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T. Uckan

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OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee 37830
operated by
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ABSTRACT

The perpendicular energy gain $\Delta W_{\perp}$ of electrons from the applied extraordinary microwave field in ELMO Bumpy Torus (EBT) is calculated by means of the stochastic model for the field-plasma cyclotron resonance interactions. In these calculations an inhomogeneous bumpy magnetic field is chosen in order to simulate the field strength of the EBT as well as to include the effects of mirror trapping.

The effects of the initial energy of the electrons and the value of the mirror ratio on the trapping are discussed, and the heating rate $\Delta W_{\perp}/\Delta t$ (where Δt is the reflection time from the mirror) is estimated. The loaded cavity quality factor Q_L is then expressed from the heating rate, the result is applied to the EBT-I plasma, and a value of $Q_L \cong 15$ is found.

1. INTRODUCTION

The basic heating mechanism in the ELMO Bumpy Torus (EBT) is based on the electron cyclotron resonance interactions between the applied microwave and plasma electrons.[1] The nonuniform magnetic field configuration of EBT provides an efficient means of energy gain for the electrons from the field. As is well known, the maximum energy increase for electrons takes place when the circularly polarized extraordinary component of the rf electric field, which is propagating along the magnetic field lines, becomes resonant with the cyclotron motion of the electrons when the local Larmor frequency Ω is close to the microwave frequency ω. The purpose of this report is mainly to consider the relation of the trapping of the electrons in a nonuniform magnetic field to their energy gain.

In the literature there are a number of papers[2] that deal with the calculation of the rate of energy gain of the electrons from the applied electromagnetic wave. Most of the researchers have used slightly increasing (or decreasing) nonuniform external magnetic fields. In this work the calculations will assume a more realistic magnetic field form that simulates the EBT field configuration. Also, the importance of the mirror trapping of the nonuniform magnetic field will be considered through the magnetic invariant of the motion μ. The approach for the computation of the energy gain of the electrons from the applied microwave field is similar to that of Grawe's,[3] except that here the particle motion is assumed to be nonrelativistic and to have the shape of the applied magnetic field.

The purpose of this paper is twofold. First, the heating rate of the electrons will be estimated with the use of the following inhomogeneous magnetic field:

$$B(z) = B_o \left[1 - \left(\frac{M - 1}{M + 1} \right) \cos \left(\frac{2\pi}{L} z \right) \right] . \tag{1}$$

This simulates the field strength reasonably well for the EBT.[4] Here M is the mirror ratio, L is the distance between the mirrors, and z is the minor axis. In this field most of the electrons will be trapped in the potential well and eventually heated at the resonance regions. It should be mentioned that the work done by Sprott and Edmonds[5] was based on Eq. (1) for the field model but that their computation was carried out numerically.

In this study the calculations employ a stochastic approach. The effects of the initial energy of the electrons and the effect of the mirror ratio on the trapped and untrapped electrons in the field will be studied. However, the Doppler effect of the wave will not be considered.

Second, after the heating rate $dW_\perp/dt$ is obtained, the loaded quality factor Q of the cavity will also be estimated. The results will then be applied to the typical EBT-I plasma.

2. THE BASIC EQUATIONS

The Lorentz force on the electron moving in a magnetic field in the presence of a uniform rf electric field $\underline{E}$ is given by

$$m \frac{d\underline{v}}{dt} = -e\underline{E} - e \frac{\underline{v} \times \underline{B}}{c} + \underline{\mu} \cdot \nabla B \ , \tag{2}$$

where the magnetic invariant of the motion $\mu = mv_{\perp}^2/2B$ and $v_{\perp}$ is the perpendicular component of the velocity with respect to the magnetic field. If we assume that the electric field is perpendicular to the magnetic field and is of the form

$$\underline{E}(\underline{r},t) = \underline{E} \exp(-i\omega t) \ ,$$

then the components of Eq. (2) in the Cartesian coordinate system are as follows:

$$\frac{dv_x}{dt} = -\frac{e}{m} E_x - \frac{e}{mc} Bv_y \ , \tag{3}$$

$$\frac{dv_y}{dt} = -\frac{e}{m} E_y + \frac{e}{mc} Bv_x \ , \tag{4}$$

and

$$\frac{dv_z}{dt} = \frac{d^2z}{dt^2} = -\frac{\mu}{m}\frac{dB}{dz} \ . \tag{5}$$

Here e (>0) and m represent the charge and the mass of the electron, respectively.

In these calculations the mirror ratio is defined as

$$M = \frac{B(z = 0)}{B(z = \pm L/2)} \ ,$$

where $z = \pm L/2$ is the location of the coil plane.

From Eqs. (3) and (4) we get

$$\frac{du}{dt} + i\Omega(t)u = A \exp(-i\omega t) \ , \tag{6}$$

where $u = v_x - iv_y$, $\Omega(t) = [eB(t)]/mc$, and $-(m/e)A = E_x - iE_y$ is the right-hand circularly polarized (extraordinary) rf field.

The solution of the differential Eq. (6) is of the form

$$u(t) = \left[u(0) + Ah(t)\right] \exp\left[-i \int_0^t \Omega(t')\, dt'\right] \ , \tag{7}$$

with

$$h(t) = \int_0^t \exp[i\Phi(t')]\, dt' \ , \tag{8}$$

and

$$\Phi(t) = -\omega t + \int_0^t \Omega(t')\, dt' \ . \tag{9}$$

Knowing $u(t)$, we may compute the perpendicular energy change of the electron in the following manner:

$$\Delta W_\perp(t) = \frac{m}{2}\,[v_\perp^2(t) - v_\perp^2(0)] = \frac{m}{2}\,[u(t)u^*(t) - u(0)u^*(0)] \ , \tag{10}$$

where $u^*(t)$ is the complex conjugate of $u(t)$.

From Eq. (7) we then find

$$\Delta W_\perp \frac{2}{m} = u(0)\, A^* h^*(t) + u^*(0)\, Ah(t) + |h(t)|^2\, |A|^2 \ . \tag{11}$$

Noting that $h(t) = |h(t)| \exp(i\psi')$ and $u(0) = v_\perp(0) \exp(i\phi')$ with an arbitrary phase ϕ', we find

$$u(0)\, A^* h^*(t) + u^*(0)\, Ah(t) = 2 \frac{e}{m} Ev_\perp(0)\ |h(t)|\ \sin(\phi' - \psi') .$$

Here, since $E_z = 0$ is assumed, $E = \sqrt{2}\, E_x = \sqrt{2}\, Ey$.

Making use of the stochastic approach,[3] Eq. (10) becomes

$$\langle \Delta W_\perp(t) \rangle = \frac{m}{2} |A|^2\ |h(t)|^2 . \tag{12}$$

Here

$$\langle f \rangle = \int_0^{2\pi} \frac{d\phi'}{2\pi} f .$$

Before proceeding further, we will study μ around the resonance region. Since $\mu = mv_\perp^2/2B$, we may write

$$\frac{\Delta\mu}{\mu} = \frac{\Delta v_\perp^2}{v_\perp^2} - \frac{\Delta B}{B}$$

or

$$\frac{\Delta\mu}{\mu} = \Delta z \left(\frac{\partial}{\partial t} \ln W_\perp - \frac{\partial}{\partial z} \ln B \right) .$$

In this study we assume that the thickness of the resonance region Δz is small enough that μ is still considered to be invariant. We should also mention that, under the weak electric field assumption, the energy gain of the electron is not large enough to violate the constancy of μ for a single pass from the resonance surface.

In Sect. 4 we will compute h(t) by making use of the axial motion of the electron z(t). Thus, we need to know the time evolution of the electron motion along the field line. This will be discussed in the next section.

3. CALCULATION OF AXIAL ORBITS z(t)

Let us recall from Eq. (5) that

$$\frac{d^2z}{dt^2} + \frac{\mu}{m}\frac{dB}{dz} = 0 \ . \tag{13}$$

Using

$$\frac{\mu}{m}\frac{dB}{dz} = \left[\frac{v_\perp^2(0)}{2}\right]^2 \alpha(M - 1) \ , \ \alpha = \frac{2\pi}{L}$$

in the above and noting that $d/dt = v_z(d/dz)$, we get

$$\frac{dz}{dt} = v_z = (d + f \cos \alpha z)^{1/2} \ . \tag{14}$$

Here

$$f = \frac{v_\perp^2(0)}{2}(M - 1) \geqslant 0 \tag{15}$$

and

$$d = v_z^2(0) - f \ . \tag{16}$$

The orbit equation z(t) then becomes

$$\int_0^\phi \frac{d\phi}{(d + f \cos \phi)^{1/2}} = \alpha t \ , \tag{17}$$

where $\phi = \alpha z(t)$.

From Graishteyn and Ryzhik[6] we have

$$
\int \frac{dx}{(a + b\cos x)^{1/2}} = \begin{cases} \dfrac{2}{\sqrt{a+b}}\, F\left(\dfrac{x}{2}, r\right) & \\ \quad \text{for } a > b > 0,\ 0 \leqslant x \leqslant \pi\,, & (18) \\ & \\ \sqrt{\dfrac{2}{B}}\, F\left(\gamma, \dfrac{1}{r}\right), \text{ for } b \geqslant |a| > 0, & \\ \quad 0 \leqslant x \leqslant \arccos(-a/b), & (19) \end{cases}
$$

where

$$
r = \sqrt{\frac{2b}{a+b}}\,,\ \gamma = \arcsin \sqrt{\frac{b(1-\cos x)}{a+b}}\,,
$$

$$
K(k) \equiv F(\psi = \pi/2, k),
$$

and the elliptic integral of the first kind

$$
F(\psi, k) = \int_0^{\psi} \frac{dx}{(1 - k^2 \sin^2 x)^{1/2}}\,.
$$

For our case $f \geqslant 0$, but $d > 0$ or $d < 0$ depending upon initial velocities and the mirror ratio. Let us study these cases separately.

3.1 THE CASE FOR $d > f > 0$

In this section, we will assume that $d > f > 0$. Since

$$
\frac{v_\perp^2(0)}{v_z^2(0)} \leqslant \frac{1}{M - 1}\,,
$$

then from Eq. (18) we write

$$F\left(\frac{\alpha z}{2}, r\right) = \frac{v_z(0)}{2} \alpha t \ , \tag{20}$$

where

$$r^2 = \frac{v_\perp^2(0)}{v_z^2(0)} \ (M - 1) < 1 \ .$$

From Ref. 6 we have

$$F(\psi, k) \cong \psi + \psi \frac{k^2}{4} - \frac{k^2}{8} \sin^2 \psi \cong \psi$$

if $k << 1$. When this value is used in Eq. (18),

$$F\left(\frac{\alpha z}{2}, r\right) \cong \frac{\alpha z}{2} = \frac{v_z(0)}{2} \alpha t$$

or

$$z(t) \cong v_z(0) t \ , \tag{21}$$

which describes the untrapped or passing electrons.

3.2 THE CASE FOR $f \geqslant |d| > 0$

Our basic assumption in this section is that $f \geqslant |d| > 0$. In this case

$$\frac{v_\perp^2(0)}{v_z^2(0)} \geqslant \frac{1}{M - 1} \ ;$$

we then find

$$F\left(\gamma,\frac{1}{r}\right) = \sqrt{\frac{f}{2}}\ \alpha t \ . \tag{22}$$

Here

$$\gamma = \text{arc sin}\left(r \sin \frac{\alpha z}{2}\right) \text{ and} \tag{23}$$

$$\frac{1}{r^2} = \frac{d + f}{2f} << 1 \ .$$

We may again approximate $f(\gamma,1/r)$ as

$$F\left(\gamma,\frac{1}{r}\right) \cong \gamma = \text{arc sin}\left(r \sin \frac{\alpha z}{2}\right) = \sqrt{\frac{f}{2}}\,\alpha t$$

or

$$z(t) = \frac{v_z(0)}{\omega_T} \sin (\omega_T t) \ , \tag{24}$$

where

$$\omega_T = \frac{\alpha}{2} v_\perp(0) \ (M - 1)^{1/2} \ . \tag{25}$$

We see from Eq. (24) that the electrons are oscillating in the magnetic trap with a frequency of $f_T \equiv \omega_T/2\pi$. We will consider this trapped electron case further:

1. From Eq. (23) we write $r \sin \alpha z/2 = \sin \gamma$. Since $r >> 1$ and $(\sin \gamma)_{max} = 1$, then $\alpha z/2 \equiv \theta < \pi/2$ or $z < L/2$, which means that the electrons are trapped in the mirrors.

2. We may now estimate $\theta_{max} \equiv \theta_c$ by writing $\sin \theta = 1/r \sin \gamma$ or

$$\theta_c = \text{arc sin}\left[\frac{v_z(0)}{v_\perp(0)\sqrt{M - 1}}\right] \equiv \frac{\alpha z_c}{2} \quad .$$

3. Let us find $t = t_c$, where $z(t_c) = z_c$. We know that $\gamma = \pi/2$ and $t = t_c$ when $\theta = \theta_c$. Thus, from Eq. (22) we may write

$$F\left(\frac{\pi}{2}, \frac{1}{r}\right) = \omega_T t_c = K\left(\frac{1}{r}\right)$$

and for $r^2 >> 1$,

$$K\left(\frac{1}{r}\right) \cong \frac{\pi}{2}\left(1 + \frac{1}{4r^2}\right) = \omega_T t_c \quad . \tag{26}$$

4. We now rewrite Eq. (14) as follows:

$$v_z(t) = v_\perp(0)(M - 1)^{1/2} \quad \sin^2 \theta_c - \sin^2 \frac{\alpha z}{2} \quad ^{1/2} \quad .$$

This gives

$$\sin^2 \theta < \frac{v_z^2(0)}{v_\perp^2(0)(M - 1)}$$

and for $\theta = \theta_c$, $v_z(t = t_c) = 0$. We may identify this as a turning point of the electron.

In summary, when $\gamma = \pi/2$, then $\theta = \theta_c$, $t = t_c$, $v_z(t_c) = 0$, $z = z_c$, and $t_c \cong \pi/2\omega_T$. Henceforth, z_c will be identified as a turning point of the electron and will be denoted as z_t, which is $1 - \cos \alpha z_t = 1/r^2$.

4. CALCULATION OF h(t) FOR THE TRAPPED ELECTRON

We start by recalling the definition of h(t), which is

$$h(t) = \int_0^t \exp\,[i\Phi(dt')]\,dt' ,$$

with

$$\Phi(t) = -\omega t + \int_0^t \Omega(t')\,dt' ,$$

where

$$\frac{\Omega(t)}{\Omega_o} = 1 - \beta \cos \alpha z(t) \quad . \tag{27}$$

Here $\beta \equiv (M - 1)/(M + 1)$ and $\Omega_o \equiv eB_o/mc$.

Let us define T(t) such that

$$\int_0^t \Omega(t')\,dt' = \Omega_o t - \Omega_o \beta T(t)$$

and

$$T(t) = \int_0^t \cos \alpha z(t')\,dt' \quad . \tag{28}$$

Using Eq. (14),

$$\alpha T(t) = \int_0^\phi \frac{\cos \phi}{(d + f \cos \phi)^{1/2}}\,d\phi ,$$

or from Gradshteyn and Ryzhik[6] we find

$$T(t) = \sqrt{\frac{2}{\alpha^2 f}}\left[2E\left(\gamma,\frac{1}{r}\right) - F\left(\gamma,\frac{1}{r}\right)\right] . \tag{29}$$

Here $E(\psi,k) = \int_0^\psi dx(1 - k^2 \sin^2 x)^{1/2}$ is the elliptical integral of the second kind and $E(k) \equiv E(\psi = \pi/2,k)$. Again from Ref. 6 we have

$$E(\psi,k) = k_1^2 F(\psi,k) + kk_1^2 \frac{\partial F}{\partial k} + \frac{k^2 \sin^2 \psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

with $k_1^2 = 1 - k^2$.

Using $F(\gamma,1/r) = \sqrt{f/2}\,\alpha t$ in the above relation, we obtain

$$E\left(\gamma,\frac{1}{r}\right) = \frac{\sin^2 \gamma}{2r^2(1 - \sin^2 \gamma/r^2)^{1/2}}$$

and therefore

$$T(t) = -t + \frac{\sin^2 \gamma}{\lambda r^3 \sqrt{1 - \sin^2 \gamma/r^2}} ,$$

where $\lambda = (\alpha/2)v_z(0)$. The case we are interested in is $r^2 >> 1$; thus Eq. (9) takes the form

$$\Phi(t) = 2\omega_T \delta t - \kappa \sin (2\omega_T t) . \tag{30}$$

Here

$$2\omega_T \delta \equiv [\Omega_o(1 + \beta) - \omega] \text{ and} \tag{31}$$

$$\kappa \equiv \frac{\Omega_o \beta}{\omega_T r^2} \quad . \tag{32}$$

We now compute the perpendicular energy gain for one reflection time, that is, when $t_R \equiv 2t_c$:

$$\langle \Delta W_\perp (t_R) \rangle = \frac{m}{2} \, |A|^2 \, |h(t_R)|^2 \ , \tag{33}$$

with

$$h(t_R) = \int_0^{t_R} \exp i \, [2\omega_T \delta t - \kappa \sin (2\omega_T t)] \, dt \ .$$

For the trapped electrons, we know that $\omega_T t_c \cong \pi/2$, which leads to

$$\Psi_R \equiv 2\omega_T t_R = 2\pi \ .$$

Thus,

$$h(t_R) = \frac{1}{2\omega_T} \int_0^{2\pi} \exp \, [i(\delta\Psi - \kappa \sin \Psi)] \, d\Psi \ ,$$

and since (see Appendix A)

$$\int_0^{2\pi} \exp \, [i(\nu\Psi - z \sin \Psi)] \, d\Psi = 2\pi \, h(\nu, z) \ ,$$

then

$$h(t_R) = \frac{\pi}{\omega_T} h(\delta, \kappa) \ . \tag{34}$$

5. ESTIMATION OF THE HEATING RATE $dW_\perp/dt$

From Appendix A, we have

$$|h(\nu,z)| = |\tilde{J}_{-\nu}(z)| \ ,$$

where

$$\tilde{J}_{-\nu}(z) = J_{-\nu}(z) - \frac{\sin \nu\pi}{\pi} \int_0^\infty dt \exp (\nu t - z\mathrm{Sh}t) \ .$$

Recalling that $|A|^2 = (e^2/m^2)E^2$ and $t_R = \pi/\omega_T$, then the energy gain of the electron may be rewritten for $\Delta t = t_R$ as

$$\langle \Delta W_\perp (t_R) \rangle = \frac{e^2E^2}{2m} \left(\frac{\pi}{\omega_T}\right)^2 |\tilde{J}_{-\delta}(\kappa)|^2 \ .$$

Hence the heating rate is

$$\frac{dW_\perp}{dt} = \frac{\langle \Delta W_\perp (t_R) \rangle}{t_R} \ .$$

Considering the density of the electrons n_e in the plasma and the presence of two resonance regions for a cavity, the heating rate for the plasma takes the form

$$\left(\frac{dW_\perp}{dt}\right)_p = \frac{\omega_p^2}{4\omega_T} E^2 |\tilde{J}_{-\delta}(\kappa)|^2 \ , \tag{35}$$

where

$$\omega_p^2 = \frac{4\pi n_e e^2}{m} \ ,$$

$$\frac{\Omega_o \beta}{\omega_T r^2} \text{ , and} \tag{36}$$

$$\delta = \frac{\Omega_o(1 + \beta) - \omega}{2\omega_T} \quad . \tag{37}$$

Since $B(z) = B_o(1 - \beta \cos \alpha z)$ and $\Omega(z) = \Omega_o(1 - \beta \cos \alpha z)$ at the resonance region [i.e., $z = z_\mu$ or $\omega = \Omega(z_\mu) = \Omega_o(1 - \beta \cos \alpha z_\mu)$], then

$$\delta = \frac{\Omega_o \beta}{2\omega_T} (1 + \cos \alpha z_\mu) \quad . \tag{38}$$

We may also define a mirror ratio M_μ such that

$$M_\mu \equiv \frac{B(z_\mu)}{B(0)} = \frac{B_o(1 - \beta \cos \alpha z_\mu)}{B_o(1 - \beta)}$$

or

$$M_\mu = \frac{M + 1}{2} (1 - \beta \cos \alpha z_\mu) \quad . \tag{39}$$

Using Eq. (39) in Eq. (38) we obtain

$$\delta = \frac{\Omega_o}{\omega_T} \frac{M - M_\mu}{M + 1} \text{ ,} \tag{40}$$

and also

$$\varepsilon \equiv \frac{\kappa}{\delta} = \frac{v_z^2(0)}{v_\perp^2(0)} \frac{1}{M - M_\mu} \; . \tag{41}$$

In terms of ε, the expression for the turning point may take the form

$$\cos \alpha z_t = 1 - 2\varepsilon \frac{M - M_\mu}{M - 1} \; . \tag{42}$$

6. THE CONDITION FOR THE RESONANCE HEATING OF TRAPPED ELECTRONS

The condition for the resonance heating is

$$z_\mu \leqslant z_t$$

or

$$\cos \alpha z_t \leqslant \cos \alpha z_\mu \; .$$

Using the expressions for z_t and z_μ we get

$$\frac{2\Delta}{\delta} \leqslant 1 + \frac{\kappa}{\delta} \; , \tag{43}$$

where $\Delta \equiv \Omega_o \beta / 2\omega_T$. Recalling that

$$\frac{2\Delta}{\delta} = \frac{M - 1}{M - M_\mu}$$

and making use of Eq. (41) in Eq. (43), we find

$$\frac{v_\perp^2(0)}{v_z^2(0)} \leqslant \frac{1}{M_\mu - 1} \; .$$

This is the necessary condition for electron cyclotron resonance interactions with the applied microwave field. On the other hand, as we discussed in Sect. 3.2, the condition necessary for mirror trapped electrons is

$$\frac{v_\perp^2(0)}{v_z^2(0)} > \frac{1}{M - 1} \; .$$

Therefore, combining these two equations, we write

$$\frac{1}{M - 1} < \frac{v_{\perp}^2(0)}{v_z^2(0)} \leq \frac{1}{M_{\mu} - 1} \quad . \tag{44}$$

7. THE LOADED QUALITY FACTOR Q_L

The experimentally observable quality factor Q_L is defined by

$$\frac{1}{Q_L} = \frac{1}{Q_{plasma}} + \frac{1}{Q_{wall}} \quad ,$$

where $Q_{wall} = F_w/\delta_s$ is the Q factor of the cavity wall due to skin effect losses. The form factor of the cavity F_w may be approximated to be the radius of the cavity R_c. The skin depth δ_s is given by

$$\delta_s = \sqrt{\frac{2}{\omega\sigma\mu_o}} \quad ,$$

where $\mu_o = 4\pi \times 10^{-7}$ H/m and σ is the conductivity of the wall. For an aluminum wall, as in the EBT for example, $\delta_s \cong 6 \times 10^{-5}$ cm at 18 GHz. Therefore, with very good approximation, we may write[7]

$$Q_L \cong Q_{plasma} = \omega \frac{W}{P_{ab}} \quad . \tag{45}$$

Here, $W = E^2/4\pi$ is the stored energy of the wave and

$$P_{ab} = \left(\frac{V_p}{V_c}\right) \left(\frac{dW_\perp}{dt}\right)_p$$

is the absorbed power in the cavity; the plasma volume is V_p and the cavity volume is V_c. Using Eq. (35) in Eq. (45), we find

$$Q_L = \left(\frac{\omega\omega_T}{\pi\kappa_p^2}\right)\left(\frac{V_c}{V_p}\right)\frac{1}{|\tilde{J}_{-\delta}(\kappa)|^2} \quad . \tag{46}$$

We may further simplify the above expression by assuming δ to be an integer, that is,

$$|\tilde{J}_{-\delta}(\kappa)|^2 = J_\delta^2(\varepsilon\delta) ,$$

and from Watson[8]

$$J_\delta(\varepsilon\delta) \cong \frac{\varepsilon^\delta \exp\left(\delta\sqrt{1-\varepsilon^2}\right)}{\sqrt{2\pi\delta}\,(1-\varepsilon^2)^{1/4}\,(1+\sqrt{1-\varepsilon^2})^\delta} ,$$

which is valid when ε lies between 0 and 1 and δ is large.

Substituting the last expression in Eq. (46) and recalling that $\delta\omega_T = \Omega_o(M - M_\mu)/(M + 1)$, we obtain

$$Q_L = 2\,\frac{V_c}{V_p}\,\frac{f_\mu f_o}{f_p^2}\,\frac{M - M_\mu}{M + 1}\,\tilde{F}(\varepsilon,\delta) \tag{47}$$

and the heating rate

$$\left(\frac{dW_\perp}{dt}\right)_p = \frac{f_p^2}{f_o}\,\frac{M + 1}{M - M_\mu}\,\frac{E^2}{4\tilde{F}(\varepsilon,\delta)} .$$

Here, we have $f_\mu = \omega/2\pi$, $f_p = \omega_p/2\pi$, $f_o = \Omega_o/2\pi$, and

$$\tilde{F}(\varepsilon,\delta) = \frac{(1-\varepsilon^2)^{1/2}\,(1+\sqrt{1-\varepsilon^2})^{2\delta}}{\varepsilon^{2\delta}\exp\left(2\delta\sqrt{1-\varepsilon^2}\right)} , \tag{48}$$

which is studied in detail in Appendix B.

We should also mention that the lower value of Q_L is advantageous because of more power absorption in the plasma; see Eq. (45).

8. APPLICATION TO EBT-I AND CONCLUSION

The orientation of the microwave power coupling in EBT is ordinary wave rather than extraordinary wave,[1] because the ordinary wave propagates readily through the plasma under normal conditions of $\omega > \omega_p$ and also provides more uniform power distribution throughout the cavity. Since the cavity walls are highly reflective, the input wave undergoes multiple low loss reflections, and hence its polarization and direction change so that the total input wave is soon converted to the extraordinary wave, which is damped heavily by the resonance interactions with the plasma.

We should also keep in mind that the input power is distributed among the various plasma components of the EBT. For example, the total power P_μ is the summation of the power loss in the transmission and distribution system P_D, the annulus power P_A (which is needed for stable toroidal plasma), the power for the surface plasma P_S, and the power for the toroidal plasma P_T. Typically, we have[9]

$$\frac{P_D}{P_\mu} = 0.35, \ \frac{P_A}{P_\mu} = 0.2, \ \frac{P_S}{P_\mu} = 0.2, \text{ and } P_T/P_\mu = 0.25.$$

We may now estimate the loaded Q value of the toroidal plasma. The typical EBT-I parameters are as follows:[9]

$N = 24$ cavities,

$M = 2$,

$B_{max} = 10$ kG,

$R_o = 150$ cm (the major radius),

$a = 10$ cm (the plasma radius),

$R_c = 26$ cm (the cavity radius),

$f_\mu = 18$ GHz

$n_e \cong 10^{12}$ cm^{-3} (the plasma density), and

$V_t = 1.35$ m^3 (the torus volume).

With this information we can compute the following quantities:

$$\alpha = \frac{2\pi}{L} = \frac{N}{R_o} ,$$

$$B_o = \left(\frac{M + 1}{2M}\right) B_{max} ,$$

$$B_\mu \cong 6.42 \text{ kG} ,$$

$$M_\mu = \frac{B_\mu}{B_{min}} = 1.28 ,$$

$$f_p = \frac{\omega_p}{2\pi} = 9 \text{ GHz} ,$$

$$f_o = \frac{\Omega_o}{2\pi} = 21 \text{ GHz} , \text{ and}$$

$$\delta = 2\Delta \left(\frac{M - M_\mu}{M - 1}\right) = \frac{3\pi}{v_\perp(0)/c} >> 1 .$$

From the resonance heating condition of Eq. (44), we have

$$1 < \frac{v_\perp^2(0)}{v_z^2(0)} < 3.5 .$$

We may now assume that the initial velocity distribution is isotropic; thus,

$$\frac{v_\perp^2(0)}{v_z^2(0)} \cong 2 \ ,$$

which satisfies the above condition. Furthermore, using this value in Eqs. (39), (41), and (42), we find

$$z_\mu = 0.177 \ L \ ,$$

$$\varepsilon = \frac{\kappa}{\delta} = 0.694 < 1 \ ,$$

and

$$z_t = 0.25 \ L > z_\mu \ .$$

Knowing the total volume of torus V_t of the EBT and estimating the plasma volume from the turning points, we get

$$\frac{V_c}{V_p} = \frac{V_t}{\pi^2 a^2 R_o} \cong 9.12 \ .$$

Using these computed quantities in Eq. (47), we find

$$Q_L \cong 2 \times 9.12 \times \frac{18 \times 21}{9^2} \times \frac{2 - 1.28}{2 + 1} \times 0.72$$

and

$$Q_L \cong 15 \ ,$$

which is close to the previously estimated[10] value for the EBT-I plasma that includes the contribution from the annulus and surface plasma. Since one may write

$$\frac{1}{Q_L} = \frac{1}{Q_T} + \frac{1}{Q_S} + \frac{1}{Q_A} ,$$

where Q_T, Q_S, and Q_A are the Q values of toroidal, surface, and annulus plasma, respectively, then one can conclude that Q_S, and $Q_A > Q_T$ and thus $Q_L \cong Q_T$.

In conclusion, in this work, we have computed the heating rate of the electron cyclotron heated EBT plasma considering a bumpy magnetic field. We then obtained a relation for the quality factor of the cavity that may be observable experimentally. The expression of Q_L [Eq. (47)] contains most of the physical and plasma parameters of the device; thus, it may be useful for parametric studies of different devices.

APPENDIX A — COMPUTATION OF $|h(\nu,z)|$

Following is the computation of $|h(\nu,z)|$:

$$h(\nu,z) = \frac{1}{2\pi}\int_0^{2\pi} \exp\,[i(\nu\Theta - z\sin\Theta)]\,d\Theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos\,(\nu\Theta - z\sin\Theta)\,d\Theta$$

$$+ \frac{i}{2\pi}\int_0^{2\pi} \sin\,(\nu\Theta - z\sin\Theta)\,d\Theta\ ;$$

$$h(\nu,z) = \tilde{J}_\nu(z)\cos^2\pi\nu + \frac{1}{2}\tilde{E}_\nu(z)\sin^2\pi\nu$$

$$+ i\left[\tilde{E}_\nu(z)\sin^2\pi\nu + \frac{1}{2}\tilde{J}_\nu(z)\sin^2\pi\nu\right].$$

Here $\tilde{J}_\nu(z)$ is the Anger[8] function,

$$\tilde{J}_\nu(z) = \frac{1}{2\pi}\int_0^{\pi} \cos\,(\nu\Theta - z\sin\Theta)\,d\Theta\ ,$$

and $\tilde{E}_\nu(z)$ is the Weber[8] function

$$\tilde{E}_\nu(z) = \frac{1}{2\pi}\int_0^{\pi} \sin\,(\nu\Theta - z\sin\Theta)\,d\Theta\ .$$

Since ν and z are the real quantities for our case, then

$$|h(\nu,z)|^2 = [\tilde{J}_\nu(z) \cos \pi\nu + \tilde{E}_\nu(z) \sin \pi\nu]^2 ,$$

and, making use of the relation

$$\tilde{E}_\nu(z) \sin \pi\nu = \tilde{J}_\nu(z) - \tilde{J}_{-\nu}(z) \cos \pi\nu ,$$

we find

$$|h(\nu,z)|^2 = |\tilde{J}_{-\nu}(z)|^2 .$$

Again from Watson,[8]

$$\tilde{J}_{-\nu}(z) = J_{-\nu}(z) - \frac{\sin \nu\pi}{\pi} \int_0^\infty \exp (\nu t - z\mathrm{Sh}t)\, dt ,$$

where $J_\nu(z)$ is the Bessel function of arbitrary order ν.

Watson[8] also gives the following:

$$\tilde{J}_\nu(z) \cong J_\nu(z) + \frac{\sin \nu\pi}{\pi z}\left(1 - \frac{1 - \nu^2}{z^2} + \ldots\right) \cdot$$

$$- \frac{\sin \nu\pi}{\pi z}\left[\frac{\nu}{z} - \frac{\nu(2^2 - \nu^2)}{z^3} + \ldots\right],$$

and assuming that $z = \varepsilon\nu$ when $\varepsilon \lesssim 1$, we find

$$\tilde{J}_\nu(z) = \tilde{J}_\nu(\varepsilon\nu) \cong J_\nu(\varepsilon\nu) + \frac{\sin \nu\pi}{\pi\nu} \left(\frac{1 - \varepsilon}{\varepsilon^3}\right).$$

Furthermore, from Watson[8] we also have

$$J_\nu(\varepsilon\nu) = \frac{(\nu\varepsilon)^\nu \exp\left(\nu\sqrt{1-\varepsilon^2}\right)\exp(-V_\nu)}{\exp(\nu)\,\Gamma(\nu+1)(1-\varepsilon^2)^{1/4}(1+\sqrt{1-\varepsilon^2})^\nu} ,$$

with

$$V_\nu \cong \frac{1}{24\nu}\left[\frac{2+3\varepsilon^2}{(1-\varepsilon^2)^{3/2}} - 2\right] .$$

APPENDIX B — COMPUTATION OF $\tilde{F}(\varepsilon,\delta)$

In this appendix we estimate $\tilde{F}(\varepsilon,\delta)$, which is Eq. (48) for large values of δ as a function of ε. Let us start by making a variable transformation in Eq. (48); we write

$$\tilde{F}(\varepsilon,\delta) = \frac{x(1 + x)^{\delta}}{(1 - x)^{\delta}} e^{-2\delta x} ,$$

where $x = (1 - \varepsilon^2)^{1/2}$. Recalling that

$$e^{\delta x} = \lim_{k \to \infty} \left(1 + \frac{\delta x}{k}\right)^{k} ,$$

and furthermore replacing k by δ, we find

$$e^{\delta x} = \lim_{\delta \to \infty} \left(1 + \frac{\delta x}{\delta}\right)^{\delta} = \lim_{\delta \to \infty} (1 + x)^{\delta} .$$

Similarly,

$$e^{-\delta x} = \lim_{\delta \to \infty} (1 - x)^{\delta} .$$

Therefore, $\tilde{F}$ becomes as

$$\lim_{\delta \to \infty} \tilde{F}(x,\delta) = x \frac{e^{\delta x}}{e^{-\delta x}} e^{-2\delta x} = x$$

or

$$\lim_{\delta \to \infty} \tilde{F}(\varepsilon,\delta) = (1 - \varepsilon^2)^{1/2} .$$

The values of $\tilde{F}$ are given below for $0 \leq \varepsilon \leq 1$.

ε	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\tilde{F}$	1	0.995	0.979	0.957	0.916	0.866	0.8	0.714	0.6	0.435	0

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