \tfrac{1}{2}\sum_\alpha u_\alpha^{\,2} \ , \tag{120}$$

is finite. Because of Eq. (119), it can readily be verified that ε is conserved by the nonlinear interaction.

5.2: Kraichnan's Original Arguments

As usual, we desire a statistical solution to Eq. (118), where statistics may enter either through initial conditions $u_\alpha(0)$ or through the external driving force f. Inasmuch as the statistics of u(0) or f(t) may be arbitrarily complicated, the statistical solution of Eq. (118) will be extremely complex and little can be said in general. However, in his pioneering work on the DIA, Kraichnan (1958a,1959) introduced two fundamental assumptions, "maximal randomness" and "weak dependence", which sufficiently limit the class of solutions so that useful analysis can be done.

Maximal randomness assumes that "the statistical dependence among the [u_α's] is induced wholly by the non-linear terms in [Eq. (118)] and not at all by the initial conditions or by the external forces which may be acting. ... The essential qualitative content of [the weak dependence] principle is that the effective dynamical coupling and statistical interdependence among any few individual ... amplitudes corresponding to different [α's] is very weak when [N] is very large." We do not attempt here to capture the full breadth of Kraichnan's original very expressive and complete discussion; we urge the serious student to read the original reference (Kraichnan 1959).

Two illustrations of weak dependence, which we adapt from Kraichnan (1959), are instructive. If α, β, and γ are distinct, then weak dependence states that

$$\frac{\langle u_\alpha u_\beta u_\gamma \rangle}{\langle u_\alpha{}^2\rangle^{\frac{1}{2}}\langle u_\beta{}^2\rangle^{\frac{1}{2}}\langle u_\gamma{}^2\rangle^{\frac{1}{2}}} \to 0 \qquad (N \to \infty), \tag{121a}$$

$$\langle u_\alpha{}^2 u_\beta{}^2\rangle \to \langle u_\alpha{}^2\rangle\langle u_\beta{}^2\rangle \quad (N \to \infty). \tag{121b}$$

The limits are those of a Gaussian distribution for the u's. However, the limits must be carefully distinguished from equalities. It can readily be shown that true equality in Eqs. (121) for all α's is incompatible with Eq. (118). (Physically, if Eq. (118) were the Navier-Stokes equation, it would be said that identically vanishing triple correlations give rise to no mean nonlinear energy transfer, which is ridiculous.) However, the right-hand sides of Eqs. (121) are compatible with the left-hand sides vanishing as some inverse power of N. For example, let us consider the "physical" (summed over all modes) quantity

$$s \equiv \sum_{\substack{\alpha+\beta\\+\gamma=0}} \langle v_\alpha u_\beta u_\gamma \rangle ,$$

which is related, via a normalization of order unity in N, to the so-called skewness factor of u. If we note that, since ε is finite, $\langle u_\alpha{}^2 \rangle \sim N^{-1}$ [see Eq. (119)], we can easily determine that s remains finite if the left-hand side of Eq. (121a) vanishes as $N^{-\frac{1}{2}}$.

Let us write $u_\alpha(t) \equiv u_\alpha$, $u_\alpha(t') \equiv u_\alpha'$, etc., and consider the evolution equation for the covariance $U_\alpha(t,t') \equiv \langle u_\alpha u_\alpha' \rangle$:

$$\left(\frac{d}{dt} + \nu_\alpha\right) U_\alpha(t,t') = \tfrac{1}{2}\sum_{\beta,\gamma} M_{\alpha\beta\gamma} \langle u_\alpha u_\beta u_\gamma' \rangle + \langle f_\alpha u_\alpha' \rangle \ . \quad (122)$$

According to maximal randomness and weak dependence, in the limit $N \to \infty$, the contribution of any <u>finite</u> number of modes to the sum in Eq. (122) must become infinitesimal (though not negligible). Kraichnan argues that the residual very weak phase correlation between modes α, β, and γ arises through the <u>direct interaction</u> of u_α, u_β, and u_γ--that is, through the term $M_{\alpha\beta\gamma} u_\beta u_\gamma$ and its cyclic permutations. To quantify the effects of the direct interaction, let us follow Kraichnan and define Δu_α as the difference between the exact solution u_α of Eq. (118) and the value $\bar{u}_\alpha$ which that solution would take if the direct-interaction term $M_{\alpha\beta\gamma} u_\beta u_\gamma$ were deleted from the right-hand side of Eq. (122)--that is,

$$\left(\frac{d}{dt} + \nu_\alpha\right) \bar{u}_\alpha = \tfrac{1}{2}\sum_{\bar{\beta},\bar{\gamma}} M_{\alpha\bar{\beta}\bar{\gamma}} \bar{u}_{\bar{\beta}} \bar{u}_{\bar{\gamma}} - M_{\alpha\beta\gamma} \bar{u}_\beta \bar{u}_\gamma . \quad (123)$$

(More precisely, we should write something like $\Delta u_{\alpha|\beta\gamma}$. However, the explicit triad under consideration will be clear from the context.) According to the principle of weak dependence, Δu_α is infinitesimal. Upon writing $\bar{u}_\alpha \equiv u_\alpha - \Delta u_\alpha$ and using Eq. (118), we find to lowest order

$$\left(\frac{d}{dt} + \nu_\alpha\right)\Delta u_\alpha - \sum_{\bar\beta,\bar\gamma} M_{\alpha\bar\beta\bar\gamma} u_{\bar\beta}\Delta u_{\bar\gamma} = M_{\alpha\beta\gamma}u_\beta u_\gamma \quad . \tag{124}$$

Equation (124) can be solved in terms of the exact infinitesimal response function $\tilde{R}_\alpha(t;t')$ which describes the change in $u_\alpha(t)$ which results from an impulsive perturbation in mode α at time t' and thus obeys

$$\left(\frac{d}{dt} + \nu_\alpha\right)\tilde{R}_\alpha(t;t') - \sum_{\beta,\gamma} M_{\alpha\beta\gamma}u_\beta(t)\tilde{R}_\gamma(t;t') = \delta(t-t') \quad . \tag{125}$$

From Eq. (124), then,

$$\Delta u_\alpha = \int_{-\infty}^{t} dt''\tilde{R}_\alpha(t;t'')M_{\alpha\beta\gamma}u_\beta''u_\gamma'' \quad . \tag{126}$$

We may insert this result into Eq. (122), noting that, to lowest order, the contribution of the direct interaction to the triple correlation is

$$\langle u_\beta u_\gamma u_\alpha'\rangle = \langle\Delta u_\beta u_\gamma u_\alpha'\rangle + \langle u_\beta \Delta u_\gamma u_\alpha'\rangle + \langle u_\beta u_\gamma \Delta u_\alpha'\rangle \quad .$$

When performing the ensemble average, we can use the principle of weak dependence to write, for example,

$$\langle\tilde{R}_\beta(t;t'')u_\gamma''u_\alpha''u_\gamma u_\alpha'\rangle \simeq R_\beta(t;t'')U_\gamma(t,t'')U_\alpha(t'',t') \quad , \tag{127}$$

where $R \equiv \langle\tilde{R}\rangle$ is the mean response function. The term of Eq. (122) involving f_α can be similarly evaluated, and the final equation for the covariance becomes

$$\begin{aligned}\left(\frac{d}{dt} + \nu_\alpha\right)U_\alpha(t,t') = &\sum_{\beta,\gamma} M_{\alpha\beta\gamma}M_{\beta\gamma\alpha}\int_{-\infty}^{t} dt''R_\beta(t;t'')U_\gamma(t,t'')U_\alpha(t'',t') \\ &+ \tfrac{1}{2}\sum_{\beta,\gamma} M^2_{\alpha\beta\gamma}\int_{-\infty}^{t'} dt''R_\alpha(t';t'')U_\beta(t,t'')U_\gamma(t,t'') \\ &+ \int_{-\infty}^{t'} dt''R_\alpha(t';t'')F_\alpha(t;t''),\end{aligned} \tag{128}$$

where we used the symmetry of M to combine two terms and where

$$F_\alpha(t;t'') \equiv \langle f_\alpha(t) f_\alpha(t'')\rangle \ . \tag{129}$$

Because of Eq. (119), the nonlinear terms of Eq. (128) conserve the total energy $U_\alpha(t,t)$--an obvious consistency condition which any sensible closure should respect.

To find an independent equation for the as-yet unknown mean response function, we turn to the exact equation for the response δu_α to an infinitesimal perturbation $\delta\hat{\eta}\delta(t-t')$:

$$\left(\frac{d}{dt} + \nu_\alpha\right)\langle\delta u_\alpha\rangle = \sum_{\beta,\gamma} M_{\alpha\beta\gamma}\langle u_\beta \delta u_\gamma\rangle = \delta\hat{\eta}\delta(t-t') \ . \tag{130}$$

Since the perturbation was introduced into mode α, weak dependence implies that δu_β and δu_γ must be infinitesimal compared to δu_α . Correct to first order in $\delta\hat{\eta}$, the equation for δu_γ becomes

$$\left(\frac{d}{dt} + \nu_\gamma\right)\delta u_\gamma = \sum_{\bar{\alpha},\bar{\beta}}{}' M_{\gamma\bar{\alpha}\bar{\beta}} u_{\bar{\beta}} \delta u_{\bar{\alpha}} + M_{\gamma\alpha\beta}(u_\beta \delta u_\alpha + u_\alpha \delta u_\beta) \ , \tag{131}$$

where the prime indicates that we have separated out the last two explicit terms in Eq. (131); of these, the term in δu_β can be neglected by weak dependence. The solution to Eq. (131) is, in obvious notation,

$$\delta u_\gamma = \int_{t'}^{t} dt'' \tilde{R}'_\gamma(t;t'') M_{\gamma\alpha\beta} u_\beta{}'' \delta u_\alpha{}'' \ . \tag{132}$$

We again use weak independence to factor the ensemble average required in Eq. (130). In doing so, we may replace $\langle\tilde{R}'\rangle$ by R because the explicit terms deleted from the sum in Eq. (131) contribute negligibly in the limit $N \to \infty$. The final equation for $R_\alpha(t;t') \equiv \langle\delta u_\alpha(t)/\delta\hat{\eta}\rangle$ becomes

$$\left(\frac{d}{dt} + \nu_\alpha\right) R_\alpha(t;t') - \sum_{\beta,\gamma} M_{\alpha\beta\gamma} M_{\gamma\beta\alpha} \int_{t'}^{t} dt'' R_\gamma(t;t'') U_\beta(t;t'') R_\alpha(t'';t') = \delta(t-t') \ . \qquad (133)$$

Equations (128) and (133) form a coupled and complete, albeit complicated, system which determines R and U.

At this point, it useful to ask why the DIA is not essentially exact, since it was derived by neglecting terms of high order in infinitesimal amplitudes. The point, of course, is that there are very many of such terms, so that their sum can give a finite contribution as $N \to \infty$ even though each term is individually small. The size of these contributions (from the indirect interactions) can be quantified in terms of the skewness parameter introduced in Sec. 6. However, in the present context the reasonableness of the DIA must be judged on other grounds. In fact, even the qualitative behavior of the solutions for R and U is not obvious from inspection of Eqs. (128) and (133). It is not clear, for example, that $U_\alpha(t,t)$ remains positive, as it must. Such a requirement is called a realizability constraint. It turns out that such constraints are notoriously hard to prove directly from the statistical equations. However, Kraichnan (1958b, 1961) noted that if an explicit model dynamical system could be found which the DIA described exactly, many of the constraints would of necessity be satisfied simply because the model system must have a statistical solution. Several such "stochastic model" systems have been discovered; we describe these in the next two sub-sections.

5.3: Generalized Langevin Representation

To motivate our discussion, let us write Eqs. (133) and (128) in the form

$$\left(\frac{d}{dt} + \nu_\alpha\right)R_\alpha(t;t') + \int_{t'}^{t} dt''\,\Sigma_\alpha(t,t'')R_\alpha(t'';t') = \delta(t-t') \ , \quad (134)$$

$$\left(\frac{d}{dt} + \nu_\alpha\right)U_\alpha(t,t') + \int_{-\infty}^{t} dt''\,\Sigma_\alpha(t;t'')U_\alpha(t'',t')$$
$$= \int_{-\infty}^{t'} dt''R_\alpha(t';t'')[F_\alpha(t,t'') + \tilde{F}_\alpha(t,t'')] \ , \quad (135)$$

where

$$\Sigma_\alpha(t,t'') \equiv -\sum_{\beta,\gamma} M_{\alpha\beta\gamma}M_{\gamma\beta\alpha}R_\gamma(t;t'')U_\beta(t,t'') \ , \quad (136)$$

$$\tilde{F}_\alpha(t,t'') \equiv \tfrac{1}{2}\sum_{\beta,\gamma} M^2_{\alpha\beta\gamma}U_\beta(t,t'')U_\gamma(t,t'') \ . \quad (137)$$

Clearly the term Σ_α in Eq. (134) describes the nonlinear relaxation of the response in mode α due to coupling to all other modes. The same term appears in Eq. (135) and describes a nonlinear energy drain on mode α. This effect is balanced by the stirring (energy input) due to f_α and described by F_α, as well as by the source term $\tilde{F}_\alpha$, which represents nonlinear stirring due to mode-mode interactions.

We shall seek a generalized Langevin equation which contains the physics described by Eqs. (134)-(137). A generalized Langevin equation (cf. Sec. 10.2) is an apparently linear, forced dynamical equation of the form (Mori 1965a, Krommes 1975)

$$\left(\frac{d}{dt} + \nu_\alpha\right)u_\alpha + \int_{-\infty}^{t} dt'\,\mathfrak{N}_\alpha(t,t')u_\alpha(t') = f_\alpha^{\dagger}(t) \ , \quad (138)$$

where $\mathfrak{N}_\alpha$ and $f^{\dagger}$ are chosen to properly represent the nonlinear effects of all the other modes on mode α, and such that its statistical solution agrees with the exact or approximate statistics of the original system. To obtain agreement with the left-hand side of Eq. (135), we must clearly choose (Kraichnan 1970a, Leith 1971)

$$\mathcal{G}_\alpha(t,t') = \Sigma_\alpha(t,t') . \tag{139}$$

The Langevin representation is completed by writing

$$f_\alpha^\dagger(t) = f_\alpha(t) + \tilde{f}_\alpha(t) , \tag{140}$$

where

$$\tilde{f}_\alpha(t) = \frac{1}{\sqrt{2}} \sum_{\beta,\gamma} M_{\alpha\beta\gamma} \xi_\beta(t) \xi_\gamma^\dagger(t) . \tag{141}$$

The random fields ξ and $\xi^\dagger$ are chosen to be statistically independent of each other, the initial velocity field $u_\alpha(0)$, and f_α, and to have the same covariance as the true velocity (Kraichnan 1970a):

$$\langle \xi_\alpha(t) \xi_\alpha(t') \rangle = \langle \xi_\alpha^\dagger(t) \xi_\alpha^\dagger(t') \rangle = U_\alpha(t,t') . \tag{142}$$

It can be readily verified that

$$\langle \tilde{f}_\alpha(t) \tilde{f}_\alpha(t') \rangle = \tilde{F}_\alpha(t,t') \tag{143}$$

and that the solution of Eq. (138) agrees with Eqs. (134) and (135).

5.4: Random Coupling Models

Historically, the first stochastic model representation of the DIA was the so-called random coupling model. In his original formulation, Kraichnan (1958b) modified Eq. (118) to read

$$\left(\frac{d}{dt} + \nu_\alpha\right) u_\alpha(t) = \tfrac{1}{2} \sum_{\substack{\alpha+\beta \\ +\gamma=0}} \phi_{\alpha\beta\gamma} M_{\alpha\beta\gamma} u_\beta(t) u_\gamma(t) + f_\alpha(t) . \tag{144}$$

Here the new factor $\phi_{\alpha\beta\gamma}$ is fully symmetric in α, β, and γ and is invariant to $\alpha \to -\alpha$, etc., but otherwise assumes the value ± 1 at random for each triad (α, β, γ). Thus, in the model the fundamental triadic nonlinear couplings are randomly phased, as opposed to those of the original system, which are coherently phased. When the perturbation series for the covariance and response functions of the model are formed, it can be shown that the additional randomness causes all terms except those of the DIA to vanish as $N \to \infty$. In later work, Kraichnan (1961) showed that the effect of $\phi_{\alpha\beta\gamma}$ can be understood in terms of fictitious, randomly phased couplings between an infinite number of copies of the true system (118).

Because the triad couplings of the model are randomly phased, it follows that the DIA cannot take into account individual flow or phase-space structures like solitons (which, by definition, can be described only by coherent mode-mode interactions). For the same reason, the DIA does not correctly describe phase space trapping or the convection of small-scale fluctuations by large-scale ones (Kraichnan 1958a,b; Sec. 7.3). The random-coupling representation also lends support to the description of the DIA as "the most Gaussian approximation consistent with nonlinearity."

For the application to Vlasov plasma, see Orszag and Kraichnan (1967).

5.5: Series Reversion

Very recently, Kraichnan (1978) has discussed yet another way of deriving the DIA equations as well as various more sophisticated approximations. The idea is based on the well-known technique of series reversion, applied here to functions rather than numbers. We illustrate with the description of response, for which the method involves expanding the unperturbed (linear) response function $R_\alpha^{(0)}(t;t')$ in powers of the renormalized function $R_\alpha(t;t')$.

One begins with the exact equation for the stochastic response function $\tilde{R}_\alpha$ [see Eq. (130)]:

$$\left[R_\alpha^{(o)}\right]^{-1}\tilde{R}_\alpha(t;t') - \sum_{\beta,\gamma} M_{\alpha\beta\gamma}u_\beta(t)\tilde{R}_\gamma(t;t') = \delta(t-t') \ . \qquad (145)$$

If we treat the .near term as formally small (in general, it is large) and assume for the moment that u_α is given, $\tilde{R}_\alpha$ can be expanded as a functional of $R_\alpha^{(o)}$ and u_α:

$$\tilde{R}_\alpha(t;t') = R_\alpha^{(o)}(t;t') + \int_{t'}^{t} d\bar{t}\, R_\alpha^{(o)}(t;\bar{t}) \sum_{\beta,\gamma} M_{\alpha\beta\gamma}u_\beta(\bar{t}) \times\left[R_\gamma^{(o)}(\bar{t};t') + \int_{t'}^{\bar{t}} d\tilde{t}\, R_\gamma^{(o)}(\bar{t};\tilde{t}) \times \sum_{\beta',\gamma'} M_{\gamma\beta'\gamma'}u_{\beta'}(\tilde{t})R_\gamma^{(o)}(\tilde{t};t') + \cdots\right]. \qquad (146)$$

Upon averaging Eq. (146) and assuming $\langle u\rangle = 0$, we find a functional series representation for R in terms of $R^{(o)}$ and the covariance U:

$$R_\alpha(t;t') = R_\alpha^{(o)}(t;t') + \int_{t'}^{t} d\bar{t}\, R_\alpha^{(o)}(t;\bar{t}) \sum_{\beta,\gamma} M_{\alpha\beta\gamma}M_{\gamma,-\beta,\alpha} \times\int_{t'}^{\bar{t}} d\tilde{t}\, R_\gamma^{(o)}(\bar{t};\tilde{t})U_\beta(\bar{t},\tilde{t})R_\alpha^{(o)}(\tilde{t};t') + \cdots \ . \qquad (147)$$

We may now revert this series to express $R_\alpha^{(o)}$ in terms of R_α:

$$R_\alpha^{(o)}(t;t') = R_\alpha(t;t') - \iint RMMRUR + \cdots \ . \qquad (148)$$

Finally, we insert Eq. (148) into each term of Eq. (146), collect terms with like powers of M, substitute the result into Eq. (145), and average. To lowest order, the procedure is trivial and gives rise to precisely the DIA equation (134). The reversion for the covariance is also straightforward, though more complicated, and to lowest order leads to Eq. (135).

As presented here, the reversion is little more than an efficient algorithm for generating the DIA and, perhaps, higher order approximations. (The latter may be badly behaved unless a further (vertex) renormalization is performed.) The physical basis for a given truncation of the reverted series must be sought elsewhere. (See the next section for more information in this respect.) Perhaps its greatest success has been to provide a systematic algorithm for generating the so-called Lagrangian-history schemes, originally proposed by Kraichnan (1965) on the basis of more or less heuristic modifications of the DIA. We shall describe the basic technique only briefly, as little has been done in applying the Lagrangian schemes to plasmas.

A problem with the DIA as applied to the Navier-Stokes fluid is its lack of so-called random Galilean invariance, which results in an incorrect inertial-range energy spectrum (see Sec. 7.3). Kraichnan traced the problem to the use of Eulerian functions in the statistical description and suggested that an improved treatment would result by considering the extended function $\vec{u}(\vec{x},t|s)$, which is the velocity at (the measuring) time s of a fluid element which passed through point $\vec{x}$ at (the labeling) time t. The quantity $\vec{u}(\vec{x},t|t)$ is the usual Eulerian velocity, while $\vec{u}(\vec{x},0|s)$ is the conventional Lagrangian velocity. Associated with $\vec{u}$ is a generalized response function $R(\vec{x},t|s;\vec{x}',t'|s')$. Now Kraichnan (1978) noted that the zeroth order generalized response function $R^{(o)}$ is in fact independent of the measuring times:

$$R^{(o)}(\vec{x},t|s;\vec{x}',t'|s') = R^{(o)}(\vec{x},t|t;\vec{x}',t'|t') \; .$$

This means that in the expansion of $R(\vec{x},t|s;\vec{x}',t'|s')$ corresponding to Eq. (146) the measuring times of each $R^{(o)}$ which appear can be changed at will without affecting the value of the series. However, the procedure of reversion followed by truncation at finite order is sensitive to the way the measuring times are altered. That is, the form of the reverted series is not unique. Kraichnan used this freedom to show how two reasonable modifications of the measuring times gave rise to the so-called Lagrangian-history DIA and the so-called abridged Lagrangian-history DIA (Kraichnan 1965) as the first terms in certain systematic expansions which are random Galilean invariant at each order--a substantial improvement over the Eulerian renormalizations. We refer the reader to Kraichnan (1978) for a considerably more detailed discussion.

Virtually nothing has been done in applying Lagrangian closures to plasmas. We suggest that this may be a fruitful area for further research, although we caution (see also Sec. 7) that random Galilean invariance seems to be less troublesome for the typical classes of laboratory plasma turbulence than for strongly turbulent Navier-Stokes flows.

6: SYSTEMATOLOGY of RENORMALIZATION

6.1: Introduction

In the preceding chapters we learned about the following techniques for renormalization:

(1) summation of perturbation theory to all orders;

(2) cumulant expansion and truncation;

(3) reversion of perturbation series;

(4) equivalent Langevin representations;

(5) random coupling models.

Methods (1)-(3) can in principle furnish an exact representation of the solution (when it is analytic; see Kraichnan (1966) for a counterexample); however, in practice the expansions are sufficiently complex and opaque that it is feasible to extract only a low order renormalization like the DIA. Methods (4) and (5) suffer the added disadvantage that no systematology is immediately apparent whereby one can improve over the lowest order result. (However, Kraichnan (1970a) has reported some work in this direction.) Thus, the nature of the renormalized approximations remains somewhat mystical--in the DIA, for example, what physics has really been neglected? In some sense, what is required is a useful representation of the exact answer--a generalized integral representation, for example--from which, by a systematic scheme, perturbative or otherwise, useful approximations emerge. We must stress at the outset that no completely satisfactory formalism has yet been proposed. Nevertheless, substantial progress has been made. In this chapter we describe the important work of Martin, Siggia, and Rose (1973), which provided a quantum leap forward in our understanding of the systematology. (Since the procedure to be described extends to the classical domain well-known techniques of quantum field theory (DeDominicis and Martin 1964), we might also say that MSR provided a quantum leap backwards.) In the limited space available to us, we will be unable to do complete justice to either the philosophy or mathematical techniques underlying the MSR work. We urge the serious reader to study the original references (Martin et al. 1973, Rose 1974; also Phythian 1975, 1976), which are very good and contain a wealth of information. We shall follow these references closely (but not exclusively).

Consider a random function ψ parametrized by a set of phase space coordinates denoted collectively by the symbol "1". The set "1" may include continuous labels like the time t_1, the position $\vec{x}_1$, and the velocity $\vec{v}_1$, as well as discrete indices like a species label s_1, vector components, etc. The notation "$\underset{\sim}{1}$" will denote the set "1" excluding the time t_1. We will adopt an integration-summation convention over repeated arguments. Let ψ obey an equation of the form

$$\partial_{t_1} \psi(1) = U(1) + U(1,2)\psi(2) + \tfrac{1}{2}\bar{U}(1,2,3)\psi(2)\psi(3) \;, \quad (149)$$

where the coupling coefficients or "bare vertices" $U_n \equiv U(1,2,\ldots,n)$ are known, non-random functions local in time [e.g., $\bar{U}(1,2,3) \propto \delta(t_1 - t_2)\delta(t_1 - t_3)$]. It is assumed that Gaussian initial conditions are imposed at $t = t_0$; the goal is to determine the statistical behavior of ψ for $t > t_0$. An important example of Eq. (149) is the Vlasov-Poisson system

$$\partial_t f + \vec{v}\cdot\vec{\nabla} f + \vec{E}\cdot\vec{\partial} f = 0, \quad (150a)$$

$$\vec{E} = -\vec{\nabla}\phi \;,\; \nabla^2\phi = 4\pi\sum_s nq\int d\vec{v}\, f \;, \quad (150b)$$

for which we can identify

$$\psi(1) = f(1) \ , \tag{151a}$$

$$U(1) = 0 \ , \tag{151b}$$

$$U(1,2) = -\vec{v}_1 \cdot \vec{\nabla}_1 \delta(1-2) \ , \tag{151c}$$

$$\overline{U}(1,2,3) = [\vec{E}(1,2) \cdot \vec{\partial}_1 \delta(1-3) + (2 \leftrightarrow 3)] \ , \tag{151d}$$

$$\vec{E}(1,2) \equiv -\vec{\nabla}_1 |\vec{x}_1 - \vec{x}_2|^{-1} (nq)_2 \delta(t_1 - t_2) \ . \tag{151e}$$

The techniques we will develop can be extended to the important practical cases of random U's (including the case of a random driving force U_1) (Deker and Haake 1975a, Phythian 1976, Deker 1979), polynomial nonlinearities of arbitrary finite order (cf. Deker and Haake 1975b), and non-Gaussian initial conditions (Deker 1979, Rose 1979).

Let us sketch the problems one encounters in developing a statistical theory of Eq. (149), and indicate the philosophy of the renormalization technique we shall use to deal with those problems. [Here we expand slightly on the discussion in Martin et al. (1973).] Let us decompose ψ into mean and fluctuating parts and use the cumulant notation of App. A. For purposes of dimensional analysis, let us assume that at $t = 0$, $\langle\langle\psi^2\rangle\rangle = 1$, $\langle\langle\psi^3\rangle\rangle = 0$, corresponding to an initially Gaussian state. Then for times $t \geq 0$, the first several members of the cumulant hierarchy for Eq. (149) are schematically

$$[\partial_t - (U_2 + \tfrac{1}{2}\overline{U}_3 \langle\psi\rangle)]\langle\psi\rangle - \tfrac{1}{2}\overline{U}_3 \langle\langle\psi^2\rangle\rangle = \langle\psi(0)\rangle\delta(t) + U_1 \ , \tag{152a}$$

$$(\partial_t - U_2{}^{(\ell)})\langle\langle\psi^2\rangle\rangle - \tfrac{1}{2}\overline{U}_3 \langle\langle\psi^3\rangle\rangle = \delta(t) \ , \tag{152b}$$

$$(\partial_t - U_2{}^{(\ell)})\langle\langle\psi^3\rangle\rangle - \overline{U}_3 \langle\langle\psi^2\rangle\rangle^2 - \tfrac{1}{2}\overline{U}_3 \langle\langle\psi^4\rangle\rangle = 0 \ , \tag{152c}$$

where we have introduced the linearized mean field operator

$$U_2^{(\ell)} \equiv U_2 + \bar{U}_3 \langle\psi\rangle \ . \tag{153}$$

Now if the system remained Gaussian for all times, the triple cumulant would always vanish and the fluctuations $\langle\langle\psi^2\rangle\rangle = \langle\delta\psi^2\rangle$ would evolve in an uninteresting (quasi-) linear fashion in the mean field. However, because $\langle\langle\psi^3\rangle\rangle$ is driven by $\langle\langle\psi^2\rangle\rangle^2$, non-Gaussian triple correlations must always build up in the course of time. Noting this, one might try the "quasinormal" hypothesis, in which the fourth cumulant is taken to vanish as in the Gaussian state; according to (152c), this determines a nonvanishing triple cumulant according to

$$\langle\langle\psi^3\rangle\rangle \simeq \bar{U} \langle\langle\psi^2\rangle\rangle^2 / (\partial_t - U_2^{(\ell)})$$

$$\equiv \langle\langle\psi^3\rangle\rangle_{qn} \ . \tag{154}$$

If $\langle\langle\psi^3\rangle\rangle_{qn}$ were small, then to lowest order in $\langle\langle\psi^3\rangle\rangle$

$$\langle\langle\psi^2\rangle\rangle \simeq (\partial_t - U_2^{(\ell)})^{-1} \equiv \langle\langle\psi^2\rangle\rangle_{ql} \ , \tag{155}$$

the quasilinear approximation. Iterating Eq. (152b) with Eqs. (154) and (155), we get

$$(\partial_t - U_2^{(\ell)}) \langle\langle\psi^2\rangle\rangle \simeq \left[1 + \tfrac{1}{2}\bar{U}_3^2/(\partial_t - U_2^{(\ell)})^3\right] \delta(t) \ , \tag{156}$$

which determines $\langle\langle\psi^2\rangle\rangle$ in terms of the dimensionless parameter $\bar{U}_3^2/(\partial_t - U_2^{(\ell)})^3$ and is a reasonable approximation if that parameter is small. This is the approach followed in the usual "random phase" approximation to weak plasma turbulence (Sagdeev and Galeev 1969). However, the expansion parameter is not small in several important situations. In the first, the nonlinear coupling is intrinsically large, $\bar{U}_3{}^2/U_2{}^3 \gg 1$, as in the Navier-Stokes equation for sufficiently large forcing and sufficiently small viscosity. In the second, the nonlinear coupling may remain small, but there are resonances so that $\partial_t - U_2{}^{(\ell)} \to 0$. This can happen either because of linear resonances, $\partial_t - U_2 = 0$, or because the full mean field, arising from the term $\bar{U}_3 \langle\psi\rangle$, supports a normal mode. As an explicit example, we may recall the linear response function for plasma (Ichimaru 1973):

$$R^{(0)}_{\vec{k},\omega}(\vec{v};\vec{v}\,') = \int d\vec{v}\, g_{\vec{k},\omega}(\vec{v};\bar{\vec{v}}) \Big[\delta(\bar{\vec{v}} - \vec{v}\,') + (4\pi i\vec{k}/k^2)\cdot\vec{\partial} f(\bar{\vec{v}}) \times \Big[\varepsilon^{(\ell)}\Big]^{-1}(\vec{k},\omega) \sum_{\bar{s}} (nq)_{\bar{s}} \int d\hat{\vec{v}}\, g_{\vec{k},\omega}(\hat{\vec{v}};\vec{v}\,') \Big] , \quad (157a)$$

$$g_{\vec{k},\omega}(\vec{v};\vec{v}\,') \equiv [-i(\omega - \vec{k}\cdot\vec{v} + i\delta)]^{-1} \delta(\vec{v} - \vec{v}\,') , \quad (157b)$$

$$\varepsilon^{(\ell)}(\vec{k},\omega) \equiv 1 - i\sum_s \frac{\omega_p^2}{k^2} \int d\vec{v}\, d\bar{\vec{v}}\, g_{\vec{k},\omega}(\vec{v};\bar{\vec{v}})\vec{k}\cdot\frac{\partial f^{(o)}}{\partial \bar{\vec{v}}} , \quad (157c)$$

where δ is a positive infinitesimal. Singularities in $R^{(o)}$ arise either from the unperturbed propagator $g^{(o)}$, which describes resonant particles, or from $\varepsilon^{(\ell)}$, the (linear) dielectric function, which (for real ω) nearly vanishes at the normal mode frequencies.

These pathologies of linear theory are vitiated, to a large extent, nonlinearly because of two effects. First, in a turbulent or stochastic state correlation and response functions decay because of nonlinear mixing; this decay is manifested as a (generalized) resonance broadening which resolves the singularities of $(\partial_t - U_2{}^{(\ell)})^{-1}$ mentioned above. Second, the proper dimensionless measure of the degree of non-Gaussianity is not $\bar{U}_3{}^2/(\partial_t - U_2^{(\ell)})^3 \equiv \langle\langle\psi^3\rangle\rangle_{qn}^2 / \langle\langle\psi^2\rangle\rangle_{q1}^3$ but rather the skewness parameter

$$\bar{\Gamma} \equiv \frac{\langle\langle\psi^3\rangle\rangle}{\langle\langle\psi^2\rangle\rangle^{3/2}}, \tag{158}$$

which is measured with respect to the "true" (fully nonlinear, interacting) fluctuations. In certain situations, the skewness and other statistical observables can be small even though naive perturbation theory fails. The renormalized theory we shall develop treats the skewness or certain higher-order statistical functions as self-consistently determined small expansion parameters.

In developing any renormalized theory, one should keep in mind the following very important technical point:

> <u>If any function can exhibit resonant behavior, then approximations should be made on the inverse of that function.</u> (159)

Thus, in the renormalized theory resonance broadening effects are handled by dealing not with $\langle\langle\psi^2\rangle\rangle$ itself, but with its inverse; we write Eq. (152b) in the form

$$\langle\langle\psi^2\rangle\rangle^{-1} - (\partial_t - U_2^{(\ell)}) \equiv \Delta\langle\langle\psi^2\rangle\rangle^{-1} \equiv \Sigma, \tag{160}$$

thus defining and shifting attention to the so-called "mass operator"

$$\Sigma \equiv -\tfrac{1}{2}\bar{U}_3\langle\langle\psi^3\rangle\rangle / \langle\langle\psi^2\rangle\rangle \quad , \tag{161}$$

the deviation of $\langle\langle\psi^2\rangle\rangle^{-1}$ from its non-interacting value. For finite Σ, $\langle\langle\psi^2\rangle\rangle$ itself is now well-defined. The simplest estimates for Σ are the quasinormal (also called quasilinear) result

$$\Sigma_{qn} = -\bar{U}_3(\partial_t - U_2^{(\ell)})^{-1}\langle\langle\psi^2\rangle\rangle\ \bar{U}_3 \quad , \tag{162}$$

which follows from Eq. (154), and the renormalized version thereof,

$$\Sigma = -\bar{U}_3 \langle\langle \psi^2 \rangle\rangle^2 \bar{U}_3 \ , \tag{163}$$

which follows by heuristically replacing $(\partial_t - U_2^{(\ell)})^{-1}$ in Eq. (154) by its renormalized value $(\partial_t - U_2^{(\ell)} + \Sigma)^{-1}$:

$$\langle\langle \psi^3 \rangle\rangle = \bar{U}_3 \langle\langle \psi^2 \rangle\rangle^3 \quad . \tag{164}$$

Equations (163) and (160), which afford a completely self-consistent determination of $\langle\langle \psi^2 \rangle\rangle$, are the DIA. (Our somewhat schematic notation does not adequately distinguish between response and correlation functions.) The heuristic procedure we used to obtain it can be formalized, as we show in great detail in the remainder of this chapter.

In a certain sense, the DIA can be thought of as the most Gaussian approximation consistent with nonlinearity. That is, we see by examining Eqs. (152c) and (164) that the fourth cumulant gives rise to two kinds of effects: resonance broadening (of the operator acting on $\langle\langle \psi^3 \rangle\rangle$), which according to (152b) ensures the complete self-consistency of the fluctuations $\langle\langle \psi^2 \rangle\rangle$; and "intrinsically non-Gaussian" effects, which represent everything else. Let us make this precise by introducing a new function K such that

$$\langle\langle \psi^3 \rangle\rangle \equiv \tfrac{1}{2} \bar{U}_3 K \langle\langle \psi^2 \rangle\rangle \ , \tag{165}$$

$$\Sigma \equiv -\tfrac{1}{2} \bar{U}_3 K \bar{U}_3 \quad . \tag{166}$$

Comparing Eqs. (165), (164), and (152b), we can write

$$K = \langle\langle \psi^2 \rangle\rangle^2 + \langle\langle \psi^4 \rangle\rangle_{ng} \equiv \langle\langle \psi^2 \rangle\rangle^2 + \Delta K \ . \tag{167}$$

The renormalization is now reduced to determining the non-Gaussian part ΔK of K. (It might have been expected that $\Delta K \equiv K - \langle\langle\psi^2\rangle\rangle^2$ should involve the factor 3 rather than 1 as the multiplier of $\langle\langle\psi^2\rangle\rangle^2$. This does not occur, in part, because when passing from the moment to the cumulant hierarchy one factor of $\langle\langle\psi^2\rangle\rangle^2$ cancels out, and, in part, because of the way K is normalized and, later, symmetrized. However, this does not vitiate the interpretation of ΔK as a non-Gaussian correction.)

In accordance with the rule (159), we shall actually app..oximate K^{-1}. To do this, let us make things dimensionless by defining

$$G \equiv \langle\langle\psi^2\rangle\rangle ,$$

$$\overline{\gamma} \equiv G^{3/2}\overline{U}_3 , \qquad (168)$$

$$\overline{K} \equiv G^{-2}K .$$

Then, from Eqs. (164) and (165),

$$\overline{\gamma} \sim \overline{\Gamma}\,\overline{K}^{-1} ,$$

$$\Sigma \sim (\overline{\Gamma}\,\overline{K}^{-1}\overline{\Gamma})\,G^{-1} , \qquad (169)$$

in which the inverse $\overline{K}^{-1}$ appears naturally. We write

$$\overline{K}^{-1} - 1 \equiv \Delta(\overline{K}^{-1})\{\overline{\Gamma}\} , \qquad (170)$$

so the fundamental problem is to reasonably determine the functional form of the non-Gaussian correction $\Delta(\overline{K}^{-1})$.

Equation (170) is called the Bethe-Salpeter equation (BSE) (Krommes 1978a). The BSE can be thought of as a self-consistent equation for $\overline{K}$, since from Eq. (169) $\overline{\Gamma}$ and $\overline{K}$ are related according to

$\bar{K} = \bar{\Gamma}/\bar{\gamma}$. Given the exact functional form of $\Delta(\bar{K}^{-1})$, then, we can hope to generate a self-consistent equation for $\bar{K}$ by expanding (or otherwise approximating) $\Delta(\bar{K}^{-1})$ in terms of $\bar{\Gamma} = \bar{K}\bar{\gamma}$. The BSE then determines $\bar{K}\{\bar{\gamma}\}$, whereupon, reverting to dimensional variables and replacing $\bar{U}_3$ by γ for later convenience,

$$\Sigma = -\tfrac{1}{2}\gamma\, K\{G,\gamma\}\,\gamma$$

$$= -\tfrac{1}{2}\gamma\, G\, G\,\gamma \;-\; \tfrac{1}{2}\gamma\Delta K\{G,\gamma\}\gamma\ . \qquad (171)$$

This completes the determination of the mass operator, including non-Gaussian corrections to the resonance broadening. Needless to say, solution of the resulting nonlinear equation for G represents a formidable task.

6.2: The Renormalized Equations of Martin, Siggia, and Rose

We now develop in some detail the renormalized equations discussed above. In this section we ask the reader to take certain technical results on faith so that we can keep the overall scheme as uncluttered as possible. We shall return in the next section to the technical justification of the approach. To begin, we must stress a vital point, apparently first stated clearly by Kraichnan (1958a,b, 1959): to effect closure of the statistical equations, it will be necessary to consider not only fluctuation functions--e.g., $C \equiv \langle\delta\psi(1)\,\delta\psi(2)\rangle$--but also response functions--e.g., the average response R to an infinitesimal external perturbation. Indeed, both kinds of quantities entered into the discussion of Sec. 6.1--for example, $(\partial_t - U_2^{(\ell)})^{-1}$ is a linear response function--although there we deliberately blurred the distinction. The point is that any instantaneous fluctuation in the medium can be considered as a source to which the medium responds. The probability of fluctuation (emission) is described, in some sense, by C, the response (absorption) by R. Because the response of the medium must be self-consistent, and because R describes infinitesimal response whereas the fluctuations are of finite size, the relation between C and R is not straightforward; we shall find nonlinearly coupled equations for C and R.

The unaveraged infinitesimal response $\tilde{R}$ to a source added to Eq. (149) is defined by

$$\tilde{R}(1;1') \equiv \delta\psi(1)/\delta\hat{\eta}(1')\big|_{\hat{\eta}=0} \tag{172}$$

and obeys

$$\partial_t \tilde{R}(1;1') = U(1,2)\tilde{R}(2;1') + \tfrac{1}{2}\overline{U}(1,2,3)[\psi(2)\tilde{R}(3;1') + \tilde{R}(2;1')\psi(3)] + \delta(1-1') \ . \tag{173}$$

(Though the two bracketed terms of Eq. (173) are identical, we choose the present form for later convenience.) Now the definition (172) is not convenient because the mean response $R \equiv \langle\tilde{R}\rangle$ does not appear manifestly conjugate to, or symmetric with, $C \equiv \langle\delta\psi\delta\psi\rangle$. Consider, however, an operator $\hat{\psi}$ which does not commute with ψ (Martin et al. 1973) and form the combination

$$\tilde{r}(1;1') \equiv H(t-t')[\psi(1),\hat{\psi}(1')] \ ,$$

where $[A,B] \equiv AB - BA$. The equation of motion for $\tilde{r}$ is

$$\partial_t \tilde{r}(1;1') = H(t-t')\{U(1,2)[\psi(2),\hat{\psi}(1')] + \tfrac{1}{2}\overline{U}(1,2,3)[\psi(2)\psi(3),\hat{\psi}(1')]\} + \delta(t-t')[\psi(\underset{\sim}{1},t),\hat{\psi}(\underset{\sim}{1}',t)] \ . \tag{174}$$

Recalling that the U's are local in time and noting the identity

$[AB,C] = A[B,C] + [A,C]B$, we get

$$\partial_t \tilde{r}(1;1') = U(1,2)\tilde{r}(2;1') + \tfrac{1}{2}\bar{U}(1,2,3)[\psi(2)\tilde{r}(3;1') + \tilde{r}(2;1')\psi(3)] + \delta(t-t')[\psi(\underset{\sim}{1},t),\hat{\psi}(\underset{\sim}{1}',t)] \quad . \tag{175}$$

Therefore, if we choose $\hat{\psi}$ such that the equal-time canonical commutation relation

$$[\psi(\underset{\sim}{1},t),\hat{\psi}(\underset{\sim}{1}',t)] = \delta(\underset{\sim}{1}-\underset{\sim}{1}') \tag{176}$$

is satisfied, then Eq. (175) is formally identical to Eq. (173). This is very suggestive; however, we cannot conclude that $\tilde{r} = \tilde{R}$. The function $\tilde{R}$ is a c-number which commutes with ψ; $\tilde{r}$, however, is an operator which generally does not commute with ψ. Nevertheless, what we will be able to show (Rose 1974) is that, with an appropriate definition of $\hat{\psi}$ and an averaging operation $<\ldots>$, the means of $\tilde{r}$ and $\tilde{R}$ agree,

$$<\tilde{r}(1;1')> = <\tilde{R}(1;1')> \equiv R(1;1') \quad . \tag{177}$$

We will also be able to show that the mean of any product of ψ's and $\hat{\psi}$'s which begins on the left with $\hat{\psi}$ vanishes,

$$<\hat{\psi}\ldots> \equiv 0 \ , \tag{178}$$

so

$$\begin{aligned} <\tilde{r}(1;1')> &= H(t-t')<\psi(1)\hat{\psi}(1')> \\ &= H(t-t') \ll\psi(1)\hat{\psi}(1')\gg \quad . \end{aligned} \tag{179}$$

The last line followed since $<\hat{\psi}> = 0$.

It is convenient to introduce the time-ordering operation T:

$$T[A(t)B(t')] \equiv [A(t)B(t')]_+$$

$$\equiv H(t-t')A(t)B(t') + H(t'-t)B(t')A(t) \quad . \qquad (180)$$

In view of Eq. (178), Eq. (179) can then be compactly written as

$$R(1;1') = \langle\!\langle \psi(1)\hat{\psi}(1') \rangle\!\rangle_+ \; . \qquad (181)$$

Furthermore, because $[\psi(t),\psi(t')] \equiv 0$, time ordering is irrelevant for functions of ψ alone and we can write

$$C(1,1') \equiv \langle \delta\psi(1)\delta\psi(1') \rangle = \langle\!\langle \psi(1)\psi(1') \rangle\!\rangle_+ \; . \qquad (182)$$

The conjugacy between C and R is now clear upon comparing (181) and (182). We can carry the symmetrization still further by collecting ψ and $\hat{\psi}$ in a two-component vector operator φ,

$$\varphi(1) \equiv \begin{pmatrix} \psi(1) \\ \hat{\psi}(1) \end{pmatrix} \equiv \begin{pmatrix} \varphi_+(1) \\ \varphi_-(1) \end{pmatrix} \quad , \qquad (183)$$

$$\langle \varphi(1) \rangle = \begin{pmatrix} \langle \psi(1) \rangle \\ 0 \end{pmatrix} \; , \qquad (184)$$

and by defining a matrix G of fluctuation and response functions:

$$G(1,1') \equiv \langle\!\langle \varphi(1)\varphi(1') \rangle\!\rangle_+$$

$$= \begin{pmatrix} C(1,1') & R(1;1') \\ R(1';1) & 0 \end{pmatrix} \quad . \qquad (185)$$

[$G_{-,-} \equiv 0$ because $\langle \hat{\psi}(1)\hat{\psi}(1') \rangle \equiv 0$ according to (178).] For functions like G(1,1') we shall extend the notation so that "1" includes the "spin" index "+" or "-". With this convention, G is totally symmetric, $G(1,1') = G(1',1)$.

It will turn out (see Sec. 6.3) that $\mathcal{Q}$ obeys a vector equation similar in form to Eq. (149):

$$-i\sigma\partial_t \mathcal{Q}(1) = \gamma(1) + \gamma(1,2)\mathcal{Q}(2) + \tfrac{1}{2}\gamma(1,2,3)\mathcal{Q}(2)\mathcal{Q}(3) \;, \qquad (186)$$

where

$$i\sigma \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad (187a)$$

$$\gamma\begin{pmatrix} 1 \\ - \end{pmatrix} = U(1) \;, \qquad (187b)$$

$$\gamma\begin{pmatrix} 1 & 2 \\ - , & + \end{pmatrix} = U(1,2) \;, \qquad (187c)$$

$$\gamma\begin{pmatrix} 1 & 2 & 3 \\ - , & + , & + \end{pmatrix} = \bar{U}(1,2,3) \qquad (187d)$$

and the matrix γ's can be taken to be totally symmetric, with non-vanishing components involving exactly one "-" spin index. Because $\mathcal{Q}$ is noncommuting, the symmetrization is not entirely trivial. The goal is now to find a convenient way of manipulating the moments of Eq. (186). By obvious extension of the cumulant generating functionals for fluctuation functions discussed in App. A, let us define

$$S \equiv \exp[\mathcal{Q}(1)\eta(1)] \qquad (188a)$$

and the extended cumulant generating functional

$$W\{\eta\} \equiv \ln\langle S\rangle_+ \;. \qquad (188b)$$

The quantity $\eta \equiv (\eta_+,\eta_-)$ is a c-number. The fundamental observables then follow as

$$\langle\psi(1)\rangle = \delta W/\delta\eta_+(1)\,\big|_{\eta=0} \quad ,$$

$$C(1,1') = \delta^2 W/\delta\eta_+(1)\,\delta\eta_+(1')\,\big|_{\eta=0} \quad , \tag{189}$$

$$R(1;1') = \delta^2 W/\delta\eta_+(1)\,\delta\eta_-(1')\,\big|_{\eta=0} \quad ,$$

and in general

$$\langle\!\langle \mathcal{Y}(1)\ldots\mathcal{Y}(n)\rangle\!\rangle = G^\eta(1,\ldots,n)\big|_{\eta=0} \quad ,$$

$$G^\eta(1,\ldots,n) \equiv \delta^n W/\delta\eta(1)\ldots\delta\eta(n)$$

$$= \delta G^\eta(1,\ldots,n-1)/\delta\eta(n) \quad . \tag{190}$$

The equation of motion for $\langle\!\langle \mathcal{Y}\rangle\!\rangle^\eta$ is

$$\partial_t \langle\!\langle \mathcal{Y}(1)\rangle\!\rangle^\eta = i\sigma\eta(1) + \langle\partial_t\mathcal{Y}(1)S\rangle_+/\langle S\rangle_+ \quad , \tag{191}$$

where the explicit term in η arises from the time derivative of the Heaviside function implicit in the time ordering. For example, through two terms,

$$S = 1 + \mathcal{Y}(\bar{1})\,\eta(\bar{1})$$

and

$$\partial_t \langle\mathcal{Y}(1)S\rangle_+ = \partial_t[\langle\mathcal{Y}(1)\rangle + \langle\mathcal{Y}(1)\mathcal{Y}(\bar{1})\rangle_+\eta(\bar{1})]$$

$$= \langle\partial_t\mathcal{Y}(1)\rangle + \partial_t[H(t-\bar{t})\langle\mathcal{Y}(1)\mathcal{Y}(\bar{1})\rangle$$

$$+ H(\bar{t}-t)\langle\mathcal{Y}(\bar{1})\mathcal{Y}(1)\rangle]\,\eta(\bar{1})$$

$$= \langle\partial_t\mathcal{Y}(1)S\rangle + \delta(t-t')[\mathcal{Y}(1),\mathcal{Y}(\bar{1})]\,\eta(\bar{1}) \quad . \tag{192}$$

For $\partial_t\mathcal{Y}$ we use the right-hand side of Eq. (186). Noting that $(i\sigma)^{-1} = -i\sigma$ and that

$$\frac{\langle \mathcal{Q}(1)\mathcal{Q}(2)S\rangle_+}{\langle S\rangle_+} = \frac{\delta}{\delta\eta(2)}\left(\frac{\langle \mathcal{Q}(1)S\rangle_+}{\langle S\rangle_+}\right) + \left(\frac{\langle \mathcal{Q}(1)S\rangle_+}{\langle S\rangle_+}\right)\left(\frac{\langle \mathcal{Q}(2)S\rangle_+}{\langle S\rangle_+}\right).$$

$$= G^\eta(1,2) + \langle\!\langle \mathcal{Q}(1)\rangle\!\rangle^\eta \langle\!\langle \mathcal{Q}(2)\rangle\!\rangle^\eta , \qquad (193)$$

we get the fundamental equation of motion as

$$-i\sigma\partial_t \langle\!\langle \mathcal{Q}(1)\rangle\!\rangle^\eta = \gamma(1) + \eta(1) + \gamma(1,2)\langle\!\langle \mathcal{Q}(2)\rangle\!\rangle^\eta$$

$$+ \tfrac{1}{2}\gamma(1,2,3)[\langle\!\langle \mathcal{Q}(2)\rangle\!\rangle^\eta \langle\!\langle \mathcal{Q}(3)\rangle\!\rangle^\eta + G^\eta(2,3)] . \qquad (194)$$

Successive functional differentiations of Eq. (194) with respect to η generate the cumulant hierarchy in the $\mathcal{Q}$ space--for example,

$$-i\sigma\partial_t G^\eta(1,1') = \delta(1,1') + \gamma(1,2)G^\eta(2,1') + \gamma(1,2,3)\langle\!\langle \mathcal{Q}(2)\rangle\!\rangle^\eta G^\eta(3,1')$$

$$+ \tfrac{1}{2}\gamma(1,2,3)G^\eta(2,3,1') . \qquad (195)$$

Were we to just proceed in this vein, we would gain only a symmetric representation of the hierarchy of equations which would follow by standard techniques directly from Eqs. (149) and (175). We wish, however, to effect closure by expressing G_3 in terms of G_2 (where the subscript indicates the number of arguments). To do this, we treat the correlation functions for $n \geq 2$ as functionals not of η but rather of $\langle\!\langle \mathcal{Q}\rangle\!\rangle^\eta$. This will enable us to eliminate η entirely from the theory. The point is that η serves as a system probe. (Formally, we will learn in Sec. 6.3 that the generating function W can be interpreted as the system's distribution functional in interaction representation, with $\mathcal{Q}(1)\eta(1)$ being the perturbing Hamiltonian.) As η is varied, all moments $\langle\!\langle \mathcal{Q}\rangle\!\rangle^\eta$, G_2^η, G_3^η, ... vary in concert. Thus, an increment $\delta\langle\!\langle \mathcal{Q}\rangle\!\rangle^\eta$ $[=(\delta\langle\!\langle \mathcal{Q}\rangle\!\rangle^\eta/\delta\eta)\delta\eta = G_2\delta\eta]$ is related to increments $\delta G_{n\geq 2}$, implying that functional differential equations exist between the observables. Since η does not appear explicitly in these equations, we can set it to zero with impunity.

Formally, the change of variables from η to $\langle\langle\varphi\rangle\rangle$ is accomplished via a Legendre transform (Deker and Haake 1975a). Define

$$L\{\langle\varphi\rangle\} \equiv W\{\eta\} - \langle\varphi(1)\rangle\eta(1) \ . \tag{196}$$

We shall define the renormalized vertices Γ by derivatives of L with respect to $\langle\varphi\rangle$:

$$\Gamma_n(1,\ldots,n) \equiv \frac{\delta^n L}{\delta\langle\varphi(1)\rangle\ldots\delta\langle\varphi(n)\rangle} \ ; \tag{197}$$

$$\Gamma(1) = -\eta(1) \ , \tag{198a}$$

$$\Gamma(1,2) = -\frac{\delta\eta(1)}{\delta\langle\varphi(2)\rangle} = -\left(\frac{\delta\langle\varphi(2)\rangle}{\delta\eta(1)}\right)^{-1} = -G^{-1}(1,2) \ , \tag{198b}$$

$$\Gamma(1,2,3) = -\frac{\delta G^{-1}(1,2)}{\delta\langle\varphi(3)\rangle} \ . \tag{198c}$$

Since $G\,G^{-1} = 1$ implies $\delta G^{-1} = -G^{-1}\delta G\,G^{-1}$ and since $\delta/\delta\langle\varphi\rangle = G^{-1}\delta/\delta\eta$, we also have

$$\Gamma(1,2,3) = G^{-1}(1,\bar{1})G^{-1}(2,\bar{2})G^{-1}(3,\bar{3})G(\bar{1},\bar{2},\bar{3}) \ . \tag{199}$$

The matrix $G^{3/2}\Gamma_3 \equiv \bar{\Gamma}_3$ generalizes the skewness parameter introduced earlier.

Inserting Eq. (199) into (195), we find the Dyson equation

$$[-i\sigma\partial_t\delta(1-2) - \gamma(1,2) - \gamma(1,2,3)\langle\varphi(3)\rangle + \Sigma(1,2)]G(2,1') = \delta(1,1') \ , \tag{200}$$

where

$$\Sigma(1,\bar{1}) \equiv -\tfrac{1}{2}\gamma(1,2,3)G(2,\bar{2})G(3,\bar{3})\Gamma(\bar{3},\bar{2},\bar{1}) \ . \tag{201}$$

According to (159), the Dyson equation should be thought of as an equation for $\Gamma_2 = -G^{-1}$. An independent equation for Γ_3 follows by differentiating the Dyson equation:

$$\Gamma_3(1,2,3) = -\frac{\delta G^{-1}(1,2)}{\delta\langle\phi(3)\rangle} = \gamma(1,2,3) - \frac{\delta\Sigma(1,2)}{\delta\langle\phi(3)\rangle} . \qquad (202)$$

In general, this is a functional differential equation which determines Γ. Detailed considerations (Rose 1974, Deker 1979) show that when the initial conditions are Gaussian Σ depends on $\langle\phi\rangle$ only implicitly, via its dependence on G_2. In this case,

$$\frac{\delta\Sigma}{\delta\langle\phi\rangle} = \left(\frac{\delta G}{\delta\langle\phi\rangle}\right)\left(\frac{\delta\Sigma}{\delta G}\right)$$

$$= G\,G\,\Gamma\left(\frac{\delta\Sigma}{\delta G}\right) = G\,G\,\Gamma\left[\gamma\,G\,\Gamma + G\,G\left(\frac{\delta\Gamma}{\delta G}\right)\right] . \qquad (203)$$

In summary, then, we have derived, for the case of Gaussian initial conditions, the set of exact equations

$$\Gamma = \gamma - (\delta\Sigma/\delta G)G\,G\,\Gamma, \qquad (204a)$$

$$\Sigma \equiv -\tfrac{1}{2}\gamma G\,G\,\Gamma , \qquad (204b)$$

$$G^{-1} = \left(G^{(0)}\right)^{-1} - \gamma\langle\phi\rangle + \Sigma , \qquad (204c)$$

$$\gamma_1 = \left(G^{(0)}\right)^{-1}\langle\phi\rangle - \tfrac{1}{2}\gamma\langle\phi\rangle^2 - \tfrac{1}{2}\gamma G , \qquad (204d)$$

where

$$\left(G^{(0)}\right)^{-1}(1,2) \equiv -i\sigma\partial_t\,\delta(1,2) - \gamma(1,2) . \qquad (205)$$

If the last term of (202) were small in some sense, we would find

$$\Gamma(1,2,3) \simeq \gamma(1,2,3) . \qquad (206)$$

This is, in fact, the DIA in matrix form:

$$\Sigma_{DIA} \simeq -\tfrac{1}{2}\gamma G\, G\, \gamma \ . \tag{207}$$

To be more specific, we can take matrix components. The equation for $R = G_{+,-}$ follows from the (-,-) component of Eq. (200):

$$[\partial_t\, \delta(1,2) - U(1,2) - \overline{U}(1,2,3)\langle\psi(3)\rangle + \Sigma_{-+}(1,2)]R(2,1') = \delta(1-1') \tag{208}$$

$$\Sigma_{-+}(1,\overline{1}) \equiv -\overline{U}(1,2,3)R(2,\overline{2})C(3,\overline{3})\overline{U}(\overline{2},\overline{3},\overline{1}) \ , \tag{209}$$

where we recalled that $G_{-,-} = 0$ and that the γ's are symmetric and contain exactly one "-". Similarly, C is obtained from the (-,+) component:

$$[\partial_t\, \delta(1,2) - U(1,2) - \overline{U}(1,2,3)\langle\psi(3)\rangle + \Sigma_{-+}(1,2)]C(2,1') + \Sigma_{--}(1,2)R^t(2,1') = 0 \ , \tag{210}$$

where

$$R^t(1,2) \equiv R(2,1) \tag{211}$$

and

$$\Sigma_{--}(1,\overline{1}) \equiv -\tfrac{1}{2}\overline{U}(1,2,3)C(2,\overline{2})C(3,\overline{3})\overline{U}(\overline{1},\overline{2},\overline{3}) \ . \tag{212}$$

We shall examine the consequences of the DIA for plasmas more specifically in Secs. 8 and 9.

If the nonlinearity is weak, the most obvious way of improving Eq. (206) is to expand Eq. (204a) in powers of γ:

$$\Gamma = \gamma + \gamma G \gamma G\, G\, \gamma + \cdots \ . \tag{213}$$

However, for strong nonlinearity it turns out (see Secs. 6.4 and 6.5) that this perturbation theory is ill-motivated and ill-behaved.

According to Martin et al. (1973) a better scheme is to expand Eq. (204a) in powers of Γ--that is, to expand the (unobservable) bare vertex in terms of the (observable) renormalized vertex:

$$\gamma = \Gamma - \Gamma G \Gamma G G \Gamma + \cdots \, . \tag{214}$$

We will discuss this procedure in more detail later. It is well-behaved through order Γ^3, but not beyond. In Sec. 6.5 we suggest a yet more sophisticated closure which generalizes Eq. (214).

6.3: The Operator $\hat{\psi}$ and Some Technical Details of the Functional Approach

Let us now discuss some of the technical details of the above scheme. A realization of the operator $\hat{\psi}$ which satisfies the canonical commutation relation (176) at some arbitrary time $t = 0$ is (Rose 1974, Phythian 1975)

$$\hat{\psi}(\underset{\sim}{1}, t=0) \equiv -\delta/\delta\psi(\underset{\sim}{1}, t=0) \, . \tag{215}$$

Since $\psi(t)$ evolves in a definite way from $\psi(t=0)$, the action of $\hat{\psi}(t=0)$ on any (differentiable) function of $\psi(t)$ is well-defined. We shall define the time dependence of $\hat{\psi}$ so that the equal-time commutation relations are satisfied for all times. We can then write the equation of motion in a Hamiltonian form which has close relatives in quantum mechanics:

$$\partial_t \psi(\underset{\sim}{1}, t) = [\psi(\underset{\sim}{1}, t), H(t)] \, , \tag{216}$$

where

$$H(t) \equiv \hat{\psi}(\underset{\sim}{\overline{t}}, t)[U_1(\underset{\sim}{\overline{1}}, t) + U_2(\underset{\sim}{\overline{1}}, t; \overline{2})\psi(\overline{2}) + \tfrac{1}{2}\overline{U}_3(\underset{\sim}{\overline{1}}, t; 2, 3)\psi(2)\psi(3)] \quad \text{(no sum on t)} \, . \tag{217}$$

The desired equation of motion for $\partial_t \hat{\psi}$ is then

$$\partial_t \hat{\psi}(\underset{\sim}{1}, t) = [\hat{\psi}(\underset{\sim}{1}, t), H(t)] \, , \tag{218}$$

as can be verified by noting that

$$\partial_t [\psi(\underset{\sim}{1},t),\hat{\psi}(\underset{\sim}{2},t)] = \Big[[\psi(\underset{\sim}{1},t),H],\hat{\psi}(\underset{\sim}{2},t)\Big] + \Big[\psi(\underset{\sim}{1},t),[\hat{\psi}(\underset{\sim}{2},t),H]\Big]$$

$$= -\Big[[\psi(\underset{\sim}{1},t),\hat{\psi}(\underset{\sim}{2},t)],H\Big]$$

$$= \delta(\underset{\sim}{1}-\underset{\sim}{2})[1,H] = 0. \tag{219}$$

Thus,

$$\partial_t \mathcal{U}(1) = [\mathcal{U}(\underset{\sim}{1},t),H(t)] \;; \tag{220}$$

in fact, the same law holds with $\mathcal{U}$ replaced by any Taylor-expandable function of $\mathcal{U}$.

To formally solve Eq. (220) in terms of the initial conditions, let us define

$$U^{-1}(t_1,t_2) \equiv T\exp[-\int_{t_2}^{t_1} dt'\,H(t')] \;, \tag{221a}$$

$$U(t_1,t_2) \equiv T^* \exp[\int_{t_2}^{t_1} dt'\,H(t')] \;, \tag{221b}$$

where T^* is the anti-time-ordering operator. These functions obey

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3) \;, \tag{222}$$

$$\partial_t U(t,t_o) = U(t,t_o)H(t) \;, \tag{223a}$$

$$\partial_t U^{-1}(t,t_o) = -H(t)U^{-1}(t,t_o) \;, \tag{223b}$$

and are inverses, as follows from

$$\partial_t [U(t,t_o)U^{-1}(t,t_o)] = (UH)U^{-1} - U(HU^{-1}) = 0 \;. \tag{224}$$

The solution of Eq. (220) is then

$$\mathcal{U}(t) = U^{-1}(t,0)\mathcal{U}(0)U(t,0) \;, \tag{225}$$

which can be verified by noting that it satisfies both the initial conditions and the equations of motion:

$$\partial_t \mathcal{Q}(t) = -[H(t)U^{-1}(t,0)]\mathcal{Q}(0)U(t,0) + U^{-1}(t,0)\mathcal{Q}(0)U(t,0)\,H(t)$$

$$= [\mathcal{Q}(t), H(t)] \ . \qquad (226)$$

Useful forms for the system response functions follow by examining the change in $\mathcal{Q}$ due to a change δH in $H \equiv H_o + \delta H$. Let us denote the solution in the absence of perturbation by $\mathcal{Q}_o(t)$,

$$\partial_t \mathcal{Q}_o(t) = [\mathcal{Q}_o(t), H_o\{\mathcal{Q}_o\}] \ , \qquad (227)$$

with associated $U_o(t,t_o)$. (Note that the "o" functions describe the fully <u>nonlinear</u> dynamics of the unperturbed system.) To describe the solution $\mathcal{Q}$ in the presence of perturbation, it is convenient to go to the interaction representation. That is, we write

$$U(t,t_o) \equiv U_o(t,t_o)U'(t,t_o) \ , \qquad (228)$$

and study U'. If we note that for any function $A\{\mathcal{Q}\}$ we have

$$A(t) = U^{-1}(t,0)A\{\mathcal{Q}(0)\}U(t,0)$$

$$= (U')^{-1}(t,0)\,[U_o^{-1}(t,0)A\{\mathcal{Q}(0)\}U_o(t,0)]U'(t,0)$$

$$= (U')^{-1}(t,0)A\{\mathcal{Q}_o(t)\}U'(t,0) \ , \qquad (229)$$

we can then determine the equation of motion for U' as follows:

$$\partial_t U = U_o H_o\{\mathcal{Q}_o\}U' + U_o \partial_t U'$$

$$= U\,H\{\mathcal{Q}\} = U_o U'(H_o\{\mathcal{Q}\} + \delta H\{\mathcal{Q}\}) \qquad (230)$$

so that, using Eq. (229),

$$\partial_t U' = -H_o\{\vartheta_o\}U' + U'[(U')^{-1}(H_o\{\vartheta_o\} + H\{\vartheta_o\})U']$$

$$= \delta H\{\vartheta_o\}U' ,$$

$$U'(t) = T\exp[\int_0^t dt'\delta H(t')] . \tag{231}$$

From Eq. (229), then,

$$\psi(t) = \{T^*\exp[-\int_0^t dt'\delta H_o(t')]\}\psi_o(t)\{T\exp[\int_0^t dt'\delta H_o(t')]\} \tag{232}$$

ıd the response functions follow by Taylor-expanding this formula to the desired order. Through second order, for example,

$$\delta\psi(t) = \int_0^t dt'[\psi(t), H(t')]_+ + \int_0^t dt'dt''\Big[[\psi(t),\delta H(t')],\delta H(t'')\Big]_+ , \tag{233}$$

where we have now dropped the "o" subscript. The case of most interest involves perturbations $\eta_-(1)$ and $\eta_{-+}(1,2)$ to H. For such perturbations,

$$\delta H(t) = \hat{\psi}(\bar{\underset{\sim}{1}},t)[\eta_-(\bar{\underset{\sim}{1}},t) + \eta_{-+}(\bar{\underset{\sim}{1}},\bar{\underset{\sim}{2}},t)\psi(\bar{\underset{\sim}{2}},t)] \tag{234}$$

and

$$\tilde{r}(1;1') \equiv \delta\psi(1)/\delta\eta_-(1') = H(t-t')[\psi(1),\hat{\psi}(1')]_+ , \tag{235}$$

$$\tilde{r}(1;1',1'') \equiv \delta^2\psi(1)/\delta\eta_-(1')\delta\eta_-(1'')$$

$$= H(t-t')H(t-t'')\Big[[\psi(1),\hat{\psi}(1')],\hat{\psi}(1'')\Big]_+ , \tag{236}$$

$$\delta\psi(\underset{\sim}{1},t)/\delta\eta_{-+}(\underset{\sim}{1}',\underset{\sim}{1}'',t') = H(t-t')[\psi(1),\hat{\psi}(1)\psi(1'')]_+ . \tag{237}$$

The n-th order response functions are functions of $\psi(t)$ and $\hat{\psi}(t)$, thus are functionals of $\psi(t=0)$ and $\hat{\psi}(t=0)$. For any function

$A\{\psi(t),\hat{\psi}(t)\}$ let us introduce the following definition of averaging:

$$\langle A\{\psi,\hat{\psi}\}\rangle \equiv \int d\psi(t=0)A\{\psi(t),\hat{\psi}(t)\}P\{\psi(t=0)\} \ , \tag{238}$$

where $P\{\psi\}$ is the distribution of initial conditions and where each $\hat{\psi}(t)$ is to be written in terms of $\hat{\psi}(t=0) = -\delta/\delta\psi(t=0)$, which then acts on everything to its right. This definition has the following important properties (Rose 1974):

(1) when A is a function of ψ only, the prescription reduces to the conventional definition of averaging;

(2) the time dependence of $\langle A\rangle$ is consistent with the equation of motion;

(3) for reasonable P's, the mean of any quantity of the form $\langle\hat{\psi}(t)A\{\psi,\hat{\psi}\}\rangle$ vanishes identically.

At time $t=0$, this latter property follows directly from

$$\int d\psi(0)\left[\frac{\delta}{\delta\psi(0)}\right] \cdots P\{\psi(0)\} = 0 \ . \tag{239}$$

At later times, it follows by induction upon noting that $\hat{\psi}(t) = U^{-1}\hat{\psi}(0)U$ and that $H(t)$ begins on the left with $\hat{\psi}$.

Thus,

$$R(t;t') = H(t-t')\langle[\psi(t),\hat{\psi}(t')]\rangle = \langle\psi(t)\hat{\psi}(t')\rangle_+ \ , \tag{240a}$$

$$R(t;t',t'') = \langle\psi(t)\hat{\psi}(t')\hat{\psi}(t'')\rangle_+ \ , \tag{240b}$$

etc. Also,

$$\langle\delta\psi(t)/\delta\eta_{-+}(t',t'')\rangle = \langle\psi(t)\hat{\psi}(t')\psi(t'')\rangle_+ \ . \tag{241}$$

The significance of this response function can be understood by noting that in the Vlasov equation an external, infinitesimal perturbing electric field $\vec{E}_e$ gives rise to an additional coupling of the form

$$\eta_{-+}(1,2) = -\vec{E}_e(1)\cdot\vec{\partial}_1\Delta(1-2) \ . \qquad (242)$$

Thus, formula (241) describes the infinitesimal response of the system to an external force <u>when ψ is a distribution function</u>. This interpretation will be useful in Sec. 8, where we compute the renormalized plasma dielectric function.

Finally, we must justify the symmetrization used in Eq. (186). We can, of course, write in general

$$H(t) = \gamma(1)\mathcal{Q}(1) + \tfrac{1}{2}\gamma(1,2)\mathcal{Q}(1)\mathcal{Q}(2)$$

$$+ \frac{1}{3!}\gamma(1,2,3)\mathcal{Q}(1)\mathcal{Q}(2)\mathcal{Q}(3) \quad \text{(no sum on t)} \qquad (243)$$

where the γ's have as yet no special symmetry, but merely follow from Eq. (217). The equation for $\langle\langle\mathcal{Q}\rangle\rangle^{\eta}$ is

$$\partial_t \langle\langle\mathcal{Q}\rangle\rangle^{\eta} = -i\sigma\eta + \langle[\mathcal{Q},H]S\rangle_+/\langle S\rangle_+ \ . \qquad (244)$$

Now $[\mathcal{Q},H]$ consists of a sum of products of $\mathcal{Q}$'s at equal times. Let us make the convention that the time argument of any $\hat{\psi}(t)$ is to be interpreted as $t+\varepsilon$, where $\varepsilon \to 0^+$. It is then easy to convince oneself that the $\mathcal{Q}$'s in $\langle[\mathcal{Q},H]S\rangle_+$ can be commuted arbitrarily, leaving the result unchanged in value. For example,

$$\langle\psi(\underset{\sim}{1},t)\hat{\psi}(\underset{\sim}{2},t+\varepsilon)S\rangle_+ = \langle\hat{\psi}(\underset{\sim}{2},t+\varepsilon)\psi(\underset{\sim}{1},t)S\rangle_+ \ . \qquad (245)$$

This implies that only the symmetric parts of the γ's contribute to the time-ordered expectations in terms of which the theory is developed, and the procedure of Sec. 6.2) is justified.

6.4: The Stochastic Oscillator Revisited

We wish to use the stochastic oscillator problem described in Sec. 2 to illustrate the renormalized equations we have been discussing. Now the formalism as developed thus far is, in fact, inapplicable to the stochastic oscillator since the functional equations describe dynamic, rather than stochastic, nonlinearity. However, it is possible to extend the functional techniques to include the latter, asymmetric case. We shall sketch this extension only briefly, since the procedure (Deker and Haake 1975a) follows that for dynamic nonlinearity closely.

For stochastic nonlinearity, the prototype equation is of the form

$$\partial_t \psi(1) = U'(1,2,3)\omega(2)\psi(3) \quad , \tag{246}$$

where we assume that ω is a Gaussianly distributed random variable. The spinor notation is extended to include a "o" index denoting the random coefficient ω. The extended state vector $\mathcal{V}$ becomes $\mathcal{V} = (\psi, \hat{\psi}, \omega)$ and the covariance matrix G becomes

$$G(1,2) = \begin{pmatrix} C(1,2) & R(1;2) & \langle\delta\psi(1)\delta\omega(2)\rangle \\ R(2;1) & 0 & 0 \\ \langle\delta\omega(1)\delta\psi(2)\rangle & 0 & \langle\delta\omega(1)\delta\omega(2)\rangle \end{pmatrix} . \tag{247}$$

The functional equations are developed from a generating functional of form identical to Eq. (188). (Now, $\mathcal{V}$ and η each have three components.) The equations have form similar to Eqs. (204) except that the matrix Dyson equation is no longer square. (Because the statistics of ω are given, there can be no independent evolution equation for ω.) Because certain three-point cumulants vanish identically, certain vertices also vanish. Using Eq. (199), it can be shown that

$$G_3(-,-,-) \equiv 0 \rightarrow \Gamma(+,+,+) \equiv 0 , \quad (248a)$$

$$G_3(-,-,0) \equiv 0 \rightarrow \Gamma(+,+,0) \equiv 0 , \quad (248b)$$

$$G_3(-,0,0) \equiv 0 \rightarrow \Gamma(+,0,0) \equiv 0 . \quad (248c)$$

If we restrict our attention to the stochastic oscillator at infinite Kubo number (a difficult regime for most statistical theories), the relations simplify further, particularly for the response function R. It can be shown that the Dyson equation for R is determined once Γ_{-o+} is known, and that the vertex equation for Γ_{-o+} is closed in terms of Γ_{-o+}, R, and the known quantity $\langle\delta\omega^2\rangle \equiv \beta^2$. Furthermore, because ω is time-independent, only the reduced quantity

$$\Gamma(t_1,t_3) = \Gamma(t_1 - t_3) \equiv \int_{-\infty}^{\infty} dt\, \Gamma_{-o+}(t_1,t_2,t_3)$$

enters and Fourier transformation with respect to time difference variables is convenient. In the DIA,

$$\Gamma_\omega \simeq \gamma_\omega = -i , \quad (249)$$

so that

$$\Sigma_\omega = -\gamma_\omega R_\omega \beta^2 \Gamma_\omega \quad (250a)$$

$$\simeq R_\omega \beta^2 . \quad \text{(DIA)} \quad (250b)$$

One can easily verify that with Eq. (250b) the Dyson equation

$$(-i\omega + \Sigma_\omega)R_\omega = 1 \tag{251}$$

agrees with the Fourier transform of Eq. (55).

To proceed beyond DIA, we must understand how to approximate Eq. (204a). This will be discussed in some generality in the next section; here, we illustrate the vertex expansions described at the end of Sec. 6.2 . Let us normalize the time to β^{-1}. Naive expansion in the bare coupling γ [cf. Eq. (213)] gives rise to

$$i\Gamma_\omega = 1 - R^2 + \cdots \ . \tag{252}$$

Since $|\gamma_\omega| = 1$, one may expect such a procedure to be unfounded. More precisely, the dimensionless parameter in the above expansion is $R_\omega \gamma_\omega$. For large ω, one has $R_\omega \sim 1/\omega \ll 1$, so that in this limit Eq. (252) may be adequate. However, large frequencies correspond to short times and are generally uninteresting. For small ω, $R_\omega \gtrsim 1$ both exactly and in the DIA, so that in this regime (corresponding to long times) the expansion (252) will be badly behaved. This is illustrated for the special case of $\omega = 0$, for which the system (252) and (250a) predicts complex roots for R--a gross violation of the realizability constraint that $R_{\omega=0}$ must be real and positive (Kraichnan 1961).

As we discussed in Sec. 6.1, the proper dimensionless parameter of Eq. (204a) is the generalized skewness, which we define here by

$$x \equiv iR_\omega \Gamma_\omega \ . \tag{253}$$

The utility of x is suggested by its behavior at $\omega = 0$. Writing Eq. (251) in the form

$$-i\omega R + xR = 1 \ , \tag{254}$$

we see that $x_{\omega=0} = 1/R_{\omega=0}$. Inasmuch as $R_{\omega=0}$ is greater than 1 in the exact solution, $x_{\omega=0}$ will be less than one and thus becomes a possible expansion parameter (albeit a rather poor one). Iterating Eq. (204a) in powers of x [formally, expanding the known function γ_ω in powers of the unknown quantity Γ_ω as in Eq. (214)], one finds

$$R_\omega = x + x^3 + \cdots , \qquad (255)$$

which is much better behaved than Eq. (252) inasmuch as $x < 1$. If we now eliminate x in favor of R by using Eq. (254), we arrive at an equation first given by Kraichnan (1961):

$$R_\omega^4 - i\omega(1-\omega^2)R_\omega^3 + (3\omega^2-1)R_\omega^2 - 3i\omega R_\omega - 1 = 0 . \qquad (256)$$

From Eq. (256) it is easy to verify that $R_{\omega=0} = [(1+\sqrt{5})/2]^{\frac{1}{2}} \simeq 1.27$, which is to be compared with $(\pi/2)^{\frac{1}{2}} \simeq 1.25$ for the exact solution and with 1 for the DIA. Thus, in the present approximation $|x_{\omega=0}| = [(\sqrt{5}-1)/2]^{\frac{1}{2}} \simeq 0.79 < 1$ (ω=0 is the worst case). The solution of Eq. (256) is compared with the exact solution and the DIA in Fig. 13 of Kraichnan (1961). It seems clear that this second renormalization is a substantial improvement over the DIA and should be considered as successful.

However, although expansion of Eq. (204a) in powers of the true skewness appears to be successful at $O(\Gamma^3)$, it turns out that truncations at $O(\Gamma^5)$ and higher are ill-behaved. The correct generalization of the sequence of renormalized approximations is more subtle. We discuss this matter in the next subsection.

6.5: The Bethe-Salpeter Equation and Higher Order Renormalizations

The geometry and intuitive content of the renormalized theory are emphasized by dealing not with Γ but with a certain four-point function K, sometimes called the two-body scattering matrix (Krommes 1978a). To discuss two-body functions conveniently, we introduce the extended generating functional

$$W\{\eta_1,\eta_2\} \equiv \ln\langle\exp[\mathcal{U}(1)\eta_1(1) + \tfrac{1}{2}\mathcal{U}(1)\mathcal{U}(2)\eta_2(1,2)]\rangle_+ \quad . \tag{257}$$

Let us write $F \equiv G_1$ and $G \equiv G_2$. The "proper" two-body generalization of the one-body function G is then

$$K(1,2;1',2') \equiv \frac{\delta G(1,2)}{\delta\eta_2(1',2')}\Big|_F \quad . \tag{258}$$

The analogy can be seen most clearly by introducing the Legendre transform (Krommes 1978a)

$$L\{F,G\} \equiv W\{\eta_1,\eta_2\} - \eta_1(1)F(1) - \eta_2(1,2)G(1,2) \tag{259}$$

and constructing the renormalized vertices according to

$$\Gamma_i(1,\ldots,n) \equiv \frac{\delta^n L}{\delta G_i(1)\cdots\delta G_i(n)} \quad . \tag{260}$$

We have in particular

$$\Gamma_1(1) = -\eta_1(1) \ , \qquad \Gamma_2(1,2) = -\eta_2(1,2) \ ; \tag{261a,b}$$

$$\Gamma_1(1,2) = -\frac{\delta\eta_1(1)}{\delta F(2)}\Big|_G = -G^{-1}(1,2) \ , \qquad \Gamma_2(1,2;1',2') = -\frac{\delta\eta_2(1,2)}{\delta G(1',2')}\Big|_F = -K^{-1}(1,2;1',2') ; \tag{262a,b}$$

$$\Gamma_1(1,2,3) = -\frac{\delta G^{-1}(1,2)}{\delta F(3)}\Big|_G , \qquad \Gamma_2(1,2;1',2';1'',2'') = -\frac{\delta K^{-1}(1,2;1',2')}{\delta G(1'',2'')}\Big|_F \tag{263a,b}$$

In particular, compare Eqs. (261a,b).

The equation for K (Martin et al. 1973, Krommes 1978a) follows by adding $-\eta_2$ to Eq. (205) and then differentiating the Dyson equation (204c) with respect to η_2:

$$K(1,2;1',2') + G(1,\overline{1})G(2,\overline{2})I(\overline{1},\overline{2};\overline{1}',\overline{2}')K(\overline{1}',\overline{2}';\overline{1}',\overline{2}')$$

$$= [G(1,1')G(2,2')]_s$$

$$\equiv \tfrac{1}{2}[G(1,1')G(2,2') + (1' \leftrightarrow 2')] \quad . \qquad (264)$$

Here

$$I(1,2;1',2') \equiv \left.\frac{\delta\Sigma(1,2)}{\delta G(1',2')}\right|_F \qquad (265)$$

and we used the chain rule $\delta\Sigma/\delta\eta_2 = (\delta\Sigma/\delta G)(\delta G/\delta\eta_2)$. Equation (263) or its alternative form

$$K^{-1} = (G^{-1}G^{-1})_s + I \qquad (266)$$

is called the Bethe-Salpeter equation (BSE) (Krommes 1978a). The interaction kernel I, which describes "intrinsic" two-body correlations , was already introduced in our study of Γ. In fact, comparing Eqs. (264) and (204a) and using (265), we see that

$$K(1,2;\overline{1},\overline{2})\gamma(\overline{1},\overline{2},\overline{3}) = G(1,\overline{1})G(2,\overline{2})\Gamma(\overline{1},\overline{2},\overline{3}) \quad , \qquad (267)$$

so we have the symmetric representation

$$\Sigma(1,\overline{1}) = -\tfrac{1}{2}\gamma(1,2,3)K(2,3;\overline{3},\overline{2})\gamma(\overline{3},\overline{2},\overline{1}) \quad . \qquad (268)$$

The DIA emerges by neglecting I entirely in Eq. (265). The solution of the resulting "unrenormalized" BSE,

$$K \simeq (GG)_s, \qquad (269)$$

is, in a certain sense, analogous to the approximation $G \simeq G^{(o)}$ obtained from the Dyson equation by neglecting Σ. A better renormalization emerges by retaining I but approximating its form--that is, we approximate the inverse of K. In particular, we write

$$I = -\frac{\delta}{\delta G}\left(\tfrac{1}{2}\gamma K\gamma\right)$$

$$= \tfrac{1}{2}\gamma K\left(\frac{\delta K^{-1}}{\delta G}\right)K\gamma$$

$$= -\tfrac{1}{2}\Gamma_1 GG\Gamma_2 GG\Gamma_1 \ , \tag{270}$$

which should be compared to Eq. (268). The equation for Γ_2 follows by differentiating Eq. (266):

$$\Gamma_2 = -\tfrac{1}{2}(G^{-1}G^{-1}G^{-1})_s - \delta I/\delta G \ . \tag{271}$$

If we ignore two-body vertex renormalization and retain only the first term of Eq. (271), we arrive at

$$I(1,2;1',2') \simeq -\Gamma(1,3,2)G(3,\bar{3})\Gamma(1',\bar{3},2') \ . \tag{272}$$

The complexity of the resulting self-consistent equation (266) for K is similar, in the space of two-body functions, to that of the DIA, which describes one-body functions.

We can make contact with our earlier work in Sec. 6.2 by multiplying Eq. (264) on the left by $G^{-1}G^{-1}$ and on the right by γ, and then using Eqs. (272) and (267). One finds thereby

$$\Gamma - \Gamma G \Gamma G G \Gamma = \gamma \ , \tag{273}$$

which is precisely the result obtained by truncating Eq. (214) at third order. On the other hand, if we ignored the apparent resonance in Eq. (266), we would write

$$K = (1 + G\,G\,I)^{-1}(G\,G)_s$$
$$\simeq (G\,G)_s - G\,G\,I\,G\,G \ . \qquad (274)$$

The appropriate expression for I in this approximation follows by differentiating Eq. (266) and using Eq. (274):

$$I \simeq \gamma\,G\,\gamma \ . \qquad (275)$$

The resulting expansion, which can be written using Eq. (267) in the form

$$\Gamma \simeq \gamma + \gamma\,G\,\gamma\,G\,G\,\gamma \ , \qquad (276)$$

is just the truncation of Eq. (213) through third order. Since it violates the fundamental rule (159), it is not surprising that it is ill-behaved.

The next consistent approximation would presumably involve the introduction of three-body functions and vertices and the neglect of three-body vertex renormalization. To our knowledge, this approximation has never been investigated; for obvious reasons of complexity, its practical utility is probably nil. It is, however, of some academic interest. The important point is that straightforward expansion of any of the "collision" operators Σ, I, ... will "expand out" a resonance in some n-body space and give rise to ill-behaved equations. The sequence of approximations we advocate is a kind of continued fraction representation of the statistics, as can be seen by writing the exact Dyson-Bethe-Salpeter system in the symbolic form

$$G = \frac{1}{\left[G^{(0)}\right]^{-1} + \tfrac{1}{2}\gamma \dfrac{1}{(G^{-1}G^{-1}) + I}\gamma} \quad . \tag{277}$$

Space does not permit a treatment of several other aspects of the MSR formalism, including the use of functional integral representations, application of the renormalization group, and treatment of non-Gaussian initial conditions. Regarding the latter, we refer the reader to a beautiful report by Rose (1979), which appeared too recently for proper discussion here. Among other things, Rose shows how to properly handle particle discreteness, which is strongly non-Gaussian because of self-correlation effects.

Finally, we caution that the MSR formalism is not a panacea. An important deficiency for some applications is that it is Eulerian-based, whereas certain statistical correlations are handled most conveniently in a Lagrangian frame (Sec. 5, Sec. 7.3). Presently,, it would seem that the formalism is the most systematic one which leads to workable closures. It remains to be seen whether these will be adequate for the practical applications encountered in laboratory and space plasmas. Some results in this direction are reviewed in Secs. 7-10.

7: HYDRODYNAMICS II

7.1: Introduction

In this section we shall explore the application of renormalization techniques to a practical problem involving the fluid dynamics of plasmas. In particular, we shall discuss aspects of the Hasegawa-Mima equation (Hasegawa and Mima 1978) which, it has been proposed, captures certain essential elements of nonlinear drift wave dynamics. It is beyond the scope of this article to comment on the physical validity of the equation, except to say that, as always, the model omits interesting physics (as the authors admit). However, this omission is overshadowed for present purposes by the simplicity of the resulting model.

Our study will not be complete, both because of space constraints and because much work has not yet been done. We shall emphasize the strong turbulence regimes of the equation and point out parallels to traditional analyses of the Navier-Stokes equation for ordinary fluids. We shall discuss the possibility of cascades of almost-conserved quantities and the possible formation of an inertial-range spectrum. In this context, we will learn that the DIA has one potentially serious flaw: it is not invariant to random Galilean transformations (which we define in Sec. 7.3) and hence fails to properly describe inertial-range spectra. This motivates a brief discussion of some of the relatives of the DIA which have been proposed.

In dimensionless form, the Hasegawa-Mima equation is

$$\left[\frac{\partial}{\partial t} + i\omega_*(\vec{k})\right]\phi_{\vec{k}}(t) = \frac{1}{2}\sum_{\vec{k}=\vec{p}+\vec{q}} M_{\vec{k}|\vec{p},\vec{q}}\,\phi_{\vec{p}}(t)\phi_{\vec{q}}(t) \ , \qquad (278)$$

where

$$M_{\vec{k}|\vec{p},\vec{q}} \equiv (1+k^2)^{-1}\,\hat{n}\cdot(\vec{p}\times\vec{q})(q^2-p^2) \ , \qquad (279)$$

$$M_{\vec{k}|\vec{p},\vec{q}} = M_{\vec{k}|\vec{q},\vec{p}} \ .$$

Here time and space coordinates are normalized to ω_{ci}^{-1} ($\omega_{ci} \equiv eB/m_i c$) and $\rho_s \equiv (T_e/m_i)^{1/2}/\omega_{ci}$, respectively, $\phi_{\vec{k}} \equiv e\varphi_{\vec{k}}/T_e$, where $\varphi_{\vec{k}}$ is the electrostatic potential, and $\omega_*(\vec{k}) \equiv k_y[cT_e/eBL_n\omega_{ci}(1+k^2)]$, L_n being the density scale length. A possible viscous or Landau dissipation term as well as a possible external forcing are not written explicitly in Eq. (278). We shall assume that M vanishes unless its last two arguments sum to the first. We then note that

$$(1+k^2)M_{\vec{k}|\vec{p},\vec{q}} + (1+p^2)M_{-\vec{p}|\vec{q},\vec{k}} + (1+q^2)M_{-\vec{q}|\vec{k},\vec{p}} = 0 \;, \quad (280a)$$

$$k^2(1+k^2)M_{\vec{k}|\vec{p},\vec{q}} + p^2(1+p^2)M_{-\vec{p}|\vec{q},\vec{k}} + q^2(1+q^2)M_{-\vec{q}|\vec{k},\vec{p}} = 0, \quad (280b)$$

Equation (278) has two quadratic constants of motion,

$$W \equiv \int d\vec{x}(\phi^2 + |\vec{\nabla}\phi|^2)$$

$$= L^2\sum_{\vec{k}}(1+k^2)|\phi_{\vec{k}}|^2 \;, \quad (281a)$$

$$U \equiv \int d\vec{x}[|\vec{\nabla}\phi|^2 + (\nabla^2\phi)^2]$$

$$= L^2\sum_{\vec{k}}(1+k^2)k^2|\phi_{\vec{k}}|^2 \;. \quad (281b)$$

Here the spatial integrals are over a two-dimensional box of side L. The factor of 1 in the term $1+k^2$ arises from the compressible parallel electron motion (Hasegawa and Mima 1978) and thus represents an important physical difference between Eq. (278) and the two-dimensional Navier-Stokes equation. Nevertheless, there are also many similarities. If compressibility is ignored in Eqs. (281), W and U reduce, respectively, to the energy E and enstrophy Ω constants of the two-dimensional inviscid Navier-Stokes equation.

7.2: Exact Consequences of the Hasegawa-Mima and Related Equations

In this subsection we discuss various exact results and physical phenomena connected with Eq. (278). Although these points have nothing to do with renormalization per se, it is essential to review then so that one can appreciate what the renormalized theories are required to compute. We first discuss equilibrium states of Eq. (278).

The significance of the quadratic constants is that they survive a truncation of Eq. (278) which removes all terms involving any wavevector whose magnitude exceeds an arbitrary upper cutoff $k_>$ or is less than an arbitrary lower cutoff $k_<$. This property can be used in conjunction with the theory of equilibrium statistical mechanics (Kraichnan 1975a and refs. therein) to predict an equilibrium ensemble for Eq. (278). Indeed, if in the truncated system one treats as independent the real and imaginary parts of $\phi_{\vec{k}}$ (or, equivalently, $\phi_{\vec{k}}$ and $\phi_{\vec{k}}^*$), an equilibrium distribution is (Hasegawa and Mima 1978)

$$P\{\phi_{\vec{k}}\} = N \exp(-\alpha W - \beta U) \ , \tag{282}$$

where N is a normalization factor. The inverse "temperatures" α and β must be restricted so that N is finite. The equilibrium spectrum follows as

$$\langle|\phi_{\vec{k}}|^2\rangle = (1+k^2)^{-1}(\alpha + \beta k^2)^{-1} \ . \tag{283}$$

The relation of α and β to the given, conserved quantities W and U follows by straightforward algebra, using Eqs. (281). The results of Kraichnan (1967, 1975a), who discussed the analogous two-dimensional Navier-Stokes equation, can be taken over directly. In particular, if we assume $U = k_I{}^2 W$ and take k_I to be near the lowest mode of the system, then the spectrum has a sharp peak at the long-wavelength end. This has implications for the nonequilibrium dynamics, to be discussed next.

A nonequilibrium state of Eq. (278) may arise because the system is prepared that way or because of forcing and dissipation. Hasegawa and Mima postulated the existence of a large amplitude, long-wavelength mode which coupled to Eq. (278) via the mode-coupling $M_{\vec{k}|\vec{p},\vec{q}}$. We shall be more general and not specify the form of the forcing, but merely assume that W and U are continuously injected at some forcing wavenumber k_1; we maintain the ordering $U \sim k_1{}^2 W$. We also assume that a dissipative term is added which becomes important at short wavelengths. A general question is then whether an asymptotically steady state can be reached in which dissipation balances forcing and, if so, what wavenumber dependence of $<|\phi_{\vec{k}}|^2>$ results.

It is useful to begin discussion of this question by reviewing the famous Kolmogorov arguments for the inertial range energy spectrum (Kolmogorov 1941, Rose and Sulem 1978 and refs. therein). To obtain the classical Navier-Stokes problem, let us ignore compressibility so that $W \to E$, $U \to \Omega$, and also ignore the collective oscilla- ·n at ω_* (which may be important at long wavelengths). We shall write

$$E/L^2 \equiv \int_0^\infty dk\, E(k) \ . \tag{284}$$

Let us ignore, for the moment, the co... ..ancy of U and assume that energy E is injected at a rate ε. If k space is partitioned into octaves, $k_n = 2^n k_<$, the energy density E_n in the n-th band scales as $E_n \sim u_n{}^2$, where $u_n \sim k_n \phi_{k_n}$ (the $\vec{E}\times\vec{B}$ velocity in dimensionless units) and is to be thought of as the velocity difference across a space scale of characteristic dimension $k_n{}^{-1}$. Assume that an inertial range exists--that is, a wavenumber regime which has neither direct external injection nor significant dissipation. Let us further assume that the energy transfer between bands is local--that is, that direct coupling exists only between nearby bands. In this case, once significant energy has coupled into the inertial range, the rate of energy transfer between adjacent bands must be constant and equal to ε, which can be estimated as $\varepsilon \sim E_n/\tau_n$, where the "eddy turnover time" $\tau_n \sim (k_n u_n)^{-1}$ is the time for the velocity shear across the scale k_n^{-1} to rotate an eddy of that scale once. Then $u_n \sim (\varepsilon/k)^{1/3}$ and, because locality implies that the energy in band n scales as kE(k),

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \quad . \tag{285}$$

This is the celebrated Kolmogorov spectrum. Note that no reference to the dimension of space was made. The two essential assumptions were the existence of a single constant of motion and the locality of cascade. (Inasmuch as the energy in band n arrives there in a finite number of steps from the forcing band, locality cannot be strictly correct. This leads to the interesting subject of intermittency. See Rose and Sulem (1978 and refs. therein) for further details.)

The direction of cascade is not determined by the Kolmogorov arguments. In the classical models, energy was injected into long wavelengths and assumed to be absorbed at an equal rate by dissipation at short wavelengths, creating a time-asymptotic steady state. This result, a ("direct") cascade toward short wavelengths, is supported by the form of the absolute equilibrium ensemble in this case, which is an equipartition spectrum E(k)/k = constant. Thus, according to Eq. (235) the short-wavelength modes are far below equilibrium level, which suggests a cascade of injected energy toward those modes.

These arguments fail when the second quadratic constant U or Ω is admitted (Kraichnan 1967). A simple energy cascade to short wavelengths would be accompanied by an enstrophy cascade; however, this is incompatible with the constancy of Ω, which is weighted most heavily by the short wavelengths. (It is easier to think about this in the context of an initial-value problem, with all energy and enstrophy concentrated at the forcing wavenumber, rather than in the context of an asymptotic steady state.) If we retain some sort of locality assumption, the only possible resolution is that energy cascades downward from k_1 with the Kolmogorov spectrum, while enstrophy cascades upward with a spectrum

$$E(k) \sim \eta^{2/3} k^{-3} \quad . \tag{286}$$

Here η is the rate of enstrophy transfer (or injection), and the result (286) follows by dimensional analysis similar to the above Kolmogorov arguments. Again, the existence of cascades can be thought of as the unsuccessful striving of the system to reach absolute statistical equilibrium. In particular, the downward energy cascade in two dimensions is suggested by the sharp peaking of the equilibrium spectrum for small forcing wavenumber k_1.

The directional properties of the dual cascade just proposed should not be modified qualitatively by the inclusion of compressibility. The "inverse" energy cascade thus has important implications for theories of saturated, steady-state drift turbulence. Though short-wavelength dissipation will successfully remove enstrophy, there is within the present model no mechanism for removing energy from the long-wavelength modes. The implication is that the model does not saturate (subject to important questions about the existence of an asymptotic cascade which we discuss in the next paragraph). A plausible mechanism for damping the long-wavelength modes is magnetic shear; however, discussion of this point is outside the scope of this article.

In fact, the simple form (286) of the enstrophy cascade is not quite correct. If we argue that the mean-square velocity difference across scale k_n is proportional to k_n^{-2} times the enstrophy Ω_n in that scale, which can be estimated, using Eq. (286), to be

$$\Omega_n \sim \int^{k_n} dp\, p^2 E(p)$$
$$\sim \eta^{2/3} \int^{k_n} dp/p \;,$$

we see that every octave below k_n contributes equally to Ω_n and, thus, the enstrophy cascade is not local. Logical self-consistency and some degree of locality can be restored by modifying Eq. (286) by a logarithmic factor (Kraichnan 1967). However, the low degree of locality of the enstrophy cascade implies that many octaves of k space must be available before the asymptotic spectrum will manifest itself. This may not be realizable for drift turbulence. At short wavelengths, the nonlinear coupling is significantly reduced by the effects of finite ion gyroradius; at long wavelengths, the oscillatory and dispersive properties of the fluctuations become important and weak turbulence theory provides the appropriate description. Nevertheless, it is important to understand how to formulate the description of cascades quantitatively, as this provides useful insight into the statistical statics and dynamics of Eq. (278). Ultimately, one must invoke a closure approximation such as the DIA, as we discuss in the next section. It is convenient, however, to begin by drawing certain exact consequences from Eq. (278).

Let us pass to the limit of a continuous, isotropic k spectrum and write Eq. (281a) in the form

$$W/L^2 = \int_0^\infty dk\, W(k) \quad , \tag{287a}$$

where

$$W(k) = \pi k(1+k^2)\langle|\phi(\vec{k})|^2\rangle \ . \tag{287b}$$

(Isotropy cannot be correct at sufficiently long wavelengths.) The balance equation for W(k) follows from Eq. (278) as

$$\begin{aligned} \frac{\partial}{\partial t} W(k) &= \pi k(1+k^2)\int \frac{d\vec{p}}{(2\pi)^2}\, d\vec{q}\,\delta(\vec{k}+\vec{p}+\vec{q})M_{\vec{k}|-\vec{p},-\vec{q}}\mathrm{Re}\langle\phi(\vec{p})\phi(\vec{q})\phi(\vec{k})\rangle L \\ &= \pi\int_\Delta dp\,dq\,T(k|p,q) \quad , \end{aligned} \tag{288}$$

where

$$T(k|p,q) \equiv \frac{k(1+k^2)}{2\pi|\sin\alpha|} M_{-\vec{k}|\vec{p},\vec{q}}\mathrm{Re}\langle\phi(\vec{p})\phi(\vec{q})\phi(\vec{k})\rangle L \ . \tag{289}$$

(Because we omit sources and sinks Eq. (288) is correct only in the inertial range.) The domain of integration Δ is defined in Fig. 1 and restricts p and q so that $\vec{k}$, $\vec{p}$, and $\vec{q}$ form a triangle. In arriving at the final form (289), we appealed to the geometry defined in Fig. 2, assumed that the triple correlation of ϕ was isotropic, noted that $M_{\vec{k}|\vec{p},\vec{q}}$ can be expressed entirely in terms of wavevector magnitudes through the law of cosines, and then used the result

$$\begin{aligned} \int d\vec{p}\,d\vec{q}\,\delta(\vec{k}+\vec{p}+\vec{q})F(k,p,q) &= \int_0^\infty p\,dp\,q\,dq\,F\int_0^{2\pi} d\beta\,d\gamma\,\delta(k-p\cos\gamma - q\cos\beta) \\ &\quad \times\delta(p\sin\gamma - q\sin\beta) \\ &= \int_\Delta dp\,dq\,F/|\sin\alpha| \quad , \end{aligned} \tag{290}$$

valid for any function F of only wavevector magnitudes. By using Eqs. (280), one can verify that

$$T(k|p,q) + T(p|q,k) + T(q|k,p) = 0 , \tag{291a}$$

$$k^2T(k|p,q) + p^2T(p|q,k) + q^2T(q|k,p) = 0 , \tag{291b}$$

which are statements of detailed--triad by triad--conservation of W and U, respectively. They lead immediately to Eqs. (281), and can also be deduced from the latter.

To discuss cascades, it is useful to define the rates of W and U transferred across a given surface in k space. Thus,

$$\frac{\partial}{\partial t}\int_0^k d\bar{k}\begin{pmatrix} W(\bar{k}) \\ U(\bar{k}) \end{pmatrix} \equiv -\begin{pmatrix} \Pi(k) \\ \Lambda(k) \end{pmatrix} + \varepsilon . \tag{292}$$

Manipulations using Eqs. (291) show that

$$\begin{pmatrix} \Pi(k) \\ \Lambda(k) \end{pmatrix} = \tfrac{1}{2}\left[\int_k^\infty d\bar{k}\begin{pmatrix} 1 \\ \bar{k}^2 \end{pmatrix}\int_0^k dp\,dq\,T(\bar{k}|p,q) - \int_0^k d\bar{k}\begin{pmatrix} 1 \\ \bar{k}^2 \end{pmatrix}\int_k^\infty dp\,dq\,T(\bar{k}|p,q)\right] . \tag{293}$$

The compressibility factor 1 complicates the cascade analysis. In the following discussion of similarity ranges, we shall ignore this factor, both for purposes of illustration and because we expect the true results to be qualitatively similar. In this case, the analysis reduces to one already given by Kraichnan (1967) for the energy and enstrophy cascades of the two-dimensional Navier-Stokes equation. For the energy cascade, we postulate a similarity solution

$$E(ak)/E(k) = a^{-n} , \tag{294}$$

where

$$E(k) = \pi k^2 \langle \phi(k)^2 \rangle \quad . \tag{295}$$

Dimensional analysis implies that

$$T(ak|ap,aq)/T(k|p,q) = a^{-(1+3n)/2} \quad . \tag{296}$$

It can then be shown that to assure that $\Pi(k)$ equals ε independently of k, one must choose $n = -5/3$, in which case

$$\varepsilon = \int_0^1 dv \int_k^\infty dw \; \mathcal{W}_1\,(v,w;5/3)\; T(1|\,v,w) \quad , \tag{297}$$

where

$$\left.\begin{matrix} \mathcal{W}_1(v,w;5/3) \\ \mathcal{W}_2(v,w;3) \end{matrix}\right\} \equiv -(w^2-v^2)^{-1} \;(1-v^2)\begin{pmatrix} 1 \\ w^2 \end{pmatrix} \ln w \;-\; (w^2-1)\begin{pmatrix} 1 \\ v^2 \end{pmatrix} \ln(v^{-1})$$

$$\gtrless 0 \; . \tag{298}$$

The corresponding expression for $\Lambda(k)$ vanishes. Similarly, for the enstrophy cascade, k-independence of $\Lambda(k) = \eta$ leads to $n = 3$ and to

$$\eta = \int_0^k dv \int_k^\infty dw\, \mathcal{W}_2(v,w;3)\; T(1|v,w) \quad . \tag{299}$$

In this case, the associated energy flux $\Pi(k)$ vanishes. The v and w integrals in Eqs. (297) and (299) express the contributions to transfer in terms of all possible triangle shapes, while the $\mathcal{W}$ factors arise from integrals over triangle sizes. The detailed form of $T(1|v,w)$ must be determined by a closure approximation.

7.3: Closure Approximations and Random Galilean Invariance

Let us consider first the DIA for Eq. (278), which can be written immediately from the formulas given in Secs. 5 and 6. The energy balance equation gives rise to the form

$$T(k|p,q) = \frac{k(1+k^2)}{2\pi|\sin\alpha|}\mathrm{Re}\int_{-\infty}^{t} d\bar{t}\ [2a_{k|p,q}\overline{C}_p(t,\bar{t})\overline{C}_q(t,\bar{t})R_k(t;\bar{t})$$

$$- b_{k|p,q}\overline{C}_q(t,\bar{t})\overline{C}_k(t,\bar{t})R_p(t;\bar{t})$$

$$- b_{k|q,p}\overline{C}_k(t,\bar{t})\overline{C}_p(t,\bar{t})R_q(t;\bar{t})] \qquad (300)$$

where

$$a_{k|p,q} = M_{\vec{k}|-\vec{p},-\vec{q}}M_{-\vec{k}|\vec{p},\vec{q}}(1+p^2)^{-1}(1+q^2)^{-1} \ , \qquad (301a)$$

$$b_{k|p,q} = -2M_{\vec{k}|-\vec{p},-\vec{q}}M_{-\vec{p}|\vec{q},\vec{k}}(1+q^2)^{-1}(1+k^2)^{-1} \ , \qquad (301b)$$

$$\overline{C}_k(t,\bar{t}) \equiv (1+k^2)\langle\phi^2\rangle_k \ . \qquad (302)$$

Note the identities

$$a_{k|p,q} = \tfrac{1}{2}(b_{k|p,q} + b_{k|q,p}) \ , \qquad (303a)$$

$$(1+k^2)b_{k|p,q} = (1+p^2)b_{p|k,q} \ . \qquad (303b)$$

In steady state, one can rigorously write

$$\overline{C}_p(t,\bar{t}) = \overline{C}_p c_p(t-\bar{t}) \ , \qquad (304)$$

for some $c_p(\tau)$ and $\overline{C}_p \equiv \overline{C}_p(0)$, so Eq. (300) reduces to

$$T(k|p,q) = \frac{k(1+k^2)}{2\pi|\sin\alpha|}\ \mathrm{Re}[2a_{k|p,q}\theta_{k,p,q}\overline{C}_p\overline{C}_q$$

$$- b_{k|p,q}\theta_{p,q,k}\overline{C}_q\overline{C}_k - b_{k|q,p}\theta_{q,k,p}\overline{C}_k\overline{C}_p] \ , \qquad (305)$$

$$\theta_{k,p,q} \equiv \int_0^{\infty} d\tau\, c_p(\tau)\, c_q(\tau)\, R_k(\tau) \ . \qquad (306)$$

In principle, the time dependence in DIA of $c(\tau)$ and $R(\tau)$ is available through direct solution of the coupled two-time equations for $\overline{C}$ and R (which we do not write in detail here). A less detailed procedure is to assume simple functional forms involving a few unknown coefficients for c and R and attempt to determine the coefficients. A common choice, exact in absolute equilibrium, is to choose $c_k(\tau) = R_k(|\tau|)$. For the response function, we can choose

$$R_k(\tau) = \exp\{[-i\omega_*(k) - \sigma_k]\tau \quad . \tag{307}$$

In the inertial range, we can assume that σ_k dominates $\omega_*(k)$, in which case

$$\theta_{k,p,q} = (\sigma_k + \sigma_p + \sigma_q)^{-1} \quad . \tag{308}$$

If the Kolmogorov arguments are valid, the nonlinear frequency σ_k must scale like the inverse eddy turnover time:

$$\sigma_k \sim \begin{cases} \varepsilon^{1/3} k^{2/3} & \text{(energy cascade)} \; , \\ \eta^{1/3} & \text{(enstrophy cascade)} \; . \end{cases} \tag{309}$$

An equation for σ_k follows by integrating the R equation over time. This equation, the form (304), and Eqs. (292) and (305) together determine the implicit dimensionless constants in (309), (285), and (286) _if_ the inertial range spectral laws we have assumed are consistent with the DIA dynamics. Indeed, they are dimensionally consistent. However, when the form (285) is substituted into the nonlinear term of the response function equation, the wavenumber integral diverges at long wavelengths. The source of this divergence is now well-understood (Kadomtsev 1965, Kraichnan 1964, Leslie 1973 and refs. therein) and represents a significant physical deficiency of the DIA.

For purposes of illustration, consider as a prototype of the nonlinear problem the guiding center equation (see Sec. 8.4)

$$\frac{\partial\tilde{\rho}}{\partial t} + \vec{v}_E \cdot \vec{\nabla}\tilde{\rho} = 0 \ , \tag{310}$$

where $\vec{v}_E$ is the $\vec{E}\times\vec{B}$ drift. Now consider the effect of adding to each member of the statistical ensemble a velocity field $\vec{v}_\Delta$ constant in both space and time. If $\vec{v}_\Delta$ is constant in magnitude and direction over the ensemble, it can be eliminated from all members of the ensemble simultaneously by a Galilean transformation, which implies that the statistical dynamics of the ensemble remain completely unchanged. However, suppose instead (Kraichnan 1964) that $\vec{v}_\Delta$ is Gaussianly distributed in magnitude and direction. Also, assume for simplicity that the original flow velocity $\vec{v}_E$ is very weak compared to $\vec{v}_\Delta$ so that approximately

$$\frac{\partial\tilde{\rho}_k}{\partial t} = -i\vec{k}\cdot\vec{v}_\Delta\tilde{\rho}_k \ . \tag{311}$$

Since Eq. (311) is just the stochastic oscillator (cf. Sec. 2), it is clear that the two-time statistical functions will be affected by the random Galilean transformation $\vec{v}_\Delta$ and be dephased at a rate proportional to $k\langle v^2\rangle^{\frac{1}{2}}$. Similarly, consider the triple correlation

$$S(\vec{k},t;\vec{p},t';\vec{q},t'') \equiv \langle\tilde{\rho}(\vec{k},t)\tilde{\rho}(\vec{p},t')\Delta\tilde{\rho}(\vec{q},t'')\rangle \ , \tag{312}$$

where $\Delta\tilde{\rho}$ also obeys Eq. (311). S is of the form encountered in the energy balance equation of actual turbulence; with the model (311), its time dependence is of the form

$$S \sim \langle\exp(-i\vec{k}\cdot\vec{v}t + i\vec{p}\cdot\vec{v}t' + i\vec{q}\cdot\vec{v}t'')$$

$$= \exp(-\tfrac{1}{2}\langle v^2\rangle|\vec{k}t - \vec{p}t'' - \vec{q}t''|^2) \ . \tag{313}$$

For unequal times, S too decays due to convective dephasing. However, the equal-time triple correlation S(t,t,t) is invariant to the random convection if we recall the triangle constraint $\vec{k} = \vec{p} + \vec{q}$. This behavior is the simplest mathematical idealization of the physical statement that small-scale flow structures should be merely convected by large-scale fluctuations, without any internal distortion (or, hence, energy transfer into other small scales). This assumption, one form of locality in k space, was implicit in our derivation of the Kolmogorov laws, and is considerably more general than the above example. However, it is violated by the DIA. The DIA expresses the energy transfer terms Π or Λ as integrals over two-time functions. Since these are (correctly) affected by the convective sweeping of small-scale structures by large-scale ones, Π and Λ are (incorrectly) also affected. Ultimately, this failure of the DIA to properly respect random Galilean invariance is responsible for the divergence in the R equation when $\omega_* = 0$. [The integral in the energy equation is convergent. Also, note that the divergence arises in the limit of zero dispersion. This approximation is not uniformly valid in k for the drift wave problem, although it seems to be adequate for certain other problems (cf. Sudan and Keskinen 1977)].

Several methods are available to "patch up" the DIA to make it invariant to random Galilean transformations. The simplest, but most ad hoc, approximation is to restrict the divergent p integral to $p > \alpha k$, where $\alpha = O(1)$ (Kadomtsev 1965, Kraichnan 1964). We invoked a similar approximation in our discussion of the guiding center plasma in Sec. 4.2, arguing on the basis of Brownian motion ideas that only scales shorter than k^{-1} should contribute to decorrelation. Kraichnan (1971) has developed the test-field model, which provides a physically motivated but quasi-systematic recipe for σ_k which effectively retains the form of the response equation but modifies the form of $b_{k,p,q}$ so that only the mean-square shear rather than the total energy in the long-wavelength fluctuations affects the inertial-range energy transfer. In the Lagrangian history DIA (Kraichnan 1965), commented upon briefly in Sec. 5.5, the problem with random Galilean invariance is eliminated by design; the approximation appears to be quantitatively quite accurate in certain applications (Kraichnan 1970b).

Fundamentally, the correlations which preserve random Galilean invariance are described by the vertex renormalizations described in Sec. 6. Kraichnan (1964) has considered the application of the first vertex renormalization (273) to the idealized convection equation (311). He concluded that the vertex renormalization significantly reduced the error in the two-time functions and reduced, but did not eliminate, the more troublesome error in the one-time functions. He pointed out that, because of the ubiquitous coupling between the two- and one-time functions, Galilean invariance would be a problem at any order in Eulerian renormalization, and argued for the Lagrangian description. Dupree (1974) showed that an appropriate Markovianization of (effectively) the Bethe-Salpeter equation led to Galilean-invariant results; however, since the Markovianization was ad hoc (as well as somewhat unphysical), it would not seem to offer quantitative improvement over either the simpler truncation of the wavenumber integral or the test-field model. Weinstock (1977) proposed a very detailed and complicated closure, based on three-point functions, which seems not to have received much attention as yet.

The possible lack of random Galilean invariance is a serious problem only when a well-developed inertial range exists. Thus, it is less troublesome in theories of, say, drift turbulence than in the theory of high Reynolds' number Navier-Stokes flows. When the scales of the turbulence are of comparable size (as is more or less true in drift turbulence), convection without distortion does not occur and it can be argued that the DIA should be quantitatively accurate. This is but one example of a commonly voiced assertion that the DIA is better-suited for plasmas than for ordinary fluids.

8: DIELECTRIC RESPONSE in VLASOV PLASMA

8.1: Introduction

The role of the dielectric function $\varepsilon(k,\omega)$ in the description of polarization effects and collective oscillations is well-established both in the general theory of dielectric media (Landau and Lifshitz 1960) and in the linear theory of plasmas (Krall and Trivelpiece 1973). In this section we develop the renormalized theory of ε and describe some of its properties and consequences. Applications in which the correct form of the renormalized dielectric has been used are notably scarce; this is very much an area of active research. We shall, however, discuss briefly two model problems--the guiding center plasma and a certain drift wave model--which serve as informative protctypes for futher calculations in strong and weak turbulence, respectively.

Historically, the first attempts at obtaining a renormalized dielectric relied on intuition developed in linear theory. In the electrostatic approximation, the linear dielectric is, for $\vec{B} = 0$,

$$\epsilon^{(\ell)}(\vec{k},\omega;\{f^{(0)}\}) = 1 + \sum_s \frac{\omega_p^2}{k^2} \int d\vec{v}\, \frac{\vec{k}\cdot\partial f^{(0)}/\partial\vec{v}}{\omega - \vec{k}\cdot\vec{v} + i\delta} . \tag{314}$$

One often interprets the causality factor $i\delta$ as representing the presence of a small amount of turbulent collisions; in the early renormalized theories (Dupree 1966, 1967, Sleeper and Weinstock 1972), it was concluded that this factor was replaced by the finite bandwidth $\delta\gamma$ associated with turbulent diffusion. Also, the factor of $f^{(0)}$ which occurs in linear theory was retained nonlinearly, but interpreted as the mean distribution $\langle f \rangle$. Thus, it was argued that the nonlinear dielectric could be obtained from the linear one by the rule

$$\epsilon(\vec{k},\omega;\{\langle f\rangle\}) = \epsilon^{(\ell)}(\vec{k},\omega+i\delta\gamma_k;\{\langle f\rangle\}) , \tag{315}$$

where two well-known limits for $\delta\gamma_k$ are

$$\delta\gamma_k \sim \begin{cases} (k^2D)^{1/3} & \text{(unmagnetized--diffusion in v space)} \\ k_\perp^{\,2}D & \text{(strongly magnetized--diffusion in x space).} \end{cases} \tag{316}$$

If the linear dispersion relation $\epsilon^{(\ell)}(\vec{k},\omega) = 0$ has the roots

$$\omega = \omega_{rk}^{(\ell)} + i\gamma_k^{(\ell)}, \tag{317}$$

Eq. (315) leads to the nonlinear dispersion relation

$$\omega = \omega_{rk}^{(\ell)} + i(\gamma_k^{(\ell)} - \delta\gamma_k) . \tag{318}$$

In this approximation the turbulence gives rise to nonlinear damping $\delta\gamma_k$ on all modes, so the system would be completely stabilized when the turbulence grew to a level such that

$$\delta\gamma_k > \gamma_k^{(\ell)} \qquad \text{(all k)}. \tag{319}$$

When cross-field diffusion in real space dominates, as in $\vec{E}\times\vec{B}$ turbulence, Eqs. (319) and (316b) lead to the saturation criterion

$$D \gtrsim \gamma_k^{(\ell)}/k_\perp^2 . \tag{320}$$

This expression can sometimes be used to find the spectral intensity at saturation if the formula for D in terms of $\langle\phi^2\rangle$ is known.

Formula (320) states that the system saturates when particles diffuse one wavelength in an e-folding time. Arguments of this kind have been invoked in estimating confinement times and scaling laws for confinement devices (Dean et al. 1974). However, although such estimates can be useful for some purposes, it must be stressed that the prescription presented above has several significant physical flaws. Most importantly, the "resonance-broadening theory" does not in general conserve energy (Galeev 1967, 1969, Dupree and Tetreault 1978). That is, the nonlinear energy drain on all modes (associated with $\delta\gamma_k$) does not show up as heating of resonant particles and cannot be accounted for otherwise; this contradicts the fact that the system is isolated. The problem arises because the propagator broadening, which leads to Eq. (315), describes the effects of the turbulent medium on the particles but does not describe the back-reaction of those particles on the medium. This back-reaction, necessary for energy conservation, must be accounted for by a renormalization of the particle distribution function which is not considered by the primitive theories which replace $f^{(o)}$ of linear theory by $\langle f\rangle$. We will shortly derive an explicit expression for the renormalized distribution.

The lack of medium response in resonance-broadening theory is also responsible for another problem: the detailed form of the resonance broadening, Eq. (316), is not correct. Equation (316) describes diffusion of bare particles; however, just as in weak turbulence theory, the polarization fields associated with dielectric shielding of those particles reduce the net diffusive effect of the turbulence. We give an explicit, if somewhat artificial, example in our study of the guiding center plasma in Sec. 8.4.

Another deficiency of the resonance-broadening theory is that, if the proposed saturation mechanism were taken literally, it would predict that at marginal stability only the linearly most unstable mode would remain, all others having damped away. Such a single-mode state is generally incompatible with the assumption of stochasticity or turbulence. Finally, it is clear from the theory of stochastic instability that a description in terms of diffusion coefficients is appropriate only in the stochastic part of phase space. Formulas such as Eq. (315), which ignore the phase space dependence of $\delta\gamma$, are, thus, at best gross approximations which must be justified. Admittedly, it is difficult to provide an analytic theory which handles correctly both adiabatic and stochastic regions as well as the transition between them; however, the problem cannot be ignored.

It should not be inferred that the early renormalized theories consisted merely of plausibility arguments. On the contrary, they were so detailed mathematically that the nature of the neglected terms was somewhat obscured (as also occurred in the early derivations of the DIA). In his study of self-consistent Vlasov turbulence, Dupree (1966) decomposed the spectrum into many "background" waves plus a small number of "test" waves. (The decomposition was somewhat arbitrary; in particular, test waves and background waves were physically indistinguishable.) He then formally integrated the exact Vlasov equation, treating the test wave terms as a source. Finally, he averaged the result and invoked a certain closure. The entire procedure was not unlike Kraichnan's original derivation of the DIA. There was, however, a very important difference: in Dupree's approach, the fields of the test waves were assumed to be given until the very end of the calculation, at which point Poisson's equation was enforced. This means that self-consistency was not handled correctly (as is easily verified by noting that the correct weak turbulence limit could not be obtained); the procedure was, in fact, more appropriate for the stochastic acceleration problem. Though we shall not dwell on this, the point is that the infinitesimal response function $\tilde{R}$ and the exact orbit integration operator U are not precisely the same entity; statistical correlations which can be ignored in a low order theory for R may not be ignorable in a theory of similar complexity for $\langle U \rangle$ (Krommes 1979c).

Weinstock (1969,1970) formulated the Vlasov turbulence problem in terms of a certain projection operator (see Sec. 10.2) which effected the statistical averaging. The low-order truncations of his expansions also mishandle self-consistency (although all of the physics is included in principle in his formal representations of the solution).

Rudakov and Tsytovich (1971) computed the mass operator by adding an unknown "effective collision operator" ν to both sides of the fluctuation Vlasov equation, then choosing ν to cancel secularities to a certain order. Though their procedure was a definite improvement, such methods (see also Horton and Choi 1979) suffer from the problem that the concept of "order" is difficult to define in a renormalized perturbation theory; indeed, they did not obtain all terms of a given order in the skewness.

The functional techniques we have developed in Sec. 6, however, are ideally suited to deal with the problems of self-consistency, as well as the other deficiencies of resonance-broadening theory. Indeed, variation of the Vlasov equation with respect to an external source gives rise to a contribution $\vec{E}\cdot\vec{\partial}\delta f + \delta\vec{E}\cdot\vec{\partial} f$, both terms of which are accounted for by the symmetrization we have imposed on the coupling coefficient $\bar{U}$ [see Eq. (151d)]. Only the first term is retained in the usual approximation of resonance-broadening theory, making its relation to the stochastic acceleration problem clear.

8.2: The Renormalized Dielectric

To define the dielectric, imagine probing a turbulent Vlasov poasma with an external, non-random field $\vec{E}_e$. This gives rise to an induced field $\vec{E}_i$,

$$\vec{E}_i \equiv \vec{\mathcal{E}}(f\{\vec{E}_e\} - f\{\vec{E}_e = \vec{0}\}) \quad . \tag{321}$$

Let us express the perturbation in the total mean plasma field as a functional power series in $\vec{E}_e$:

$$\delta\langle\vec{E}(\underline{1})\rangle = \vec{E}_e(\underline{1}) + \vec{\mathcal{E}}(\underline{1},\bar{1})\left\langle\frac{\delta f(\bar{1})}{\delta\vec{E}_e(\underline{2})}\right\rangle\cdot \vec{E}_e(\underline{2}) + \vec{\mathcal{E}}(\underline{1},\bar{1})\left\langle\frac{\delta^2 f(\bar{1})}{\delta\vec{E}_e(\underline{2})\,\delta\vec{E}_e(\underline{3})}\right\rangle : \vec{E}_e(\underline{2})\vec{E}_e(\underline{3}) + \cdots \tag{322}$$

The notation "$\underline{1}$" denotes the set "1" with velocity $\vec{v}_1$ excluded.

We can then define the dielectric response function ε as the proportionality factor between the external field and first order response:

$$\varepsilon^{-1}(\underline{1},\underline{2}) = \delta(\underline{1}-\underline{2}) + \vec{E}(\underline{1},\overline{1}) \cdot \left\langle \frac{\delta f(\overline{1})}{\delta \vec{E}_e(\underline{2})} \right\rangle . \tag{323}$$

This observable contains effects of all orders in the background turbulence.

We can compute $\delta f/\delta \vec{E}_e$ using the response function formalism of Sec. 6. From formulas (241) and (242),

$$\delta\langle f(1)\rangle = -\langle f(1)\hat{f}(2)f(3)\rangle_+ \vec{E}_e(\underline{2})\cdot\vec{\partial}_2\delta(2-3) , \tag{324a}$$

$$\delta\langle f(1)\rangle/\delta\vec{E}_e(\underline{2}) = -\int d\vec{v}_2 \langle f(1)\hat{f}(2)\vec{\partial}_3 f(3)\rangle_{+|3=2} . \tag{324b}$$

Writing the triple moment in terms of the triple cumulant and recalling that $\langle\hat{f}\rangle \equiv 0$, we get

$$\begin{aligned}\langle f(1)\hat{f}(2)f(3)\rangle_+ &= \langle\delta f(1)\delta\hat{f}(2)\rangle_+\langle f(3)\rangle + \langle\delta f(1)\delta\hat{f}(2)\delta f(3)\rangle_+ \\ &= R(1;2)\langle f(3)\rangle + G\begin{pmatrix}1 & 2 & 3\\ +, & -, & +\end{pmatrix} .\end{aligned} \tag{325}$$

[For an alternative derivation, see Krommes and Kleva (1979).] Thus,

$$\begin{aligned}\varepsilon^{-1}(\underline{1};\underline{2}) = \delta(\underline{1}-\underline{2}) - \vec{E}(\underline{1},\overline{1})\int d\vec{v}_2 \Big[& R(\overline{1};2)\vec{\partial}_2\langle f(2)\rangle \\ & + \vec{\partial}_3 G\begin{pmatrix}\overline{1} & 2 & 3\\ +, & -, & +\end{pmatrix}_{|3=2}\Big] .\end{aligned} \tag{326}$$

This expression can be rewritten in terms of the renormalized vertices using Eq. (199). Using obvious operator notation, we find

$$\varepsilon^{-1} = 1 - \vec{E}R\cdot\vec{\partial}F , \tag{327}$$

where

$$\bar{f} = \langle f \rangle + \delta\bar{f} \;, \tag{328a}$$

$$\vec{\partial}\delta\bar{f}(\bar{1},\underline{2}) \equiv \int d\vec{v}_2 \left[\vec{\partial}_2 C(2,\bar{2})\Gamma\begin{pmatrix}\bar{1}, & \bar{2}, & \bar{3} \\ -, & +, & +\end{pmatrix} + \vec{\partial}_2 R(2;\bar{2})\Gamma\begin{pmatrix}\bar{1}, & \bar{2}, & \bar{3} \\ -, & -, & +\end{pmatrix}\right] R(\bar{3};2) \;. \tag{328b}$$

In arriving at Eq. (328b), we noted that a term

$$C(1,\bar{3})\vec{\partial}_3 R(3;\bar{1})\Gamma\begin{pmatrix}\bar{1}, & \bar{2}, & \bar{3} \\ -, & +, & +\end{pmatrix} R(\bar{2};2)\Big|_{3=2} \tag{329}$$

vanishes identically because of causality. That is, since

$$G\begin{pmatrix}1, & 2, & 3 \\ +, & -, & -\end{pmatrix} = R(1;\bar{1})\Gamma\begin{pmatrix}\bar{1}, & \bar{2}, & \bar{3} \\ -, & +, & +\end{pmatrix} R(\bar{2};2)R(\bar{3};3) \;, \tag{330}$$

we have

$$(CR\Gamma)(1,2) = G\begin{pmatrix}3, & 2, & 3' \\ +, & -, & -\end{pmatrix}\Big|_{3=2} R^{-1}(3';\bar{3})C(\bar{3},1) = 0 \;, \tag{331}$$

recalling the time ordering convention we adopted in Sec. 6.3 .

We can further simplify Eq. (327) by replacing the response function by another function, the so-called particle propagator g. Note that since

$$\Sigma(1,\bar{1}) = \Sigma(\bar{1},1) = -\tfrac{1}{2}\Gamma(1,2,3)G(2,\bar{2})G(3,\bar{3})\gamma(\bar{3},\bar{2},\bar{1}) \;, \tag{332}$$

we have

$$\Sigma_{-+}(1,\bar{1}) = -\left[\Gamma\begin{pmatrix}1, & 2, & 3 \\ -, & +, & +\end{pmatrix} C(\bar{2},2) + \Gamma\begin{pmatrix}1, & 2, & 3 \\ -, & -, & +\end{pmatrix} R(\bar{2};2)\right] R(3;\bar{3})\bar{U}(\bar{3},\bar{2},\bar{1}) \;. \tag{333}$$

Let us use the form of $\bar{U}$ given in Eq. (151d) and define Σ' to be that part of Σ which does not contain the $\vec{E}$ operator acting to the right:

$$\Sigma'(1',\bar{1}) \equiv \left[\Gamma\begin{pmatrix}1, & 2, & 3 \\ -, & +, & +\end{pmatrix} \vec{E}(\bar{3},\bar{2})C(\bar{2},2) + \Gamma\begin{pmatrix}1, & 2, & 3 \\ -, & -, & +\end{pmatrix} \vec{E}(\bar{3},\bar{2})R(\bar{2};2)\right] R(3;\bar{3})\cdot\vec{\partial}_{\bar{3}}\delta(\bar{3}-\bar{1}) \;. \tag{334}$$

The particle propagator g is then defined to obey

$$\left[(g^{(o)})^{-1} + \Sigma'\right] g = 1 \ . \tag{335}$$

The response function equation can then be written in terms of g:

$$g^{-1}R + \vec{\partial}\overline{f}\cdot\vec{E}R = 1 \ . \tag{336}$$

From Eq. (336),

$$\vec{E}R = (1 + \vec{E}g\cdot\vec{\partial}\overline{f})^{-1}\vec{E}g \ , \tag{337}$$

so we can rewrite Eq. (327) as

$$\varepsilon^{-1} = 1 - (1 + \vec{E}g\cdot\vec{\partial}\overline{f})^{-1}\,\vec{E}g\cdot\vec{\partial}\overline{f}$$

or

$$\varepsilon = 1 + \vec{E}g\cdot\vec{\partial}\overline{f} \ , \tag{338}$$

whereupon

$$\vec{E}R = \varepsilon^{-1}\vec{E}g \ , \tag{339}$$

$$R = g(1 - \vec{\partial}\overline{f}\cdot\varepsilon^{-1}\vec{E}g) \ . \tag{340}$$

The forms (338)-(340) have structure identical to the corresponding results of linear theory . [Compare Eq. (340) with Eq. (157a)]. We see, in fact, that the correct mapping which takes the linear dielectric into the renormalized one is

$$g^{(o)} \to g \ , \tag{341a}$$

$$i\delta \to i\Sigma' \ , \tag{341b}$$

$$f^{(o)} \to \overline{f} \ . \tag{341c}$$

8.3: Dielectric and Propagator in the Direct-Interaction Approximation, and Reduction to Weak Turbulence Theory

In the DIA we have $\Gamma = \gamma$. By construction, the only nonvanishing γ's contain exactly one "-" index, so that in the DIA

$$\Gamma\begin{pmatrix}1 & 2 & 3\\ - , & + , & +\end{pmatrix} = \gamma\begin{pmatrix}1 & 2 & 3\\ - , & + , & +\end{pmatrix} = \overline{U}(1,2,3) \quad , \tag{342a}$$

$$\Gamma(-,-,+) = 0 \; . \tag{342b}$$

Then, if we define the field correlation

$$\mathcal{E}(\underline{1},\underline{2}) \equiv \langle \delta\vec{E}(\underline{1})\,\delta\vec{E}(\underline{2})\rangle = \vec{\varepsilon}(\underline{1},\overline{1})\,\vec{\varepsilon}(\underline{2},\overline{2})\,C(\overline{1},\overline{2}) \quad ,$$

we find

$$\Sigma'(1,\overline{1}) = -\vec{\partial}_1\cdot[R(1;\overline{3})\,\mathcal{E}(\overline{3},1) + \vec{\varepsilon}R(1;\overline{3})\langle\delta\vec{E}(\overline{3})\,\delta f(1)\rangle]\cdot\vec{\partial}_{\overline{3}}\,\delta(\overline{3}-\overline{1}) \quad , \tag{343}$$

$$\vec{\partial}\,\delta\overline{F}(1,\underline{2}) = \int d\vec{v}_2\,[\vec{\partial}_2\langle\delta f(2)\,\delta\vec{E}(\underline{\overline{1}})\rangle\cdot\vec{\partial}_1 R(\overline{1};2) + \vec{\partial}_2\vec{\partial}_{\overline{1}}C(2,\overline{1})\cdot\vec{\varepsilon}R(\overline{1};2)] \tag{344}$$

(DuBois and Espedal 1978). Recalling Eqs. (339) and (340), we recognize two kinds of terms in Eqs. (343) and (344): those containing an explicit factor of ε^{-1} and those missing such a factor. For reasons which will become clear shortly, we call the latter terms "diffusion" terms and denote them by a superscript "d"; we call the former terms "polarization" terms and denote them by "p". We thus write (Krommes and Kleva 1979)

$$\Sigma' = \Sigma^{(d)} + \Sigma^{(p)} \quad , \tag{345}$$

$$\delta\overline{F} = \delta\overline{F}^{(d)} + \delta\overline{F}^{(p)} \quad , \tag{346}$$

where

$$\Sigma^{(d)}(1,\bar{1}) = -\vec{\partial}_1 \cdot [g(1,2)\mathcal{E}(\underline{2},\underline{1})] \cdot \vec{\partial}_2 \delta(2-\bar{1}) \;, \tag{347a}$$

$$\Sigma^{(p)}(1,\bar{1}) = -\vec{\partial}_1 \cdot [(\epsilon^{-1}\vec{E}g)(1,2)<\delta\vec{E}(\underline{2})\delta f(1)>$$

$$- (g\vec{\underline{\partial}}\bar{f}\epsilon^{-1}\vec{E}g)(1,2)\mathcal{E}(\underline{2},\underline{1})] \cdot \vec{\partial}_2 \delta(2-\bar{1}) \;, \tag{347b}$$

$$\vec{\partial}\delta\bar{f}^{(d)}(\bar{1},\underline{2}) = -\int d\vec{v}_2 [\vec{\partial}_2 <\delta f(2)\delta\vec{E}(\underline{\bar{1}})>] \cdot \vec{\partial}_1 g(\bar{1},2) \;, \tag{348a}$$

$$\vec{\partial}\delta\bar{f}^{(p)}(\bar{1},\underline{2}) = -\int d\vec{v}_2 ([\vec{\partial}_2 \vec{\partial}_{\bar{1}} C(2,\bar{1})] \cdot (\epsilon^{-1}\vec{E}g)(\bar{1},2)$$

$$- [\vec{\partial}_2 <\delta f(2)\delta\vec{E}(\underline{\bar{1}})>] \cdot \vec{\partial}_1 (g\vec{\partial}\bar{f}\epsilon^{-1} \cdot \vec{E}g)(\bar{1},2)] \;. \tag{348b}$$

To see that $\Sigma^{(d)}$ deserves its diffusion attribute, consider the action of $\Sigma^{(d)}$ on g:

$$(\Sigma g)(1,1') = \int_0^{\tau} d\bar{\tau} \int_{-\infty}^{\infty} d\bar{\rho} \int_{-\infty}^{\infty} d\vec{\bar{v}} \Sigma^{(d)}(\vec{\bar{\rho}},\bar{\tau},\vec{v};\vec{\bar{v}}) g(\rho-\bar{\rho},\tau-\bar{\tau},\vec{\bar{v}};\vec{v}') \;. \tag{349}$$

If $\Sigma^{(d)}$ is peaked in $\bar{\rho}$ and $\bar{\tau}$ relative to g and if the g in $\Sigma^{(d)}$ is taken to be approximately proportional to $\delta(\vec{v}_2 - \vec{v}')$, we can Markovianize Eq. (349) according to

$$\Sigma g \approx -\vec{\partial} \cdot D(\vec{v}) \cdot \vec{\partial} g(\vec{\rho},\tau,\vec{v};\vec{v}') \;, \tag{350}$$

$$D(\vec{v}) \equiv \int_0^{\infty} d\tau \int_{-\infty}^{\infty} d\vec{\rho} \int_{-\infty}^{\infty} d\vec{v}_2 g(\vec{\rho},\tau,\vec{v};\vec{v}_2)\mathcal{E}(\vec{\rho},\tau)$$

$$= \int_0^{\infty} d\tau \int_{-\infty}^{\infty} d\vec{v}_2 \sum_{\vec{q}} g_{\vec{k}-\vec{q}}(\tau,\vec{v};\vec{v}_2)\mathcal{E}_{\vec{q}}(\tau) \;. \tag{351}$$

The $\vec{k}\to\vec{0}$ limit of expression (351) is the diffusion coefficient used in resonance-broadening theory. $\Sigma^{(d)}$ is the only term which would survive in the stochastic acceleration problem where $\vec{E}$ is specified statistically. However, for the self-consistent problem, the terms $\Sigma^{(p)}$, $\delta\bar{f}^{(d)}$, and $\delta\bar{f}^{(p)}$ are equally important.

To demonstrate the relation of formulas (338) and (345)-(348) to the conventional weak turbulence theory, we expand Eq. (338) in powers of the fluctuation intensity (Krommes 1978a, DuBois and Espedal 1978, Krommes and Kleva 1979). From Eq. (335),

$$g = \left[(g^{(o)})^{-1} + \Sigma'\right]^{-1} = g^{(o)} - g^{(o)}\Sigma' g^{(o)} + \cdots , \qquad (352)$$

where this expansion is valid away from a wave-particle resonance. Equation (338) then becomes

$$\epsilon = 1 + \vec{\mathcal{E}} g^{(o)} \cdot \vec{\partial} \langle f \rangle + \vec{\mathcal{E}} g^{(o)} \cdot (\vec{\partial} \delta \overline{f} - \Sigma' g^{(o)} \vec{\partial} \langle f \rangle) + \cdots , \qquad (353)$$

where $\delta\overline{f}$ and Σ' must be computed to lowest nontrivial order in $\langle \delta E^2 \rangle$. To this end, we use the linear version of the equation for $\langle \delta f \delta f \rangle$,

$$(g^{(o)})^{-1} C + \vec{\partial} \langle f \rangle \cdot \vec{\mathcal{E}} C \simeq 0 , \qquad (354)$$

to find

$$\langle \delta f(1) \delta \vec{E}(1') \rangle = -g^{(o)}(1,\overline{1}) \vec{\partial}_{\overline{1}} \langle f \rangle \mathcal{E}(\underline{\overline{1}},\underline{1}') , \qquad (355a)$$

$$\langle \delta f(1) \delta f(1') \rangle = g^{(o)}(1,\overline{1}) \vec{\partial}_{\overline{1}} \langle f \rangle g^{(o)}(1',\overline{2}) \vec{\partial}_{\overline{2}} \langle f \rangle \mathcal{E}(\underline{\overline{1}},\underline{\overline{2}}) . \qquad (355b)$$

Substituting these results into Eqs. (347) and (348), thence into (353), and finally symmetrizing, we arrive at an expression for the dielectric correct to order E^2. If this expression is Fourier-transformed (assuming quasi-stationary turbulence), one finds the usual result of weak turbulence theory

$$\epsilon = \epsilon^{(\ell)} + \epsilon^{(n)} , \qquad (356)$$

where, if $\Phi_k(\overline{1}) \equiv (4\pi/k^2)(nq)_{\overline{s}}$,

$$\varepsilon^{(\ell)} \equiv 1 - i\Phi_k g_k^{(o)}\vec{k}\cdot\vec{\partial}\langle f\rangle \tag{357}$$

is the linear dielectric and

$$\varepsilon_k^{(n)} \equiv 2\sum_q[\varepsilon^{(3)}(k|q,k,-q) - 2\varepsilon^{(2)}(k|q,k-q)\varepsilon_{k-q}^{-1}\varepsilon^{(2)}(k-q|-q,k)]I_q \tag{358}$$

is the nonlinear dielectric correct to $O(E^2)$;

$$\varepsilon^{(2)}(k|k_1,k_2) \equiv \tfrac{1}{2}\Phi_k(1)g_k^{(o)}(1;2)[\vec{k}\cdot\vec{\partial}_2\, g_{k_2}^{(o)}(2;3)\vec{k}_2\cdot\vec{\partial}_3\langle f(3)\rangle + (k_1\leftrightarrow k_2)] \;, \tag{359}$$

$$\varepsilon^{(3)}(k|k_1,k_2,k_3) \equiv \tfrac{1}{2}i\Phi_k(1)g_k^{(o)}(1;2)\vec{k}_1\cdot\vec{\partial}_2 g_{k-k_1}^{(o)}(2;3)$$
$$\times[\vec{k}_2\cdot\vec{\partial}_3 g_{k_3}^{(o)}(3;4)\vec{k}_3\cdot\vec{\partial}_4\langle f(4)\rangle + (k_2\leftrightarrow k_3)] \;. \tag{360}$$

That contact with weak turbulence theory can be established so easily is an important virtue of the present formalism.

In Eqs. (358)-(360), $\Sigma^{(d)}$ gives rise to the first term of $\varepsilon^{(3)}$, while $\delta\bar{f}^{(d)}$ accounts for the proper symmetrization of $\varepsilon^{(3)}$. The polarization terms $\Sigma^{(p)}$ and $\delta\bar{f}^{(p)}$ together account for the terms in $\varepsilon^{(2)}$. It is, of course, well-known (Tsytovich 1972, 1977) that in weak turbulence theory the term in $\varepsilon^{(3)}$, describing Compton scattering (from bare test particles) is partly cancelled, at long wavelengths, by the term in $\varepsilon^{(2)}$, describing nonlinear scattering (from the shielding clouds). In the next subsection we show that a similar cancellation can occur in the renormalized theory.

8.4: Dielectric Function of Guiding Center Plasma

In general, expressions (347) and (348) are extremely complicated because of their nonlocal velocity dependence. To illustrate the importance, in the renormalized theory, of the additional terms $\Sigma^{(p)}$, $\delta\bar{f}^{(d)}$, and $\delta\bar{f}^{(p)}$ introduced by self-consistency, we consider a nontrivial example in which this velocity dependence is absent.

The example is the Liouville equation for the fluctuating charge density $\tilde{\rho}$ of guiding center plasma,

$$\partial_t \tilde{\rho} + \tilde{\vec{v}}_E \cdot \vec{\nabla} \tilde{\rho} = 0 \ , \tag{361a}$$

$$\tilde{\vec{v}}_E = (c/B) \hat{n} \times \vec{\nabla} \tilde{\phi} \ , \tag{361b}$$

$$\nabla^2 \tilde{\phi} = -4\pi \tilde{\rho} \ . \tag{361c}$$

If we make the transcriptions

$$f \to \tilde{\rho} \ , \tag{362a}$$

$$\Phi_k \to 4\pi/k^2 \ , \tag{362b}$$

$$\vec{\partial} \to (c/B) \hat{n} \times \vec{\nabla} = (c/B) \hat{n} \times (i\vec{k}) \ , \tag{362c}$$

Eqs. (347) and (348) can be taken over directly. The details are presented by Krommes and Similon (1979). They show that $\Sigma^{(d)}$ and $\Sigma^{(p)}$ combine in such a way that the order of Σ' is raised by $(k\lambda_D)^2$ from the order of $\Sigma^{(d)}$. Similarly, $\delta\overline{f}^{(d)}$ and $\delta\overline{f}^{(p)}$ combine. In the special but interesting case of thermal equilibrium, the result can be put into the form

$$\varepsilon = 1 + \left(\frac{1}{k^2 \lambda_D{}^2}\right) \left[1 + \frac{i\omega}{-i\omega + \left(\frac{k^2 \lambda_D{}^2}{k^2 \lambda_D{}^2 + 1}\right) k^2 D_k}\right] \ , \tag{363}$$

where D_k is a weakly-varying function of k which scales as the Bohm-like result (103). The dispersion relation associated with Eq. (363) is

$$\omega = -ik^2 D_k \ , \tag{364}$$

reasonably indicating a diffusive-like damping of disturbances. (The transport is not local, however; see Sec. 4.2 .)

Had $\delta\bar{f}$ been ignored, only the vacuum term "1" would have survived. Formula (363) agrees in form with a result of Taylor (1974), who used very different techniques. In a resonance-broadening theory applied to the non-adiabatic part of the distribution (Lee and Liu 1973), the factor $k^2\lambda_D{}^2/(k^2\lambda_D{}^2+1)$ would be replaced by unity. The associated dispersion relation would be

$$\omega = -i(k^2 + k_D{}^2)D|_{k=0}, \tag{365}$$

which approaches a constant as $k \to 0$, in contradiction to both numerical experiment and intuitive arguments about eddy diffusion.

8.5: Aspects of the Drift Wave Dielectric

Renormalization techniques, mostly along the lines of resonance-broadening theory, have been applied for many years to the important problem of the nonlinear theory of drift waves. Because it can be estimated (Dupree 1967) that three-wave coupling is relatively unimportant as a saturation mechanism, attention has focussed on the dielectric response. We shall describe a few of the salient features of these calculations. Confusion in the literature over the proper formulation of the renormalized drift wave theory precludes a detailed calculation here.

For the slab model with no magnetic shear, Dupree (1967) noted that, to avoid the strongly stabilizing effects of wave-particle interactions, the waves are confined in linear theory to the range $v_{ti} \ll \omega/k \ll v_{te}$. However, in the presence of turbulence, the phase velocity spectrum can broaden into the ion distribution function, at which point substantial ion stochasticity and diffusion will ensue and, possibly, ultimately saturate the waves. The required broadening is of order $\delta v \sim v_d - v_{ti} \sim v_d$, where $v_d \equiv (cT/eBL_n)$ is the diamagnetic velocity. Inasmuch as the velocity fluctuations arise from $\vec{E}\times\vec{B}$ drifts, one estimates that

$$\langle (e\phi/T_e)^2 \rangle^{\frac{1}{2}} \sim (k_\perp L_n)^{-1} , \tag{366}$$

an oft-quoted estimate for the fluctuation level at saturation, which also corresponds to density gradient fluctuations of order the background gradient.

Dupree (1968) attempted to estimate the ion diffusion coefficient associated with Eq. (366). In a Markovian theory which neglected the polarization terms, he found

$$k_\perp{}^2 D/\omega \sim \begin{cases} (R^2 - 1)^{\frac{1}{2}} & (R \gtrsim 1), \\ 0 & (R < 1), \end{cases} \tag{367}$$

where the effective Reynolds number R, essentially the ratio of linear to nonlinear coherence time, scaled like $ck_\perp{}^2 \langle \phi^2 \rangle^{\frac{1}{2}}/\omega B$. For $\gamma^{(\ell)} \ll \omega$, then, arguments based on $\gamma = \gamma^{(\ell)} - k_\perp{}^2 D$ led to $R \sim 1$ [which corresponds to (376)] and a very small D. However, for $R \gg 1$, $D \sim \omega R/k_\perp{}^2 \sim (cT/eB)\langle (e\phi/T_e)^2 \rangle^{\frac{1}{2}}$, a Bohm-like or strong turbulence result. The transition occurs at $R \sim 2$.

For $\gamma^{(\ell)} \lesssim \omega$ we will have $1 < R \lesssim 2$ at saturation. A potential problem with the above estimates is, then, that the system may be too close to the transition to turbulence for the simple statistical theories to be valid. We cannot enter here into this matter, largely unexplored for plasmas. Krommes (1979a) attempted to salvage the results by arguing that for $R < 1$ the beat resonances of induced scattering provide the necessary stochasticity in the ion distribution, but the issue seems to be still in doubt.

The Dupree (1967,1968) theories were not energy-conserving. This problem was first addressed by Galeev (1967), who effectively showed that $\delta\overline{f}$ was necessary for a proper theory. Dupree and Tetreault (1978) rederived this result for the drift wave problem and effectively computed $\delta\overline{f}^{(d)}$. They concluded that the nonlinear damping on each mode was reduced by a factor $(k_{||}v_{ti}/\omega)^2$ from the simple $k_\perp^{\,2}D$ estimate. Krommes (1979a) reconsidered the problem, including more carefully the wavenumber dependence of $\Sigma^{(d)}$, and concluded that the individual modal damping was not as predicted by Dupree and Tetreault. Rather, some γ_k's must of necessity be positive while others are negative in order that energy conservation hold. However, the net wave-particle energy transfer, $-\sum(\mathrm{Im}\,\varepsilon)\times k^2\langle|\phi_k|^2\rangle/4\pi$, was reduced by $(\overline{k}_{||}v_{ti}/\omega)^2$, $\overline{k}_{||}$ being a typical parallel wavenumber, in accordance with the prediction of Dupree and Tetreault. Krommes' calculation also indicated a tendency for energy transfer to long wavelengths, causing concern about the possible inability of the system to reach saturation. However, Krommes, as well as Dupree and Tetreault, ignored both the polarization terms, the noise or mode-coupling terms in $\tilde{F}$ (see Sec. 9), and the electron dynamics.

Hirshman and Molvig (1979) considered the universal instability in sheared slab geometry, known to be stable in linear theory. They followed Catto (1978) [and, implicitly, Galeev (1967)] by applying resonance-broadening theory to the non-adiabatic part of the electron distribution. They concluded that small amounts of turbulence would destabilize the modes, but also that larger amounts would stabilize it. A similar theory (Molvig et al. 1979) applied to the finite-β drift wave predicted rather remarkable agreement with a number of distinctive features of several confinement experiments.

We shall not discuss here the detailed relation between our renormalized equations and the prescription of resonance-broadening only the non-adiabatic distribution. However, it can readily be seen from the example of the guiding center plasma that the latter arises from the former only by including $\delta\bar{f}$. Now it can be shown that Hirshman and Molvig included only part of $\delta\bar{f}^{(d)}$, giving rise to substantial violation of energy and enstrophy conservation (Similon and Krommes 1979); they completely ignored the polarization terms. At the time of writing, then, it remains a challenge to justify in detail their tempting predictions.

9: SELF-CONSISTENT FLUCTUATIONS and the BALANCE EQUATION

9.1: Incoherent Noise and the Balance Equation

The dielectric response we discussed in the last section was defined in terms of an externally imposed infinitesimal electric field source. However, as we emphasized in Sec. 6, an internal fluctuation can also play the role of the source to which the medium responds. This response will be both self-consistent and nonlinear. The equation which describes the resulting fluctuations is sometimes called the balance equation. In this section we derive the balance equation and interpret it physically. Unfortunately, we cannot be complete, since this equation remains quite poorly understood for plasmas; most of the interesting applications remain to be made.

It turns out to be very useful to discuss the self-consistent fluctuations in terms of concepts familiar from the theory of test particle response and superposition (Oberman, Chap. 2.3). We shall think of the fluctuation δf as consisting of two parts, a "coherent" part $\delta f^{(c)}$ and an "incoherent noise" part $\delta\tilde{f}$:

$$\delta f = \delta f^{(c)} + \delta\tilde{f} \tag{368}$$

(Dupree 1972, Krommes 1978a, DuBois and Espedal 1978, Krommes 1979b). The coherent fluctuation describes the induced response to an internal fluctuation $\delta\vec{E}$ and is defined by

$$\delta f^{(c)} \equiv -g\vec{\partial}\bar{f}\cdot\delta\vec{E} \tag{369}$$

(Dupree 1972). (Dupree actually uses $\langle f \rangle$ instead of $\bar{f}$.) The incoherent fluctuation is to be thought of as a "bare" fluid element, with which is associated an incoherent field $\delta\tilde{\vec{E}}$ related to the total plasma field by dielectric shielding. That is, according to Eq. (368) the total field is

$$\delta\vec{E} = \vec{E}\,\delta f^{(c)} + \delta\tilde{\vec{E}} \; ; \tag{370}$$

upon noting the definition (338) of the dielectric, we find

$$\delta\vec{E} = \delta\tilde{\vec{E}}/\varepsilon \, . \tag{371}$$

We shall further postulate that the fluctuation $\delta\tilde{f}$ propagates with the single particle propagator g (the natural generalization of test particle streaming in the simplest linear theories), so that $g^{-1}\delta\tilde{f}$ represents the "initial" incoherent disturbance. With this interpretation, we can use Eqs. (369) and (370) to find

$$\delta f = R(g^{-1}\delta\tilde{f}) \ ; \tag{372}$$

that is, the plasma response to an incoherent disturbance develops with the full response function R. Of course, we have yet to give an explicit formula for $\delta\tilde{f}$.

It must be stressed that the above formulas are in some sense meaningless, since we are mixing observable quantities (g, R, ε) with random variables. However, the relations acquire precise meaning when correlation functions are formed. Thus, for example. Eq. (371) leads to

$$\mathcal{E} \equiv \langle\delta\vec{E}\delta\vec{E}\rangle = \frac{\langle\delta\tilde{\vec{E}}\delta\tilde{\vec{E}}\rangle}{|\varepsilon|^2} \ , \tag{373}$$

which states that the true (measurable) spectrum is the shielded incoherent spectrum. The fundamental correlation function in this approach is $\tilde{C} \equiv \langle\delta\tilde{f}\delta\tilde{f}\rangle$. The medium responds to $\tilde{C}$ according to Eq. (372), giving the total fluctuation spectrum as

$$C = R\left[g^{-1}\tilde{C}(g^{-1})^t\right]R^t \ . \tag{374}$$

We can determine an explicit formula for $\tilde{C}$ by comparing Eq. (374) to the formal solution of the Dyson equations we have derived previously. Indeed. Eq. (210) can be written

$$C = R\tilde{F}R^t, \tag{375}$$

where $\tilde{F} \equiv -\Sigma_{--}$. [Compare Eq. (375) with Eq. (135).] Upon comparing Eq. (374) with Eq. (375), we then identify

$$\tilde{C} = g\tilde{F}g^t \ . \tag{376}$$

Such a relation was first given by Krommes (1978a). Because the entire functional apparatus is available to determine the form of $\tilde{F}$, the relation (376) together with Eqs. (375) and (373) forms a nonperturbative and formally exact statement of the original heuristic ideas about coherent and incoherent response advanced by Dupree.

To emphasize the consistency of our definitions, let us construct the mixed correlation $\langle\delta f\delta\vec{E}\rangle$:

$$\langle\delta f\delta\vec{E}\rangle = R\tilde{F}(\epsilon^{-1}\vec{E}g)^t$$

$$= (g\tilde{F}g^t)(\epsilon^{-1}\vec{E})^t - g\vec{\partial}\bar{f}\cdot\epsilon^{-1}\vec{E}(g\tilde{F}g^t)(\epsilon^{-1}\vec{E})^t$$

$$\equiv \langle\delta\tilde{f}\delta\vec{E}\rangle + \langle\delta f^{(c)}\delta\vec{E}\rangle \ . \quad (377)$$

This is consistent with Eq. (368). The second term can also be written as

$$\langle\delta\vec{E}\delta f^{(c)}\rangle = \epsilon^{-1}\vec{E}\langle\delta\tilde{f}\delta f^{(c)}\rangle \ . \quad (378)$$

Dupree (1972) argued that $\langle\delta\tilde{f}\delta f^{(c)}\rangle$ should vanish because of his interpretation of $\delta\tilde{f}$ as a highly random function. This argument is incorrect (Krommes 1978a, 1979b) in view of the self-consistency: because the coherent response is related to the shielded incoherent response,

$$\delta f^{(c)} = -g\vec{\partial}\bar{f}\cdot\epsilon^{-1}\vec{E}\delta\tilde{f} \ , \quad (379)$$

$\langle\delta\tilde{f}\delta f^{(c)}\rangle$ is proportional to $\langle\delta\tilde{f}\delta\tilde{f}\rangle \neq 0$. This fact has implications for certain of the "clump" theories (Krommes 1979c).

The incoherent noise correlation $\tilde{C}$ can be determined only through explicit solution of the coupled Dyson equations for C and R.

However, one can gain further insight by considering various approximations to the form of $\tilde{F}$. In the DIA, one has

$$\tilde{F} = \mathcal{E}:\vec{\partial}\vec{\partial}C + \vec{\partial}\cdot\langle\delta f\delta\vec{E}\rangle\cdot\vec{\partial}\langle\delta\vec{E}\delta f\rangle \ . \tag{380}$$

Inserting Eq. (375) in the form

$$C = \langle\delta f^{(c)}\delta f^{(c)}\rangle + \langle\delta f^{(c)}\delta\tilde{f}\rangle + \langle\delta\tilde{f}\delta f^{(c)}\rangle + \tilde{C} \ , \tag{381}$$

we obtain (DuBois and Espedal 1978)

$$\begin{aligned}\tilde{F} = {} & \mathcal{E}:\vec{\partial}\vec{\partial}\langle\delta f^{(c)}\delta f^{(c)}\rangle + \vec{\partial}\cdot\langle\delta f^{(c)}\delta\vec{E}\rangle\cdot\vec{\partial}\langle\delta\vec{E}\delta f^{(c)}\rangle \\ & + \mathcal{E}:\vec{\partial}\vec{\partial}\langle\delta t^{(c)}\delta\tilde{f}\rangle + \vec{\partial}\cdot\langle\delta f^{(c)}\delta\vec{E}\rangle\cdot\vec{\partial}\langle\delta\vec{E}\delta\tilde{f}\rangle \\ & + \mathcal{E}:\vec{\partial}\vec{\partial}\langle\delta\tilde{f}\delta f^{(c)}\rangle + \vec{\partial}\cdot\langle\delta\tilde{f}\delta\vec{E}\rangle\cdot\vec{\partial}\langle\delta\vec{E}\delta f^{(c)}\rangle \\ & + \mathcal{E}:\vec{\partial}\vec{\partial}\tilde{C} + \vec{\partial}\cdot\langle\delta\tilde{f}\delta\vec{E}\rangle\cdot\vec{\partial}\langle\delta\vec{E}\delta\tilde{f}\rangle \ .\end{aligned} \tag{382}$$

If we ignore the possibility that the velocity derivatives may change the nominal order of the terms, then the coherent parts of Eq. (381) dominate, for then $\tilde{F}\sim\mathcal{E}^2$, $\tilde{C}\sim\mathcal{E}^2$. Retaining only the coherent parts, one finds

$$\begin{aligned}\tilde{F}(1,\bar{1}) \simeq \tilde{F}_m \equiv {} & [\vec{\partial}_1(g\,\vec{\partial}\bar{f}\,)(1,2)][\vec{\partial}_{\bar{1}}(g\,\vec{\partial}\bar{f}\,)(\bar{1},\bar{2})] \\ & \times[\mathcal{E}(1,\bar{1})\mathcal{E}(2,\bar{2}) + \mathcal{E}(1,\bar{2})\mathcal{E}(2,\bar{1})] \ .\end{aligned} \tag{383}$$

This expression is most easily recognized by constructing the associated incoherent field and by assuming stationary turbulence, so that one can rigorously Fourier-transform. Introducing an obvious symmetrization, one finds

$$\langle\delta\tilde{\phi}^2\rangle_k = 2 \sum_{k_1+k_2=k} |\varepsilon^{(2)}(k|k_1,k_2)|^2 I_{k_1} I_{k_2} \tag{384}$$

which, when I_k is approximated as $2\pi I_{\vec{k}}\delta[\omega-\omega(\vec{k})]$, reduces readily to the usual three-wave decay term of weak turbulence theory. The corresponding wave-kinetic equation, appropriate for a medium weakly inhomogeneous in space-time, is written by Krommes and Kleva (1979).

9.2: Phase Space Granulation

The mode-coupling approximation considered in the last subsection contains no hint of the stochastic instability of particle orbits--that is, one sees in Eqs. (383) or (384) no exponential divergence of neighboring phase space elements. As Dupree (1972) has pointed out (in different language), this information is contained in the terms of $\tilde{F}$ [Eq. (382)] which explicitly involve incoherent fluctuations. Let us write $\tilde{F} \equiv \tilde{F}_m + \tilde{F}'$ and consider the equation for $\tilde{C}$ in the form

$$g^{-1}\tilde{C} - \tilde{F}'g^t = \tilde{F}_m g^t \ . \tag{385}$$

For purposes of dimensional analysis, let us approximate

$$\tilde{F}' \approx \mathcal{E}:\vec{\partial}\vec{\partial}\tilde{C} \ . \tag{386}$$

(The other term of $\tilde{F}'$ in $\delta\tilde{f}^2$ is equally important; the terms in $\langle\delta\tilde{f}\delta f^{(c)}\rangle$ represent polarization effects which may also be important, especially at long wavelengths.) For stationary turbulence, Dupree has written in a Markovian approximation

$$(\tilde{F}'g^t)(\vec{x},\vec{v},\tau;\overline{\vec{v}}) = -\vec{\partial}\cdot D(\vec{x},\vec{v})\cdot\vec{\partial}C(\vec{x},\vec{v},\tau;\overline{\vec{v}}) , \tag{387}$$

$$D(\vec{x},\vec{v}) \equiv \sum_q \exp(i\vec{q}\cdot\vec{x})q^2 I_q g_q^*. \tag{388}$$

If one now reinstates a weak dependence on the mean time T and considers the evolution of $\tilde{C}(\tau=0|T)$, he finds an equation of the form

$$g_2^{-1}(T)\tilde{C}(0|T) = (\tilde{F}_m g^t)(T) , \tag{389}$$

where g_2 is a two-particle propagator, approximated here by

$$\left(\frac{\partial}{\partial T} + \vec{v}_1\cdot\vec{\nabla}_1 + \vec{v}_2\cdot\vec{\nabla}_2 + \sum_{i,j=1,2}\frac{\partial}{\partial\vec{v}_i}\cdot D_{ij}\cdot\frac{\partial}{\partial\vec{v}_j}\right) g_2(\underset{\sim}{1},\underset{\sim}{2},T;\underset{\sim}{1}',\underset{\sim}{2}',T')$$
$$= \tfrac{1}{2}[\delta(\underset{\sim}{1}-\underset{\sim}{1}')\delta(\underset{\sim}{2}-\underset{\sim}{2}') + \delta(\underset{\sim}{1}-\underset{\sim}{2}')\delta(\underset{\sim}{2}-\underset{\sim}{1}')]\delta(T-T') , \tag{390}$$

where $D_{ij} \equiv D(\vec{x}_i - \vec{x}_j)$.

To understand the physics of g_2, it is convenient to introduce relative and centrix coordinates in both velocity and position space:

$$\vec{v}_- \equiv \vec{v}_1 - \vec{v}_2 , \qquad \vec{x}_- \equiv \vec{x}_1 - \vec{x}_2 ,$$
$$\vec{v}_+ \equiv \tfrac{1}{2}(\vec{v}_1 + \vec{v}_2) , \qquad \vec{x}_+ \equiv \tfrac{1}{2}(\vec{x}_1 + \vec{x}_2) . \tag{391}$$

Equation (390) then becomes

$$\left(\frac{\partial}{\partial T} + \vec{v}_-\cdot\frac{\partial}{\partial\vec{x}_-} + \vec{v}_+\cdot\frac{\partial}{\partial\vec{x}_+} - \sum_{\nu,\mu=+,-}\frac{\partial}{\partial\vec{v}_\nu}\cdot D_{\nu\mu}\cdot\frac{\partial}{\partial\vec{v}_\mu}\right) g$$
$$= \tfrac{1}{2}[\delta(X_- - X_-') + \delta(X_- + X_-')]\delta(X_+ - X_+')\delta(T-T') , \tag{392}$$

where $X \equiv \{\vec{x},\vec{v}\}$ and

$$D_- \equiv D_{11} + D_{22} - (D_{12} + D_{21}) \,, \tag{393a}$$

$$D_{\pm\mp} = \tfrac{1}{2}[D_{11} - D_{22} \mp (D_{12} - D_{21})] \,, \tag{393b}$$

$$D_+ = \tfrac{1}{4}(D_{11} + D_{12} + D_{21} + D_{22}) \,. \tag{393c}$$

Because D_{12} depends on x_-, the diffusion coefficients behave differently in the limits of large and of small x_-. If we ignore the velocity dependence of the diffusion coefficients, and if k_0 is a typical wavenumber of the turbulence, then we find the limiting forms

$$\left.\begin{aligned} &D_{12} \to 0 \,, && (394a) \\ &D_- \to 2D \,, && (394b) \\ &D_{+-},\, D_{-+} \to 0 \,, && (394c) \\ &D_+ \to \tfrac{1}{2}D \,; && (394d) \end{aligned}\right\} \quad (k_0 x_- \gg 1)$$

and

$$\left.\begin{aligned} &D_{12} \to D \,, && (395a) \\ &D_- \to 0 \,, && (395b) \\ &D_{+-},\, D_{-+} \to 0 \,, && (395c) \\ &D_+ \to D \,. && (395d) \end{aligned}\right\} \quad (k_0 x_- \ll 1)$$

Because of property (394a), in the limit of large relative separation the solution of Eq. (391) factors into the product of one-body propagators, signifying independent diffusion of the two phase space elements:

$$g_2(\underset{\sim}{1},\underset{\sim}{2},T;\underset{\sim}{1}',\underset{\sim}{2}') \to \tfrac{1}{2}[U_+(\underset{\sim}{1},T;\underset{\sim}{1}')U_+(\underset{\sim}{2},T;\underset{\sim}{2}') + (\underset{\sim}{1}\leftrightarrow\underset{\sim}{2})]H(T) \quad , \quad (396)$$

where U_+ is defined by

$$\left(\frac{\partial}{\partial T} + \vec{v}_1\cdot\vec{\nabla}_1 - \frac{\partial}{\partial\vec{v}_1}\cdot D\cdot\frac{\partial}{\partial\vec{v}_1}\right) U_+(\underset{\sim}{1},T;\underset{\sim}{1}') = 0 \ , \quad (397a)$$

$$U_+(\underset{\sim}{1},T=0;\underset{\sim}{1}') = \delta(\underset{\sim}{1}-\underset{\sim}{1}') \ . \quad (397b)$$

However, at small x_- the physics is more interesting and describes correlated diffusion.

For $k_0x_-<<1$ we may ignore D_{+-} and D_{-+} in Eq. (392). The solution of the resulting equation is

$$g_2 = U_+(X_+,T;X_+')U_-(X_-,T;X_-')H(T) \ , \quad (398)$$

where the relative propagator U_- obeys

$$\left(\frac{\partial}{\partial T} + \vec{v}_-\cdot\frac{\partial}{\partial\vec{x}_-} - \frac{\partial}{\partial\vec{v}_-}\cdot D_-(\vec{x}_-)\cdot\frac{\partial}{\partial\vec{v}_-}\right) U_-(X_-,T;X_-') = 0 \ , \quad (399a)$$

$$U_-(X_-,T=0;X_-') = \tfrac{1}{2}[\delta(X_- - X_-') + \delta(X_- + X_-')] \ . \quad (399b)$$

The propagator U_+ describes ordinary turbulent diffusion of the centrix coordinates. Observe that if the two fluid elements are initially coincident ($X_-'=0$), they remain so indefinitely since they have no initial relative motion ($\vec{v}_-=0$) and feel the same forces ($\vec{x}_-=0$)--mathematically,

$$U_-(X_-,T;X_-') = \tfrac{1}{2}[\delta(\vec{x}_- - \vec{v}_-T - \vec{x}_-')\delta(\vec{v}_- - \vec{v}_-') + \delta(\vec{x}_- - \vec{v}_-T + \vec{x}_-')\delta(\vec{v}_- + \vec{v}_-')] \quad (D_-=0)$$

$$= \delta(\vec{x}_-)\delta(\vec{v}_-) \qquad (\vec{v}_- = \vec{x}_- = 0) \ . \quad (400)$$

To understand the motion when the two elements are slightly separated, we approximate D_- by its first nonvanishing term in a Taylor expansion about $x_- = 0$,

$$D_-(x_-) \simeq (k_0 x_-)^2 D'' \ , \quad k_0{}^2 D'' \equiv \partial^2 D_-/\partial x_-{}^2 |_{x_-=0} \ , \tag{401}$$

and consider the first and second moments of the resulting equation for U_-. To this end, define an averaging operation by

$$\langle a(\vec{x}_-,\vec{v}_-,T)\rangle \equiv \int d\vec{x}_- d\vec{v}_- a(\vec{x}_-,\vec{v}_-) U_-(\vec{x}_-,\vec{v}_-,T;\vec{x}_-{}',\vec{v}_-{}') \ . \tag{402}$$

One then finds, from Eq. (399a) with Eq. (401), that $\langle 1\rangle = 1$ --that is, probability is properly conserved--and that $\langle\vec{x}_-\rangle = \langle\vec{v}_-\rangle = 0$. Specializing to one dimension for simplicity, one finds also that

$$\frac{\partial}{\partial T}\langle x_-{}^2\rangle - 2\langle x_- v_-\rangle = 0 \ , \tag{403a}$$

$$\frac{\partial}{\partial T}\langle x_- v_-\rangle - \langle v_-{}^2\rangle = 0 \ , \tag{403b}$$

$$\frac{\partial}{\partial T}\langle v_-{}^2\rangle - 2k_0{}^2 D''\langle x_-{}^2\rangle = 0 \ , \tag{403c}$$

which combine to give

$$\frac{\partial^3}{\partial T^3}\langle x_-{}^2\rangle - \left(\frac{2}{\tau_K}\right)^3\langle x_-{}^2\rangle = 0 \ , \tag{404}$$

where

$$\tau_K \equiv (\tfrac{1}{2}k_0{}^2 D'')^{-1/3} . \tag{405}$$

Equation (404) was first given by Dupree (1972). With $\langle x_-{}^2\rangle \sim \exp(sT)$, the characteristic growth rates for Eq. (404) are

$$s = 2\tau_K{}^{-1}\{1, \ -\tfrac{1}{2}(1 \pm i\sqrt{3})\} \ . \tag{406}$$

We identify the growing solution $<x_-{}^2> \sim \exp(2T/\tau_K)$ with the exponential orbit divergence we expect, and thus identify $\tau_K{}^{-1}$ with the K-entropy. Exact solution of Eq. (404) is straightforward but uninteresting. Asymptotically (Dupree 1972),

$$<x_-{}^2> \sim {}^1\!/_3[(x_-')^2 + 2x_-'v_-'\tau_K + 2(v_-')^2\tau_K{}^2]\exp(2T/\tau_K) \quad . \qquad (407)$$

This solution is valid for $k_o{}^2<x_-{}^2><<1$. Let us follow Dupree (1972) and define the "clump lifetime" τ_{c1} as the time for two orbits initially separated by (x_-',v_-') to diverge one wavelength, $k_o{}^2<x_-{}^2> \sim 1$. Then

$$\tau_{c1} = \begin{cases} \tau_K \ \ln\left(\{3k_o{}^{-2}[(x_-')^2 + 2x_-'v_-'\tau_K + 2(v_-')^2\tau_K{}^2]^{-1}\}^{1/2}\right) & (\tau_{c1} > \tau_K) \\ 0 & (\tau_{c1} < \tau_K) \ . \end{cases} \qquad (408)$$

Equations (403) have a ready physical derivation. (See Krommes et al. (1978), where an analogous calculation is performed in detail for the problem of stochastic magnetic fields.) Consider the exact equations for the infinitesimal separations Δx, Δv between two adjacent orbits:

$$\frac{d}{dt}\Delta x = \Delta v \ , \qquad (409a)$$

$$\frac{d}{dt}\Delta v = (q/m)\left[\frac{\partial E[x(t),t]}{\partial x(t)}\right]\Delta x \ . \qquad (409b)$$

The second moments of Eqs. (409) obey

$$\frac{d}{dt}<\Delta x^2> = 2<\Delta x\Delta v> \ , \qquad (410a)$$

$$\frac{d}{dt}<\Delta x\Delta v> = <\Delta v^2> + <(q/m)E'\Delta x^2> \ , \qquad (410b)$$

$$\frac{d}{dt}<\Delta v^2> = 2<(q/m)E'\Delta x\Delta v> \ , \qquad (410c)$$

where $E'(t) \equiv \partial E[x(t),t]/\partial x(t)$. If Δv in Eq. (410c) is expressed in terms of $E'\Delta x$ by integration of Eq. (409b), and if a quasinormal approximation is made, Eqs. (403) emerge by neglecting $\langle E'\Delta x^2 \rangle$ (which is small in τ_{ac}/τ_K) and with

$$D'' \equiv (q/m)^2 \int_0^\infty d\tau \left\langle \left(\frac{\partial E[x(\tau),\tau]}{\partial [k_0 x(\tau)]}\right)\left(\frac{\partial E[x(0),0]}{\partial [k_0 x(0)]}\right)\right\rangle \quad . \tag{411}$$

We can thus summarize our investigations of two-point propagation as follows. Phase space elements more than a characteristic wavelength apart move independently, each undergoing turbulent diffusion away from a free-streaming trajectory. Phase space elements closer than a characteristic wavelength diverge exponentially rapidly from each other, because of the stochastic instability, in a characteristic time $\tau_{cl} \sim \tau_K$.

We shall not discuss here the attempts at practical computations based on Eq. (385), in part because of some apparent disagreement between the points of view presented by us (Krommes 1979b, DuBois and Espedal 1978) and by Dupree (1972) concerning the proper way of computing the incoherent noise. Dupree has examined the role of clumps in ion acoustic turbulence (Dupree 1970) and in drift turbulence (Dupree 1978). Numerical simulations support a number of the physical assertions about clumps and granulation (Hui and Dupree 1975, Dupree et al. 1975).

10: MISCELLANEOUS

10.1: Renormalization and Stochastic Magnetic Fields

We comment briefly here on an application of apparent importance in both fusion and astrophysics research: particle transport in stochastic magnetic fields (Krommes et al. 1978 and refs. therein, Krommes 1978b). It is assumed that a sheared stable equilibrium magnetic field exists in some region of space. In the simplest problem, this equilibrium is disturbed by very small directional perturbations, not self-consistent with the equilibrium, which resonate at many neighboring rational surfaces. If the perturbation amplitude is large enough, the resonances will overlap and, as is well-known, the fields will become stochastic. It is of interest to determine the diffusion rate of the fields as well as of test particles placed into the field configuration.

In more complicated situations, the particle diffusion must be made self-consistent with the magnetic perturbations. Such problems can be attacked using the statistical methods we have already developed. However, because little work has been done along these lines, we shall not discuss such applications here. [For some preliminary calculations, see Kleva (1979) and Kleva et al. (1979).]

A quasilinear description of the field lines is often adopted (Rosenbluth et al. 1966). The justification is similar to that given for the problem of test particle diffusion in a stochastic wave field discussed in Sec. 3. The magnetic field is described by a Hamiltonian whose canonical equations are $d\vec{x}\times\vec{B} = 0$. The perturbation Hamiltonian is expressed in terms of a summation over parallel wavenumber $k_{\parallel} \equiv \vec{k}\cdot\vec{B}/B$. The trapping width around each rational surface $[k_{\parallel}(x) = 0]$ is found, and the stochasticity criterion identified. The K-entropy of the field is determined by techniques similar to those described in Sec. 9--namely, equations are found for the separations between two infinitesimally separated lines, and statistically analyzed for the behavior of the second moment of the separation as a function of distance along an unperturbed line. A difference between the test particle and field line problems emerges in the computation of the linear decay of correlations. In both cases, this is determined by the width Δk of the wavenumber spectrum. However, for the electrostatic wave case, Δk is ultimately determined by the dispersive properties of the waves, which limit the spectrum in k space. In the field-line case, however, Δk is determined by the degree of spatial localization of the Fourier modes aroung $k_{\parallel}(x) = 0$, through an uncertainty relation between distances perpendicular and parallel to the lines. Quasilinear diffusion theory for the lines is then valid if the correlation distance $2\pi/\Delta k_{\parallel}$ is less than the exponentiation length. We write $\langle \Delta x^2 \rangle = 2D_m|z|$.

Now let test particles be added to the stochastic field configuration, and let the guiding center approximation be enforced in which the particles exactly follow the lines in the absence of collisions. If the particles are completely collisionless, then time and parallel distance are related according to $z = v_{\parallel}t$ and the test particle diffusion coefficient is $D \sim D_m |v_{\parallel}|$ (Jokipii and Parker 1969). However, if the particles are collisional so that $z^2 \sim 2D_{\parallel}t$, a naive estimate would predict that $\Delta x^2/t \sim D_m (D_{\parallel}/t)^{1/2}$, so that no asymptotic diffusion coefficient would exist. However, particles can also become decorrelated from a given line by cross-field diffusion. When this is taken into account, an asymptotic diffusion coefficient can be defined (Rechester and Rosenbluth 1978). However, the decorrelation time τ_c is not simply $\tau_{\perp} \equiv (k_{\perp}^2 D_{\perp})^{-1}$; because of the stochastic instability, exponential divergence dominates and τ_c scales as τ_K, being only logarithmically sensitive to $\tau_{\perp}$ (Rechester and Rosenbluth 1978, Krommes et al. 1978).

The DIA will incorrectly predict $\tau_c \sim \tau_\perp$ because of the underlying Gaussian factorization of K. A theory which predicts $\tau_c \sim \tau_K$ must retain intrinsic two-body correlations in order to describe the correlation of a particle with a specific line (which is stochastically unstable). Put more generally, any problem whose physics rests on finite K-entropy must retain intrinsic two-body effects because the K-entropy is a two-body quantity. Nontrivial closure at the level of the Bethe-Salpeter equation is required. The basic principles were described by Krommes et al. (1978); detailed quantitative work continues (Kotschenreuther and Krommes, unpublished).

10.2: Projection Operator Techniques

In this and the next subsection, we mention some of the alternative techniques which have been used to effect renormalization of plasma dynamics. We cannot hope to do justice here to the technical details of these approaches; our main purpose is to provide a guide to the literature.

Consider first a dynamical equation of the form

$$\partial_t \psi(t) = L(t)\psi(t) \ , \tag{412}$$

where the "evolution operator" L is linear and ψ may be a vector parametrized, for example, by space, velocity, and species indices. Equations (150) and (220) are of the form (412), with $LF \equiv [F, H]$ for arbitrary F. Let us divide ψ into two parts, a wanted or "observable" part $P\psi$ and an unwanted or "hidden" part $Q\psi$:

$$\psi = P\psi + Q\psi \ . \tag{413}$$

We assume that P and Q are orthogonal projection operators satisfying

$$P + Q = 1 , \tag{414}$$

$$P^2 = P , \quad Q^2 = Q , \quad PQ = QP = 0 . \tag{415}$$

Two important realizations of P are

$$PF(t) \equiv \langle F(t) \rangle , \tag{416}$$

which when applied to ψ generates its mean, and

$$PF(t) \equiv \psi(0)\langle \psi(0)\psi(0)\rangle^{-1}\langle \psi(0)F(t)\rangle , \tag{417}$$

which when applied to $\psi(t)$ generates its covariance. It is easy to verify that both definitions (416) and (417) define a time-independent ($[\partial_t , P] = 0$), linear projection operator P.

Our goal is to find an equation for the observable $P\psi$. To this end, we apply both P and Q successively to Eq. (412), finding thereby

$$P\dot{\psi} = PL(P+Q)\psi \quad , \tag{418}$$

$$Q\dot{\psi} = QL(Q+P)\psi \quad , \tag{419}$$

where we inserted the identity (414). Equation (419) can be formally solved for the hidden part $Q\psi$:

$$Q\psi = U^{\ddagger}(t,0)Q\psi(0) + \int_0^t dt' U^{\ddagger}(t,t')QL(t')P\psi(t') , \tag{420}$$

where

$$U^{\ddagger}(t,t') \equiv \exp_+\left(\int_{t'}^{t} dt'' QL(t'')Q\right) . \tag{421}$$

Substituting Eq. (420) into Eq. (418), we find

$$\partial_t P\psi(t) = -i\Omega(t)P\psi(t) + \int_0^t dt'\Phi(t,t')P\psi(t') + PL(t)U^{\ddagger}(t,0)Q\psi(0) , \tag{422}$$

where the operators Ω and Φ are defined by

$$-i\Omega(t) \equiv PL(t)P , \tag{423a}$$

$$\Phi(t,t') \equiv PL(t)QU^{\ddagger}(t,t')QL(t')P . \tag{423b}$$

Often the system can be prepared so that $Q\psi(0) \equiv 0$. [This is automatically satisfied with the construction (417).] In this case, Eq. (423) is a formally closed equation for the observable $P\psi$.

To better understand the content of Eq. (423), define

$$f^{\ddagger}(t) \equiv QL(t)\psi(0) \tag{424}$$

and use definition (417). Brief manipulations based on Eqs. (413) and (420) then lead to

$$[\partial_t + i\omega(t)]\psi(t) + \int_0^t dt'\nu(t,t')\psi(t') = f^{\ddagger}(t) , \tag{425}$$

where

$$\omega(t) \equiv i<\psi(0)L(t)\psi(0)><\psi(0)\psi(0)>^{-1} , \tag{426a}$$

$$\nu(t,t') \equiv -<f^{\ddagger}(t)f^{\ddagger}(t')><\psi(0)\psi(0)>^{-1} . \tag{426b}$$

Equation (425) is called the generalized Langevin equation (Mori 1965a), as its form is a simple generalization of the Langevin equation encountered in the elementary theory of Brownian motion. A good review is given by Kubo (1974). Other familiar-looking relations are

$$\langle f^{\ddagger}(t)\psi(0)\rangle \equiv 0 \ , \tag{427}$$

which follows from Eq. (424), and

$$[\partial_t + i\omega(t)]\langle\psi(t)\psi(0)\rangle + \int_0^t dt'\nu(t,t')\langle\psi(t')\psi(0)\rangle = 0 \ , \tag{428}$$

which follows from Eqs. (425) and (427). Equation (426b) is a generalized Einstein relation.

The problem with Eqs. (428), (422), or (425) is that they are extremely formal. The action of the modified evolution operator U is very difficult to evaluate explicitly because of the presence of the orthogonal projector Q. However, formal power series expansions can be given (Birmingham and Bornatici 1971). Mori (1965b) has developed an interesting continued fraction representation of the solution. The generalized Langevin equation has been used successfully to develop dynamical models in many-body theory. (For a review, see Berne 1971.) Using definition (416), Weinstock (1969,1970) has couched the problem of Vlasov turbulence in the projection operator formalism.

Workers of the Brussels school have developed projection operator techniques far beyond the level described here. Their approach has been described in detail elsewhere (Balescu 1975), so we do not comment further here. Misguich has written a number of very detailed papers describing their application to plasma turbulence (Misguich and Balescu 1975a,b, 1977, 1978a,b).

10.3: Canonical Transformations

A very stimulating and novel approach to statistical theories has been advocated by Dewar (1976). The idea is to introduce the Hamiltonian corresponding to the (conservative) system under discussion, then to seek a canonical transformation to new variables such that the statistical theory of the transformed system is simpler and perhaps more convergent. (We are referring to the ordinary Hamiltonian, a function of the canonical variables, not to the operator-valued Hamiltonian function of the extended fields in the MSR procedure.) For example, one might try to remove the oscillatory motions of non-resonant particles. Dewar has argued that the approach may be particularly useful for problems involving partial trapping. He has arrived at a number of interesting conclusions, but much further work remains to be done. One problem with the technique is that some information is "hidden" in the generating function for the canonical transformation and must be retrieved if results are desired in laboratory coordinates. Another is that the way of choosing the "best" canonical transformation has not yet been adequately systematized. However, it would seem that renormalized canonical transformation theory is a fruitful area for further research.

10.4: Additional References

We list here a few additional references on subjects not adequately covered in the text. A good review of fluid turbulence theory is given by Kraichnan (1975b). A review of many aspects of resonance-broadening theory is given by Pelletier (1977). Discussions of renormalizations applied to high density plasmas and liquids are given by Ichimaru (1977) and by Cook (1978a). The use of resonance-broadening theory to describe certain tokamak instabilities is discussed by Ehst (1977).

11: AFTERWORD

Hopefully, it is by now clear that renormalization is important in a number of interesting and relevant problems of plasma physics. However, though we have indicated the general framework within which these problems may be handled, in relatively few cases did we succeed in carrying the problem to a definitive solution. Thus, we cannot say that we quantitatively understand drift wave turbulence, with Reynolds number of order unity. Likewise, the detailed phase space dynamics associated with stochastically unstable systems are only partly understood qualitatively, and even less well quantitatively. Further exploration of these questions should be both challenging and exciting. As problems in nonlinear statistical physics, they are scientifically important in their own right. More practically, a quantitative (renormalized) theory of plasma turbulence is one prerequisite to successful detailed computer modeling of laboratory discharges and, ultimately, to the design of a successful fusion reactor. Perhaps in no other example is the symbiosis between fundamental and practical so necessary--and so near.

ACKNOWLEDGMENTS

This work was jointly supported by the U.S. Air Force Office of Scientific Research Contract No. F 44620-75-C-0037 and by the U.S. Department of Energy Contract No. EY-76-C-02-3073.

APPENDIX A: CUMULANTS

Consider a one-dimensional probability distribution $P(x)$ of the random variable x. The moments of P are defined as

$$\langle x^n \rangle \equiv \int_{-\infty}^{\infty} dx\, x^n\, P(x)\,. \tag{A1}$$

Assume that P has a Fourier transform P_k:

$$P_k = \int_{-\infty}^{\infty} dx\, \exp(-ikx)\, P(x) \tag{A2a}$$

$$= \langle \exp(-ikx) \rangle \tag{A2b}$$

$$= \int dx \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!}\, x^n\, P(x) \tag{A2c}$$

$$= \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle\,. \tag{A2d}$$

The cavalier interchange of integration and summation we have performed is usually justified in practice. Because of Eq. (A2d), we say that P_k is the "moment generating function" for P:

$$\langle x^n \rangle = \frac{\partial^n}{\partial(-ik)^n} P_k \Big|_{k=0}. \tag{A3}$$

In general, even physically simple distributions contain moments of all orders. For example, the Gaussian distribution

$$P(x) = (2\pi\Delta^2)^{-\frac{1}{2}} \exp[-(x-x_o)^2/2\Delta^2]$$

has all even centered moments nonvanishing:

$$\langle (x-x_o)^{2n} \rangle = (2n-1)!!\, \Delta^{2n} \quad \text{(Gaussian)}. \tag{A4}$$

Furthermore, in dimensionless units the moments become, in general, rapidly large with order and are, thus, useless as expansion parameters. This motivates an alternative description of the distribution in terms of parameters which may often be small.

Let us define the "cumulant generating function" W_k, whose significance will be made clear shortly, by

$$W_k \equiv \ln P_k \quad . \tag{A5}$$

W_k has its own formal Taylor expansion, the ℓ-th coefficient of which we call the ℓ-th cumulant and denote by $\langle\langle x^\ell \rangle\rangle$:

$$W_k = \sum_{\ell=1}^{\infty} \frac{(-ik)^\ell}{\ell!} \langle\langle x^\ell \rangle\rangle \; ; \tag{A6}$$

$$\langle\langle x^\ell \rangle\rangle = \frac{\partial^\ell}{\partial(-ik)^\ell} W_k \Big|_{k=0} = \frac{\partial^\ell}{\partial(-ik)^\ell} \ln P_k \Big|_{k=0} \quad . \tag{A7}$$

We can use Eqs. (A2b), (A5), and (A6) to write

$$\langle \exp(-ikx) \rangle = \exp\left(\sum_{\ell=1}^{\infty} \frac{(-ik)^\ell}{\ell!} \langle\langle x^\ell \rangle\rangle \right), \tag{A8}$$

so the cumulant expansion affords a way--often useful in practice--of interchanging averaging and exponentiation. Furthermore, because the Fourier transform of a Gaussian is again a Gaussian,

$$P_k = \exp(-ikx_0 - \tfrac{1}{2}k^2\Delta^2) \qquad \text{(Gaussian)}, \tag{A9}$$

we see that

$$\begin{aligned} \langle\langle x \rangle\rangle &= x_0 \; , \\ \langle\langle x^2 \rangle\rangle &= \Delta^2 \; , \qquad\qquad \text{(Gaussian)} \\ \langle\langle x^\ell \rangle\rangle &= 0 \qquad (\ell \geq 3) \; . \end{aligned} \tag{A10}$$

In general, if the distribution is nearly Gaussian--a result often guaranteed by the central limit theorem--the third and higher cumulants will be small and thus become potentially useful expansion parameters.

The general law relating moments and cumulants follows by expanding Eq. (A9) in its formal Taylor series and then comparing the result to Eq. (A2d). Clearly, the ℓ-th cumulant is determined by moments of order no higher than ℓ, and vice versa. The general law states that the n-th moment is determined by all possible sums of products of cumulants of order n or lower, such that the total number of x's in the product is n, and with each term weighted with a combinatoric factor which gives the number of distinct combinations of x's in the product:

$$\langle x^n \rangle = \sum_{m=1}^{n} \frac{n!}{\mu_m!} \prod_{\substack{i=1 \\ \Sigma \ell_i = n}}^{m} \left[\frac{\langle\langle x^{\ell_i} \rangle\rangle}{\ell_i!} \right] , \tag{A11}$$

where μ_m is the number of identical ℓ's in the set $\{\ell_1,\ldots,\ell_m\}$. Thus, through order four,

$$\langle x \rangle = \langle\langle x \rangle\rangle , \tag{A12a}$$

$$\langle x^2 \rangle = \langle x \rangle^2 + \langle\langle x^2 \rangle\rangle , \tag{A12b}$$

$$\langle x^3 \rangle = \langle x \rangle^3 + 3\langle x \rangle\langle\langle x^2 \rangle\rangle + \langle\langle x^3 \rangle\rangle , \tag{A12c}$$

$$\langle x^4 \rangle = \langle x \rangle^4 + 3\langle x \rangle\langle\langle x^3 \rangle\rangle + 6\langle x \rangle^2 \langle\langle x^2 \rangle\rangle + 3\langle\langle x^2 \rangle\rangle^2 + \langle\langle x^4 \rangle\rangle . \tag{A12d}$$

From Eq. (A12b), observe the important result that the second cumulant is the covariance of the fluctuation:

$$\langle\langle x^2 \rangle\rangle = \langle (x - \langle x \rangle)^2 \rangle \equiv \langle \delta x^2 \rangle . \tag{A13}$$

Also, note that the explicit decomposition for a centered Gaussian,

$$\langle x^{2n} \rangle = \frac{(2n)!}{n!(2!)^n} \langle\langle x^2 \rangle\rangle^n = (2n-1)!!\Delta^{2n} , \tag{A14}$$

agrees with Eq. (A4).

The cumulant expansion can be extended in an obvious way to n-variate distributions:

$$\langle \exp(-i\vec{k}\cdot\vec{x}) \rangle = \exp\left(\sum_{\substack{\ell_1 \ldots \ell_n \\ \text{all } \ell\text{'s not } 0}}^{\infty} \frac{(-ik_1)^{\ell_1}}{\ell_1!} \cdots \frac{(-ik_n)^{\ell_n}}{\ell_n!} \langle\langle x_1^{\ell_1} \, x_2^{\ell_2} \ldots x_n^{\ell_n} \rangle\rangle \right) . \tag{A15}$$

For distinct x's, the moment expansion in terms of cumulants becomes particularly transparent because all combinatoric factors are identically unity. For example, the fourth moment of a centered process becomes

$$\langle x_1x_2x_3x_4 \rangle = \langle\langle x_1x_2 \rangle\rangle\langle\langle x_3x_4 \rangle\rangle + \langle\langle x_1x_3 \rangle\rangle\langle\langle x_2x_4 \rangle\rangle + \langle\langle x_1x_4 \rangle\rangle\langle\langle x_2x_3 \rangle\rangle + \langle\langle x_1x_2x_3x_4 \rangle\rangle .$$

Finally, we can describe random functions $\mathfrak{A}(1)$ of continuous parameters "1" by changing the discrete sums in Eq. (A15) to integrations in the usual way. We may thus define a generating <u>functional</u> by

$$W\{\eta\} \equiv \ln \langle \exp\int d1 \, \eta(1) \mathfrak{A}(1) \rangle \tag{A17}$$

(it is conventional to drop the factor "$-i$"), in terms of which

$$\langle\langle \mathfrak{A}(1) \ldots \mathfrak{A}(n) \rangle\rangle = \frac{\delta^n W\{\eta\}}{\delta\eta(1) \ldots \delta\eta(n)} \bigg|_{\eta=0} . \tag{A18}$$

Here $\delta/\delta\eta$ denotes a functional derivative $[\delta\eta(1)/\delta\eta(2) = \delta(1-2)]$, familiar from variational calculus and nicely described by Beran (1968). The forms (A16) and (A17) will be used extensively in Sec. 6.

A useful discussion of cumulants can be found in Kubo (1962).

APPENDIX B: FOURIER TRANSFORM CONVENTIONS

The fundamental Fourier transform convention is

$$E(x) = \sum_k E_k \exp(ikx) \quad . \tag{B1}$$

We write amplitudes for discrete spectra as E_k , for continuous spectra as $E(k)$. For a one-dimensional discrete spectrum with mode spacing $\delta k \equiv 2\pi/L$, the following relations hold, in addition to (B1):

$$E_k = \int_0^L \frac{dx}{L} \exp(-ikx)\, E(x) \ ,$$

$$\langle E_k E_{k'} \rangle = \delta_{k+k'} \langle E\,E \rangle_k \ ,$$

$$\langle |E_k|^2 \rangle = \langle E\,E \rangle_k \ .$$

The notation $\langle E\,E \rangle_k$ means the transform with respect to ρ of the stationary correlation $\langle E(x+\rho) E(x) \rangle$. Also,

$$\int_0^L dx\, A(x)\, B(x) = L \sum_k A_k B_{-k} \ .$$

Transition to a one-dimensional continuous spectrum is effected by

$$E(k) = L\, E_k \ ,$$

$$\sum_k \rightarrow \int dk/\delta k \ .$$

In this case, the following relations hold:

$$E(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(ikx)\, E(k) \ ,$$

$$E(k) = \int_{-\infty}^{\infty} dx \exp(-ikx)\, E(x) \ ,$$

$$\langle E(k)\, E(k')\rangle = 2\pi\delta(k+k')\ \langle E\,E\rangle(k) \ ,$$

$$\langle |E(k)|^2\rangle = L\langle E\,E\rangle(k) \ ,$$

$$\int_{-\infty}^{\infty} dx\, A(x)\, B(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, A(k)\, B(-k) \ .$$

REFERENCES

Balescu, R., 1975, Equilibrium and Nonequilibrium Statistical Mechanics (Wiley-Interscience, New York), Chaps. 14-21.

Benettin, G., Galgani, L., and Strelcyn, J.-M., 1976, "Kolmogorov Entropy and Numerical Experiments", Phys. Rev. A14, 2338.

Beran, M.J., 1968, Statistical Continuum Theories, (Interscience Publishers, New York).

Berne, B.J., 1971, "Time-Dependent Properties of Condensed Media", in Physical Chemistry, Vol. 8B, edited by D. Henderson (Academic Press, New York), p. 539.

Birmingham, T.J. and Bornatici, M., 1971, "Propagators in Strong Plasma Turbulence", Phys. Fluids 14, 2239.

Bourret, R.C., 1962, "Stochastically Perturbed Fields, with Applications to Wave Propagation in Random Media", Nuovo Cimento 26, 1.

Catto, P.J., 1978, "Adiabatic Modifications to PLasma Turbulence Theories", Phys. Fluids 21, 147.

Cheng, C.Z. and Okuda, H., 1977, "Formation of Convective Cells, Anomalous Diffusion, and Strong Plasma Turbulence due to Drift Instabilities", Phys. Rev. Lett. 38, 708.

, 1978, "Theory and Numerical Simulations on Collisionless Drift Instabilities", Nucl. Fusion 18, 587.

Chirikov, B.V., 1969, "Research Concerning the Theory of Non-linear Resonance and Stochasticity", Report 267, Nuclear Physics Institute of the Siberian Section of the USSR Academy of Sciences, Novosibirsk, translated by A.T. Sanders (CERN-Trans. 71-40, Geneva, 1971).

, 1979, "A Universal Instability of Many-Dimensional Oscillator Systems", Phys. Rep. 52, 263.

Cook, I., 1978a, "Effect of Fluctuations on the Response Function of Plasmas", Plasma Phys. 20, 73.

, 1978b, "Application of the Novikov-Furutsu Theorem to the Random Acceleration Problem", Plasma Phys. 20, 349.

Cook, I. and Sanderson. A.D., 1974, "On Dupree's Theory of Strong Plasma Turbulence", Plasma Phys. 16, 977.

Dawson, J.M., Okuda, H., and Carlile, R.N., 1971, "Numerical Simulation of Plasma Diffusion Across a Magnetic Field in Two Dimensions", Phys. Rev. Lett. 27, 491.

Dean, S.O., Callen, J.D., Furth, H.P., Clarke, J.F., Ohkawa, T., and Rutherford, P.H. 1974, "Status and Objectives of Tokamak Systems for Fusion Research", USAEC Report WASH-1295 (U.S. Government Printing Office, Washington).

DeDominicis, C. and Martin, P.C., 1964, "Stationary Entropy Principle and Renormalization in Normal and Superfluid Systems. I. Algebraic Formulation", J. Math. Phys. 5, 14.

Deker, U., 1979, "Perturbation Theory for Classical Random Processes with Arb[illegible]y Preparation", Phys. Rev. A19, 846.

Deker, U. and Haake, F., 1975a, "Fluctuation-Dissipation Theorems for Classical Processes", Phys. Rev. A11, 2043.

, 1975b, "Mass and Charge Renormalization for Classical Processes with a Fluctuation-Dissipation Theorem", Phys. Rev. A12, 1629.

Dewar, R.L., 1976, "Renormalized Canonical Perturbation Theory for Stochastic Propagators", J. Phys. A9, 2043.

Drummond, W.E. and Pines, D., 1962, "Non-Linear Stability of Plasma Oscillations", Nucl. Fusion Suppl., Part 3, 1049.

DuBois, D. and Espedal, M., 1978, "Direct-Interaction Approximation and Plasma Turbulence Theory", Plasma Phys. 20, 1209.

Dupree, T.H., 1966, "A Perturbation Theory for Strong Plasma Turbulence", Phys. Fluids 9, 1773.

, 1967, "Nonlinear Theory of Drift-Wave Turbulence and Enhanced Diffusion", Phys. Fluids 10, 1049.

, 1968, "Nonlinear Theory of Low-Frequency Instabilities", Phys. Fluids 11, 2680.

, 1970, "Theory of Resistivity in Collisionless Plasma", Phys. Rev. Lett. 25, 789.

, 1972, "Theory of Phase Space Density Granulation in Plasma", Phys. Fluids 15, 334.

Dupree, T.H., 1974, "Theory of Two-Dimensional Turbulence", Phys. Fluids 17, 100.

_______, 1978, "Role of Clumps in Drift Wave Turbulence", Phys. Fluids 21, 783.

Dupree, T.H. and Tetreault, D.J., 1978, "Renormalized Dielectric Function for Collisionless Drift Wave Turbulence", Phys. Fluids 21, 425.

Dupree, T.H., Wagner, C.E., and Manheimer, W.M., 1975, "Observation of Clumps in Simulation Plasma", Phys. Fluids 18, 1167.

Ehst, D.A., 1977, "Quasi-Linear and Nonlinear Theory of Dissipative Trapped Particle Instabilities", Phys. Fluids 20, 2076.

Frisch, U. and Bourret, R., 1970, "Parastochastics", J. Math. Phys. 11, 364.

Galeev, A.A., 1967, "New Approach to Plasma Turbulence", Phys. Fluids 10, 1041.

_______, 1969, "Turbulent Relaxation of a Plasma Far from the Stability Threshold", Zh. Eksp. Teor. Fiz. 57, 1361 [Sov. Phys.-JETP 30, 737 (1970)].

Goldstein, H., 1950, Classical Mechanics (Addison-Wesley, Reading, Massachusetts).

Greene, J., 1979, "A Method for Determining a Stochastic Transition", J. Math. Phys. 20, 1183.

Hasegawa, A. and Mima, K., 1978, "Pseudo-Three-Dimensional Turbulence in Magnetized Nonuniform Plasma", Phys. Fluids 21, 87.

Hirshman, S.P. and Molvig, K., 1979, "Turbulent Destabilization and Saturation of the Universal Drift Mode in a Sheared Magnetic Field", Phys. Rev. Lett. 42, 648.

Horton, W. and Choi, D.-I., 1979, "Renormalized Turbulence Theory for the Ion Acoustic Problem", Phys. Rep. 49, 273.

Hui, B.H. and Dupree, T.H., 1975, "Computer Experiments on Clump Formation in Plasma", Phys. Fluids 18, 235.

Ichimaru, S., 1973, Basic Principles of Plasma Physics--A Statistical Approach (Benjamin, Reading, Massachusetts).

, 1977, "Theory of Strongly Correlated Charged-Particle Systems: Plasma Turbulence and High-Density Electron Liquids", Phys. Rev. A15, 744.

Jokipii, J.R. and Parker, E.N., 1969, "Stochastic Aspects of Magnetic Lines of Force with Application to Cosmic-Ray Propagation", Astrophys. J. 155, 777.

Kadomtsev, B.B., 1965, Plasma Turbulence, translated by C.C. Ronson, edited by M.G. Rusbridge (Academic Press, Reading, Massachusetts).

Kleva, R.G., 1979, Aspects of the Statistical Theory of Stochastic Magnetic Fields: Test Particle Transport and Turbulent Collisionless Tearing Mode, Ph. D. Thesis, Princeton University (unpublished).

Kleva, R.G., Krommes, J.A., and Oberman, C., 1979, "Turbulent Evolution of the Collisionless Tearing Mode in the Presence of Stochastic Magnetic Fields", Princeton Plasma Physics Laboratory Report 1574.

Kolmogorov, A.N., 1941, "The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds Number", Dokl. Akad. Nauk SSSR 30, 301.

Kraichnan, R.H., 1958a, "Irreversible Statistical Mechanics of Incompressible Hydromagnetic Turbulence", Phys. Rev. 108, 1407.

, 1958b, "A Theory of Turbulence Dynamics", in Second Symposium on Naval Hydrodynamics, Office of Naval Research Report ACR-38, p. 29.

, 1959, "The Structure of Isotropic Turbulence in the Direct-Interaction Approximation", J. Fluid Mech. 5, 497.

, 1961, "Dynamics of Nonlinear Stochastic Systems", J. Math. Phys, 2, 124.

, 1964, "Kolmogorov's Hypotheses and Eulerian Turbulence Theory", Phys. Fluids 7, 1723.

, 1965, "Lagrangian-History Closure Approximation for Turbulence", Phys. Fluids 8, 575.

, 1966, "Invariance Principles and Approximation in Turbulence Dynamics", in Dynamics of Fluids and Plasmas, edited by S.I. Pai (Academic Press, London, 1966), p. 239.

Kraichnan, R.H., 1967, "Inertial Ranges and Two-Dimensional Turbulence", Phys. Fluids 10, 1417.

, 1970a, "Convergents to Turbulence Functions", J. Fluid Mech. 41, 189.

, 1970b, "Inertial-Range Transfer in Two- and Three-Dimensional Turbulence", J. Fluid Mech. 47, 525.

, 1971, "An Almost-Markovian Galilean-Invariant Turbulence Model", J. Fluid Mech. 47, 513.

, 1975a, "Statistical Dynamics of Two-Dimensional Flow", J. Fluid Mech. 67, 155.

, 1975b, "Remarks on Turbulence Theory", Adv. in Math. 16, 305.

, 1978, "Eulerian and Lagrangian Renormalization in Turbulence Theory", J. Fluid Mech. 83, 349.

Krall, N.A. and Trivelpiece, A.W., 1973, Principles of Plasma Physics (McGraw-Hill, New York).

Krommes, J.A., 1975, On Renormalized Theories of Transport Due to Hydrodynamic Fluctuations in Strongly Magnetized Plasma, Ph.D. Thesis, Princeton University (unpublished).

, 1978a, "Turbulence, Clumps, and the Bethe-Salpeter Equation", in Theoretical and Computational Plasma Physics (International Atomic Energy Agency, Vienna), p. 405.

Krommes, J.A., 1978b, "Plasma Transport in Stochastic Magnetic Fields II: Principles and Problems of Test Electron Transport", Suppl. Prog. Theor. Phys. 64, 137.

, 1979a, "Renormalized Compton Scattering and Non-linear Damping of Collisionless Drift Waves", Princeton Plasma Physics Laboratory Report PPPL-1543.

, 1979b, "Incoherent Noise and Self-Consistency in Stochastically Unstable Plasmas", Princeton Plasma Physics Laboratory Report PPPL- 1603.

, 1979c, "Fallacies in Some Theories of the Renormalized Dielectric", Princeton Plasma Physics Laboratory Report PPPL-1605.

Krommes, J.A. and Kleva, R.G., 1979, "Aspects of a Renormalized Weak Plasma Turbulence Theory", Phys. Fluids 22, 2168.

Krommes, J.A., Kleva, R.G., and Oberman, C., 1978, "Plasma Transport in Stochastic Magnetic Fields I: General Considerations and Test Particle Transport", Princeton Plasma Physics Laboratory Report PPPL-1389 (to be published).

Krommes, J.A. and Oberman, C., 1976a, "Anomalous Transport Due to Long-lived Fluctuations in Plasma. Part I: A General Formalism for Two-Time Fluctuations", J. Plasma Phys. 16, 193.

, 1976b, "Anomalous Transport Due to Long-Lived Fluctuations in Plasma. Part II: Hydrodynamic Contributions to Transport in Two-Dimensional, Strongly Magnetized Systems", J. Plasma Phys. 16, 229.

Krommes, J.A. and Similon, P., 1979, "Dielectric Function of Guiding Center Plasma", Princeton Plasma Physics Laboratory Report PPPL-1621.

Kubo, R., 1962, "Generalized Cumulant Expansion Method", J. Phys. Soc. Japan 17, 1100.

, 1963, "Stochastic Liouville Equations", J. Math. Phys. 4, 174, and refs. therein.

, 1974, "Response, Relaxation, and Fluctuation", in Lecture Notes in Physics: Vol. 31, Transport Phenomena, edited by J. Ehlers et al. (Springer-Verlag, Berlin), p. 74.

Landau, L.D. and Lifshitz, E.M., 1960, Electrodynamics of Continuous Media, translated by J.B. Sykes and J.S. Bell (Addison-Wesley, Reading, Massachusetts).

Lee, Y.C. and Liu, C.S., 1973, "Plasma Diffusion across a Magnetic Field due to Thermal Vortices", Phys. Rev. Lett. 30, 361.

Leith, C.E., 1971, "Atmospheric Predictability and Two-Dimensional Turbulence", J. Atm. Sci. 28, 145.

Leslie, D.C., 1973, Developments in the Theory of Turbulence (Clarendon Press, Oxford).

Martin, P.C., Siggia, E.D., and Rose, H.A., 1973, "Statistical Dynamics of Classical Systems", Phys. Rev. A8, 423.

Misguich, J.H. and Balescu, R., 1975a, "Stochastic Turbulence in Vlasov Plasmas", Physica 79C, 373.

, 1975b, "Renormalized Quasilinear Approximation of Plasma Turbulence. Part I. Modification of the Weinstock Weak-Coupling Limit", J. Plasma Phys. 13, 385;

"Part II. Validity of Short-Time Expansion and Markovian Truncation of the Propagator", J. Plasma Phys. 13, 419; "Part III. Collective Binary Correlations and Turbulent Detrapping", J. Plasma Phys. 13, 429.

Misguich, J.H. and Balescu. R., 1977, "Kinetic Description of Average Trajectories in Turbulent Plasmas", Plasma Phys. 19, 611.

, 1978a, "Kinetic Theory of Binary Correlations in Turbulent Plasmas", J. Plasma Phys. 19, 147.

, 1978b, "'Clumps' as Enhanced Correlations and Turbulent Debye Screening", Plasma Phys. 20, 781.

Molvig, K., Hirshman, S.P. and Whitson, J.C., 1979, "Finite-β_e Universal Mode Turbulence and Alcator Scaling", Phys. Rev. Lett. 43, 582.

Montgomery, D., Liu, C.-S., and Vahala, G., 1972, "Three-Dimensional Plasma Diffusion in a Very Strong Magnetic Field", Phys. Fluids 15, 815.

Mori, H., 1965a, "Transport, Collective Motion, and Brownian Motion", Prog. Theor. Phys. 33, 423.

, 1965b, "A Continued-Fraction Representation of the Time-Correlation Functions", Prog. Theor. Phys. 34, 399.

Okuda, H. and Dawson, J.M., 1973, "Theory and Numerical Simulation on Plasma Diffusion Across a Magnetic Field", Phys. Fluids 16, 408.

Orszag, S.A. and Kraichnan, R. H., 1967, "Model Equations for Strong Turbulence in a Vlasov Plasma", Phys, Fluids 10, 1720.

Pelletier, G., 1977, "Renormalization Method and Singularities in the Theory of Langmuir Turbulence", J. Plasma Phys. 18, 49.

Phythian, R., 1975, The Operator Formalism of Classical Statistical Dynamics", J. Phys. A8, 1423.

, 1976, "Further Application of the Martin, Siggia, Rose Formalism", J. Phys. A9, 269.

Rechester, A.B. and Rosenbluth, M.N., 1978, "Electron Heat Transport in a Tokamak with Destroyed Magnetic Surfaces", Phys. Rev. Lett. 40, 38.

Rose, H.A., 1974, Aspects of the Statistical Dynamics of Classical Systems, Ph.D. Thesis, Harvard University (unpublished).

, 1979, "Renormalized Kinetic Theory of Nonequilibrium Many-Particle Classical Systems", J. Stat. Phys. 20, 415.

Rose, H.A. and Sulem, P.L., 1978, "Fully Developed Turbulence and Statistical Mechanics", J. Phys. (France) 39, 441.

Rosenbluth, M.N., Sagdeev, R.Z., Taylor, J.B., and Zaslavskii, G.M., 1966, "Destruction of Magnetic Surfaces by Magnetic Field Irregularities", Nucl. Fusion 6, 297.

Rudakov, L.I. and Tsytovich, V.N., 1971, "The Theory of Plasma Turbulence for Strong Wave-Particle Interaction", Plasma Phys. 13, 213.

Sagdeev, R.Z. and Galeev, A.A., 1969, Nonlinear Plasma Theory, revised and edited by T.M. O'Neil and D.L. Book (Benjamin, New York).

Similon, P. and Krommes, J.A., 1979, "The Direct-Interaction Approximation and Drift Wave Turbulence", Bull. Am. Phys. Soc. 24, 1040.

Sleeper, A.M. and Weinstock, J., 1972, "Nonlinear Theory of Density Fluctuations in Turbulent Plasmas", Phys. Fluids 15, 1507.

Sudan, R.N. and Keskinen, M., 1977, "Theory of Strongly Turbulent, Two-Dimensional Convection of Low-Pressure Plasma", Phys. Rev. Lett. 38, 966.

Taylor, J.B., 1974, "Dielectric Function and Diffusion of a Guiding-Center Plasma", Phys. Rev. Lett. 32, 199.

Taylor, J.B. and McNamara, B., 1971, "Plasma Diffusion in Two Dimensions", Phys. Fluids 14, 1492.

Tetreault, D.J., 1976, Renormalization of the Wave-Particle Resonance in Turbulent Plasma, Ph.D. Thesis, Massachusetts Institute of Technology (unpublished).

Tsytovich, V.N., 1972, "Development of the Concepts of Plasma Turbulence", Usp. Fiz. Nauk 108, 143 [Sov. Phys.-Usp. 15, 632 (1973)].

, 1977, Theory of Turbulent Plasma, translated by D.C. Burdock (Consultants Bureau, New York).

Van Kampen, N.G., 1976, "Stochastic Differential Equations", Phys. Rep. 24, 171.

Vedenov, A.A., Velikhov, E.P. and Sagdeev, R.Z., 1961, "Nonlinear Oscillations of Rarified Plasma" (in Russian), Nucl. Fusion 1 82.

Wang, M.C. and Uhlenbeck. G.E., 1945, "On the Theory of Brownian Motion II", Rev. Mod. Phys. 17, 323; reprinted in Selected Papers on Noise and Stochastic Processes, edited by N. Wax (Dover, New York, 1954), p. 113.

Weinstock, J., 1969, "Formulation of a Statistical Theory of Strong Plasma Turbulence", Phys. Fluids 12, 1045.

, 1970, "Turbulent Plasmas in a Magnetic Field--A Statistical Theory", Phys. Fluids 13, 2308.

, 1976, "Lagrangian-Eulerian Relation and the Independence Hypothesis", Phys. Fluids 19, 1702.

, 1977, "Three-Point Method in the Theory of Homogeneous Turbulence", Phys. Fluids 20, 1631.

Zaslavskii, G.M. and Chirikov, B.V., 1971, "Stochastic Instability of Non-Linear Oscillations", Usp. Fiz. Nauk 105, 31 [Sov. Phys.-Usp. 14, 549 (1972)].

Zaslavskii, G.M. and Filonenko, N.N., 1968, "Stochastic Instability of Trapped Particles and Conditions of Validity of the Quasi-Linear Approximation", Zh. Eksp. Teor. Fiz. 54, 1590 [Sov. Phys.-JETP 27, 854 (1968)].

	Response Function: dimensionless	Response Function: dimensional
$\bar{\tau} < 1$ $(\tau < \tau_{ac})$	$\exp(-\frac{1}{2}K^2\ \bar{\tau}^2)$	$\exp(-\frac{1}{2}\beta^2\tau^2)$
$\bar{\tau} > 1$ $(\tau > \tau_{ac})$	$\exp(-K^2\bar{\tau})$	$\exp(-\beta^2\tau_{ac}\tau)$

Table I. Limits of the response function of the stochastic oscillator with Gaussian coefficient.

FIGURE CAPTIONS

Fig. 1. Integration domain in the space of wavevector magnitudes.

Fig. 2. Geometry for the fundamental wavevector triad.

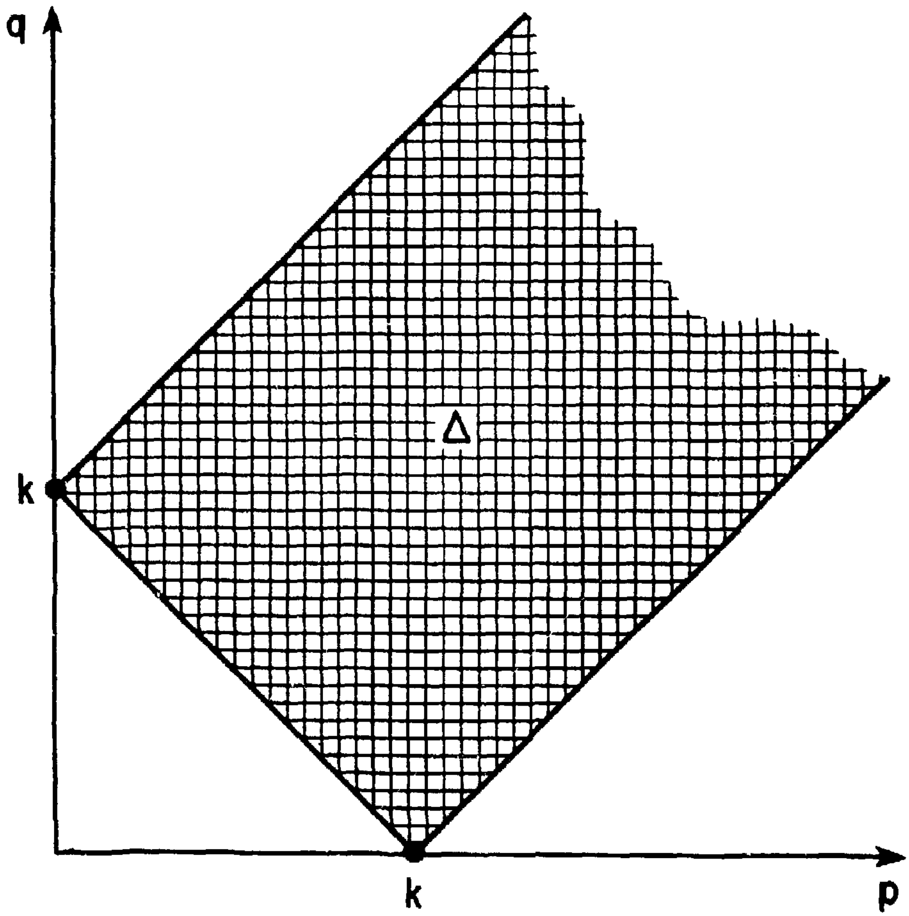

Fig. 1. 802038

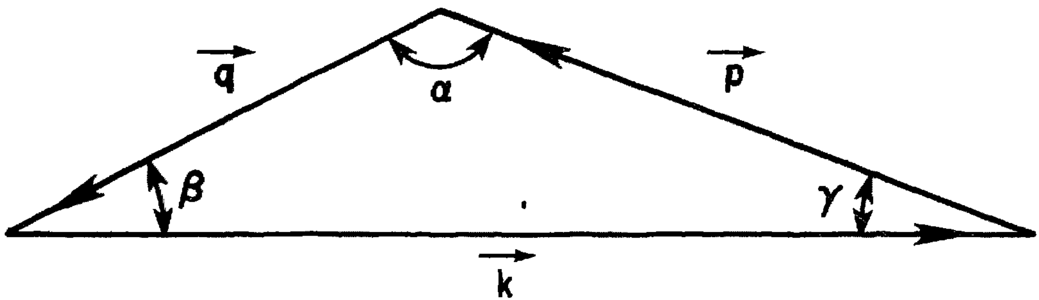

Fig. 2. 802037

SYMBOLS

<u>Roman and Script</u>

a_{kpq}	—	mode-coupling coefficient for emission
b_{kpq}	—	mode-coupling coefficient for absorption
B	—	magnetic field
BSE	—	Bethe-Salpeter equation
c	—	speed of light
$c(\tau)$	—	two-point correlation function normalized to unity
C	—	two-point correlation function
C_3'	—	residual third-order cumulant
$C(\tau)$	—	Lagrangian correlation function: $C[\vec{x}(\tau),\tau]$
$C(\vec{x},\tau)$	—	Eulerian correlation function
d	—	number of space dimensions
D	—	diffusion coefficient

DIA — direct-interaction approximation

D_m — diffusion coefficient for stochastic magnetic field lines

D_p — "plateau" diffusion coefficient for convective cells

e — absolute value of electronic charge

E — energy

$\vec{E}$ — electric field

$\vec{\mathcal{E}}$ — electric field operator (Eq. 151e)

$\vec{E}_e$ — external electric field perturbation

$\vec{E}_i$ — electric field induced in plasma by external field

f — external random force; distribution function

$f^{(o)}$ — zero-th order distribution function

f — renormalized distribution function

$f^{\neq}$ — "random" force in generalized Langevin equation

F — covariance of either random coefficient or random external force

$\tilde{F}$ — "incoherent noise" term in equation for C

$\tilde{F}_m$ — mode-coupling part of $\tilde{F}$

$\tilde{F}'$ — $\tilde{F} - \tilde{F}_m$

g — particle propagator

g_2 — two-particle propagator

$g^{(o)}$ — particle propagator in linear theory

G — two-point symmetric matrix of correlation and response functions

H — Hamiltonian

H(t) — Heaviside function: $H(t < 0) = 0$, $H(0) = 1/2$, $H(t > 0) = 1$

H — Hamiltonian functional

I — four-point interaction matrix in BSE

I_k — potential spectrum

Im — imaginary

$J_\nu(z)$ — ordinary Bessel function of first kind, order ν, and argument z

k — $\{\vec{k}, \omega_k\}$; sometimes, $|\vec{k}|$

$\vec{k}$ — wavenumber

k_o — typical wavenumber of turbulence

k_1 — forcing wavenumber

k_D — Debye wavenumber: $[\Sigma_s 4\pi(nq^2/T)_s]^{\frac{1}{2}}$

$k_<$ — long wavelength cutoff

$k_\parallel$ — wavenumber parallel to magnetic field

K — Kubo number $\beta\tau_{ac}$: the two-body scattering function in the BSE

$K(Q,P)$ — transformed Hamiltonian

L — system size; linear operator (Sec. 10)

L_n — density scale length: $[\nabla_x \ln(n)]^{-1}$

$\mathcal{L}$ — legendre transform

m_s — particle mass (species s)

$M_{\alpha\beta\gamma}$ — mode-coupling coefficient

n — density

N — number of modes in wavepacket

p — momentum: original momentum in canonical transformation

$\vec{p}$ — wavevector

P — new momentum in canonical transformation; projection operator (Sec. 10)

$P\{\psi\}$ — probability functional of ψ

q_s — particle charge (species s)

Q — new coordinate in canonical transformation; projection operator orthogonal to P (Sec. 10); test charge

$\bar{Q}$ — renormalized charge

r — radial coordinate; perpendicular dielectric factor (Eq. 117b)

$r(\tau)$ — nonlinear envelope of correlation function

R — (infinitesimal) mean response function: effective Reynolds number

$R^{(o)}$ — zeroth order (linear) response function

$\tilde{R}$ — fluctuating response function

Re — real

sgn(t) — signum function: $t/|t|$

S — stochasticity parameter

$S\{\eta\}$ — generating functional

$S[x,P]$ — generating function

t — time

T — temperature (energy units); time-ordering operator (Sec. 6); mean time: $\frac{1}{2}(t+t')$

T^* — anti-time-ordering operator

$T(k|p,q)$ — triple correlation responsible for energy transfer

u — dynamical fluid-like variable

u_0 — initial condition for stochastic ascillator

U — covaviance of u; time-ordered evolution operator (Sec. 6); "enstrophy" constant of motion (Sec. 7)

$\breve{U}$ — third order bare vertex

$U_2^{(\ell)}$ — linearized mean field operator

U_3' — three-point coupling coefficient for stochastic acceleration problem

$U^{\mp}$ — modified evolution operator in generalized Langevin theory

v — particle velocity

v_g — group velocity: $\partial\omega(k)/\partial k$

v_t — thermal velocity: $(T/m)^{\frac{1}{2}}$

v_{tr} — trapping velocity

V — velocity in frame of v_ϕ

$\vec{V}_E$ — guiding center electric drift: $c\vec{E}\times\vec{B}/B^2$

$\vec{v}_\Delta$ — velocity of random Galilean transformation

v_ϕ — phase velocity: $\omega(k)/k$

W — "energy" constant of motion (sec. 7)

$W\{\eta\}$ — cumulant generating functional (Eq. 188d)

$\mathcal{W}$ — shape factor for energy or enstrophy transfer

x — spatial coordinate; genralized skewness for stochastic oscillator

Greek

α — inverse "temperature"; angle opposite $\vec{k}$ in triangle $\vec{k} = \vec{p} + \vec{q}$; normalized wavenumber cut-off in DIA

β — r.m.s. value of random coefficient; inverse "temperature"; angle opposite $\vec{p}$ in triangle $\vec{k} = \vec{p} + \vec{q}$

γ — bare vertex matrix; angle opposite $\vec{q}$ in triangle $\vec{k} = \vec{p} + \vec{q}$

$\gamma^{(\ell)}$ — linear growth rate

Γ — renormalized vertex

$\overline{\Gamma}$ — skewness parameter: $\langle\langle\psi^3\rangle\rangle/\langle\langle\psi^2\rangle\rangle^{3/2}$

δ — positive infinitesimal

$\delta f^{(c)}$ — "coherent" part of δf

$\delta\tilde{f}$ — "incoherent" part of δf

$\delta\dot{f}$ — nonlinear part of $\vec{f}$

δk — separation between adjacent wavenumbers: $2\pi/L$

δv	—	velocity spacing between adjacent resonances
$\delta\gamma$	—	nonlinear bandwidth
$\delta\omega_o$	—	line shift
$\delta(t)$	—	Dirac function
Δ	—	frequency spacing between bare resonances; integration domain for wavevector magnitudes which satisfy $\vec{k} = \vec{p} + \vec{q}$; width of Gaussian
Δk	—	wavepacket width in k space
Δt	—	fundamental time step in random walk
Δu	—	u minus direct-interaction contribution (Sec. 5)
Δv	—	island width in velocity space
Δx	—	fundamental space step in randon walk
$(\Delta\omega)_\phi$	—	frequency width of a bare resonance
$(\Delta\omega)_\Sigma$	—	frequency width of a renormalized resonance

ε — dielectric operator: $\varepsilon(\vec{x},t;\vec{x}',t')$;

$\varepsilon^{(\ell)}$ — linear dielectric

$\varepsilon(\vec{k},\omega)$ — dielectric function: Fourier transform of $\varepsilon(\vec{x}-\vec{x}',t-t')$

ε_p — plasma parameter: $(n\lambda_D^d)^{-1}$

$\mathcal{E}$ — covariance of electric field

η — matrix of external functional "probes"; rate of enstrophy injection or transfer

$\hat{\eta}$ — external pertuibation or functional "probe"

θ_{kpq} — effective correlation time between fluctuations $\vec{k}$, $\vec{p}$, and $\vec{q}$

λ_D — Debye wavenumber: k_D^{-1}

$\Lambda(k)$ — rate of enstrophy transferred across k

μ — viscosity

ν — damping rate

ξ — random field

$\pi(k)$ — rate of energy transferred across k

ρ — gyroradius; spatial separation

ρ_s — ion gyroradius at electron temperature

$\tilde{\rho}$ — fluctuating charge density

σ — Pauli spin matrix (Eq. 187a)

σ_k — nonlinear damping decrement

Σ — mass operator; generalized resonance-broadening factor

Σ' — resonance-broadening factor

Σ_{qn} — quasinormal or quasilinear approximation to Σ

$\Sigma^{(d)}$ — "diffusion" part of Σ'

$\Sigma^{(p)}$ — "polarization" part of Σ'

τ — time difference

τ_{ac} — autocorrelation time

τ_c — decorrelation time

τ_{cl} — clump lifetime

τ_d — diffusion time: $(\frac{1}{3}k^2D)^{1/3}$

τ_k — inverse of Kolmogorev entrophy

τ_n — energy transfer time for n-th wavenumber band

τ_r — recurrence time: $N\tau_{ac}$

$\tau_\perp$ — perpendicular correlation time

ϕ — electrostatic potential

$\phi_{\alpha\beta\gamma}$ — random multiplier in random coupling model

φ — memory kernel in generalized Langevin equation; vector of dynamical variables: $\varphi = (\psi, \hat{\psi})$; electrostatic potential

Φ — memory kernel in generalized Langevin equation; potential operator

ψ — random variable

$\hat{\psi}$ — operator conjugate (in the sense of Martin et al.)

ω — frequency; often, a random coefficient

ω_o — bare oscillator frequency

$\omega(k)$ — mode frequency; $\varepsilon[k,\omega(k)] = 0$

ω_{cs} — gyrofrequency of species s: $(qB/mc)_s$

ω_{ps} — plasma frequency of species s: $[\sum_s 4\pi(nq^2/m)_s]^{1/2}$

ω_* — drift frequency

Ω — frequency matrix in generalized Langevin theory; enstrophy

Ω_o — renormalized oscillator frequency

Ω_ϕ — trapping frequency in bare resonance

Ω_Σ — trapping frequency in renormalized resonance

<u>Operations on Arbitrary Functions or Operators A, B</u>

A^* — complex conjugate of A

$A_+(t)$ — $H(t)A(t)$

$A_-(t)$ — $H(-t)A(t)$

A_k — discrete Fourier transform of A

$A(k)$	—	continuous Fourier transform of A
$\langle A \rangle$	—	ensemble average of A
$\langle\langle A \rangle\rangle$	—	cumulant average of A
δA	—	fluctuation in A: $A - \langle A \rangle$
$\delta A/\delta\hat{\eta}$	—	functional derivative of A with respect to $\hat{\eta}$
$[A,B]$	—	commutator of A with B: $AB-BA$
∂A	—	$(q/m)\,\partial A/\partial v$