Zero-Point Energy in Bag Models

by

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MASTER

Abstract

The zero-point (Casimir) energy of free vector (gluon)

fields confined to a spherical cavity (bag) is computed.

With a suitable renormalization the result for eight gluons

is

$$E \approx + \frac{0.51}{a} \ .$$

This result is substantially larger than that for a spher-

ical shell (where both interior and exterior modes are pre-

sent), and so affects Johnson's model of the QCD vacuum.

It is also smaller than, and of opposite sign to the value

used in bag model phenomenology, so it will have important

implications there.

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I. Introduction

Quantum chromodynamics (QCD) may well be the appropriate theory of hadronic matter. However, the theory is not at all well understood. It may turn out that color confinement is roughly approximated by the phenomenologically successful bag model [1,2]. In this model, the normal vacuum is a perfect color magnetic conductor, that is, the color magnetic permeability μ is infinite, while the vacuum in the interior of the bag is characterized by $\mu = 1$. This implies that the color electric and magnetic fields are confined to the interior of the bag, and that they satisfy the following boundary conditions on its surface S:

$$\vec{n} \cdot \vec{E} \big|_S = 0, \quad \vec{n} \times \vec{B} \big|_S = 0, \tag{1}$$

where $\vec{n}$ is normal to S. Now, even in an "empty" bag (i.e., one containing no quarks) there will be non-zero fields present because of quantum fluctuations. This gives rise to a zero-point or Casimir energy [3,4]. It would be anticipated that this energy would have the form $-Z/a$, where a is the radius of a (spherical) bag and Z is some pure number. Indeed, such a term has been put in bag model calculations, and a good fit has been obtained for $Z = 1.84$ [1,2]. It is my purpose here to insist that the Casimir energy must be determined by the underlying dynamics, presumably QCD. I will calculate Z in the approximation that the gluons are free inside the bag (which is roughly justified by asymptotic freedom), with a result that appears to be quite incompatible with the phenomenological value.

A related, but somewhat different motivation for this work comes from Johnson's recent model for the QCD ground state wave function [5]. Effectively, he supposes that space is filled with bags, the boundaries of which confine

color to small, asymptotically free, regions. He uses the classic result of Boyer [6], as subsequently improved [7,8], for the electrodynamic Casimir energy of a perfectly conducting spherical shell, together with various guesses for the higher-order effects, to estimate the parameters of the bag model. But the QED calculations cannot be properly extrapolated to this situation, for they refer to a single shell in otherwise empty space. There is a delicate balance between interior and exterior energy contributions, so that only the sum is cutoff independent. The closely packed bags in Johnson's model present a quite different situation. In fact, the energy density Johnson requires will be provided by the result of the calculation presented here, since the energy of space filled with contiguous bags is simply the sum of the field energies contained within each bag.

II. Calculation of Zero-Point Energy

Our discussion follows closely on the formalism developed in [8], as extended to the cases of dielectric and conducting balls in [9]. For electrodynamics the situation we consider is as shown in Fig. 1a. Duality $(\vec{E} \rightarrow \vec{H}, \vec{H} \rightarrow -\vec{E})$ then allows us to extend the result to the QCD case, Fig. 1b, where one also must allow for the fact that there are 8 vector gluon fields.

There are several methods of proceeding. One can compute the energy (or the stress on the surface) when the dielectric constant ϵ is finite in the exterior region, letting $\epsilon \rightarrow \infty$ at the end of the calculation. This is the procedure followed in [9]. Alternatively one can calculate the result directly for a spherical cavity in an infinite conductor. Since all methods agree, we simply derive here the expression for the zero-point energy in the latter case. It may be obtained from the interior contribution of Eq.(3.9) of [8]:

$$E = \frac{1}{2i} \sum_{\ell=1}^{\infty} (2\ell+1) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \int_{0}^{a} r^2 dr \left[2k^2 (\tilde{F}_\ell + \tilde{G}_\ell)(r,r) \right.$$

$$\left. + \frac{1}{r^2} \frac{d}{dr} r \left\{ \frac{d}{dr'} r' \left(\tilde{F}_\ell(r,r') + \tilde{G}_\ell(r,r') \right) \right\}_{r=r'} \right]. \tag{2}$$

We should emphasize that the cutoff $\tau \rightarrow 0$ emerges naturally from the overlap of field points, with no reference to the properties of the shell. Here $k = |\omega|$, and $\tilde{F}_\ell$ and $\tilde{G}_\ell$ are the transverse electric and magnetic Green's functions from which the vacuum parts have been removed:

$$r,r' < a: \quad \begin{cases} \tilde{F}_\ell(r,r') \\ \\ \tilde{G}_\ell(r,r') \end{cases} = -A_{F,G} \, ik \, j_\ell(kr) j_\ell(kr'), \tag{3}$$

where

$$A_F = h_\ell^{(1)}(ka)/j_\ell(ka),$$

$$A_G = [ka\, h_\ell^{(1)}(ka)]'/[ka\, j_\ell(ka)]' . \tag{4}$$

In the spherical shell calculation of [8] there were both interior and exterior contributions to the energy, and as a consequence the surface term [the second term in (2)] vanished. This is not the case here; in fact, the surface term cancels a portion of the first term in (2), leaving us with

$$E = -\frac{1}{i}\sum_{\ell=1}^{\infty}(2\ell+1)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}(A_F+A_G)$$

$$\times ka\left\{\left([ka\,j_\ell(ka)]'\right)^2 + ((ka)^2 - \ell(\ell+1))(j_\ell(ka))^2\right\}$$

$$= -\frac{1}{2\pi a}\sum_{\ell=1}^{\infty}(2\ell+1)\int_{0}^{\infty}dx\cos x\delta\; x\left\{\frac{s'_\ell}{s_\ell} + \frac{s''_\ell}{s'_\ell} + 2(e'_\ell s'_\ell - e_\ell s''_\ell)\right\}(x), \tag{5}$$

where we have performed a Euclidean rotation [8],

$$\omega \to i k_4, \quad k \to i|k_4|, \quad \tau \to i(x_4 - x'_4), \tag{6}$$

and let

$$x = |k_4|a, \quad \delta = (x_4 - x'_4)/a. \tag{7}$$

The Bessel functions of imaginary argument here are

$$s_\ell(x) = (\pi x/2)^{\frac{1}{2}}I_{\ell+\frac{1}{2}}(x),$$

$$e_\ell(x) = (2/\pi)(\pi x/2)^{\frac{1}{2}}K_{\ell+\frac{1}{2}}(x). \tag{8}$$

Expression (5) is nearly, but not quite, the same as that found by Bender and Hays [10]. Apart from an overall sign, their formula has an extra term which arises precisely from the neglect of the surface term in (2).

The result (5) is exactly what one would anticipate from the earlier work in [8] and [9]. The term proportional to

$$\frac{s_\ell'}{s_\ell} + \frac{s_\ell''}{s_\ell'} = \frac{d}{dx} \log s_\ell s_\ell'$$

is just the inside part of the spherical shell result, Eq.(3.15) of [8], while the remaining term is just the corresponding "contact term," Eq.(31) of [9], which cancels for a shell. [It is the negative of what one would obtain from use of the free space Green's function in (2).] What we have here is simply the "mirror image" of the exterior result of [9], Eq.(51). That means we can with no labor obtain a numerical result, since

$$E_{interior} = E_{shell} - E_{exterior} \cdot \tag{9}$$

Now E_{shell} is cutoff (δ) independent, and has been numerically evaluated to 4 significant figures (Eq.(5.21) of [8]):

$$E_{shell} = 0.04618/a. \tag{10}$$

On the other hand, $E_{exterior}$ has a term depending on δ; when the finite part was extracted using uniform asymptotic expansions it was found that (cf. Eqs.(55) and (56) of [9], but here included are $O((\ell+\tfrac{1}{2})^{-3})$ terms in the expansion of the curly bracket of (5) as well)

$$E_{exterior} \cong \frac{1}{\pi a}\left(\frac{4}{3\delta^2} - \frac{1}{8}\right) + \frac{3}{128a} \cdot \tag{11}$$

For consistency we should use the same, very good approximation for E_{shell} (see Eq.(5.17) of [8]):

$$E_{shell} \cong \frac{3}{64a} \ .$$

(12)

(Note that half this value appears as the second term in (11).) Then the zero-point energy of a single, free, vector field confined by a spherical cavity of radius a is

$$E_{interior} \cong - \frac{4}{3\pi a \ \delta^2} + \frac{1}{128a}\left(\frac{16}{\pi} + 3\right) \ .$$

(13)

Note that the cutoff-dependent term in (13) has the form, but not the sign, predicted by Balian and Bloch [11]. (The discrepancy may lie in our "volume" energy subtraction, see [8] and [9].)

III. Conclusions

The cutoff-dependent term in (13) reflects our continuing ignorance about field theory. Renormalization remains a recipe for dealing with difficult physics that we do not really understand. However, let us suppose that the constant force term (recall $\delta = \tau/ia$) involving δ can be absorbed by a suitable counterterm [2]. One is left then with

$$E^{Ren,QED}_{interior} \cong \frac{1}{128a}\left(\frac{16}{\pi}+3\right) = \frac{0.063}{a} \ . \tag{14}$$

The QCD bag value is eight times larger:

$$E^{Ren,QCD}_{interior} \cong \frac{1}{16a}\left(\frac{16}{\pi}+3\right) = \frac{0.51}{a} \ . \tag{15}$$

We can now make various comments.

(i) The numerical value in (15) is substantially smaller than, and of opposite sign to, -Z used in bag model parameterizations. (A typical value there is Z = 1.84 [1,2]. However, part of this is a center of mass effect [12], so the appropriate number to compare with ours is $Z' \sim 1$.) The force here, like for the shell, is repulsive, contrary to one's naive expectations.

(ii) The value here presented is of the same sign, but 40 percent larger than that appropriated by Johnson in his model of the vacuum [5]. This may have significant numerical implications for the phenomenological applications; however, since his parameter b (representing higher order effects) remains uncomputed, no decisive statement can yet be made.

(iii) The presence of the cutoff-dependent term in the zero-point energy (after all standard volume energy subtractions have been performed) is a serious matter which must be understood, and may prove to be quite important. The validity of the asymptotic expansion used in obtaining these results might also be questioned. In particular, it is disturbing to note that the $O\left((\ell+\frac{1}{2})^{-4}\right)$

terms appears to introduce an additional, logarithmic dependence on δ. [It appears fortuitous that the shell result (10) is independent of δ.]
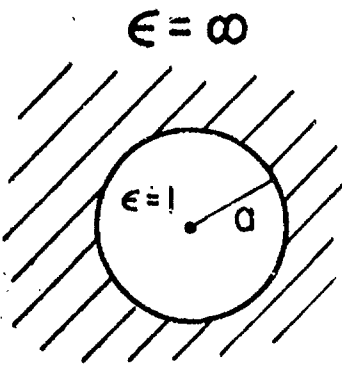
(iv) I am presently recomputing the Casimir energy due to massless quarks in this model. (The previous calculation is in [10].) The results will be presented elsewhere.
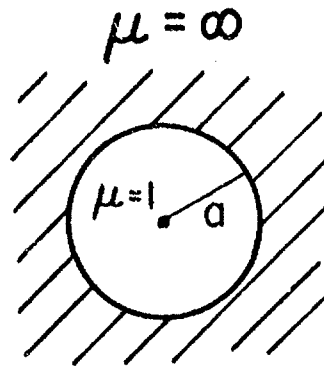
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Figure Caption

(a) The geometry of a spherical cavity imbedded in a perfect conductor.

(b) The dual geometry of the bag model.

$$\epsilon = \infty \qquad \mu = \infty$$

$$\epsilon = 1 \quad a \qquad \mu = 1 \quad a$$

(a) (b)