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ROLAND EÖTVÖS UNIVERSITY

1088 Budapest, VIII. Puskin u. 5-7.

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Zalán Horváth and Tibor Kiss-Tóth

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ON STATIC, AXIALLY SYMMETRIC SOLUTIONS OF SU/2/ MONOPOLES

Z. Horváth and T. Kiss-Tóth

Institute for Theoretical Physics, Roland Eötvös University, Budapest

Abstract

The equations for the most general axially symmetric ansatz are derived for SU/2/ monopoles in the Bogomol'ny--Prasad-Sommerfield limit. The symmetries of these equations are described. The existence of an infinite parameter invariance group is pointed out.

1. Introduction

To this date, no one has found a multimonopole solution with acceptable physical properties /e.g. finite energy/. There exist so-called "no-go" theorems which prove that it is impossible to get such solutions in the case of spherically symmetric ansätze. These theorems compel one to work with spherically nonsymmetric ansätze which leads to complicated coupled differential equations. All efforts to circumvent these difficulties until now have been unsuccessful [1].

The "no-go" theorems say nothing about the existence of solutions with several monopoles at different locations, because such field configurations are not rotationally symmetric.

We are looking for axially symmetric SU/2/ monopoles in the Bogomol'ny-Prasad-Sommerfield /BPS/ limit in a classical non-Abelian gauge theory with a Higgs field in the adjoint representation, such that we can treat it as the A_o component of the potential A_ℓ . In the BPS limit the scalar becomes massless and can mediate a long range force which can cancel the magnetic forces. It has been shown [2,3] that the force between monopoles decreases faster than any inverse power of the separation. This encourages one to hope that these forces are exactly zero and static, noninteracting, finite energy multimonopoles may exist.

In section 2 we study the Bogomol'ny equations which may describe axially symmetric, static monopoles. We will also point out the equivalence of the Bogomol'ny equations to the static selfduality equations /SDE/. Using the R gauge [4] form of SDE we derive the equations which describe the most general axially symmetric solutions. We also show that these equations are equivalent to the complex generalized Ernst equations. It will be proved that it is possible to define for these equations a similar mapping as the Neugebauer-Kramer mapping 5 for the Ernst equations, which is an important feature because it makes it possible to generate new solutions from given solutions of the equations.

Our equations have an SL/2,C/ invariance group which is the extension of the SL/2,R/ group of the Ehlers transformation [6] . These two transformations, i.e. Neugebauer-Kramer and SL/2,C/, used together generate an infinite parameter invariance group which is the generalisation of the so-called Geroch group of the Ernst equation [7].

The discovery of these facts is important because the group theory methods may be useful for generating new solutions.

In section 3 it will be shown that these equations can be written in σ-model form, which is invariant under an SO /3,1/ group. This is important because recently some new methods were presented for generating exact solutions of -models. These are on one hand the inverse scattering methods and the Bäcklund transformation on the other hand. It is certain that these methods will be applicable to these σ-models.

2. The Description of the Most General, Static Axially Symmetric Ansatz

In this section we consider sourceless gauge fields in Yang's R gauge [4] and describe the most general static, axially symmetric ansatz.

It is extremely convenient to introduce complex variables with the following convention:

$$\sqrt{2}\, y = x_1 + i x_2, \qquad \sqrt{2}\, \bar{y} = x_1 - i x_2,$$
$$\sqrt{2}\, z = x_3 - i x_4, \qquad \sqrt{2}\, \bar{z} = x_3 + i x_4. \qquad /1/$$

The Jacobian of the transformation $/x_1, x_2, x_3, x_4/ \longrightarrow /y, \bar{y}, z, \bar{z}/$ is 1. The metric in the new variables has only the following nonvanishing elements:

$$g_{y\bar{y}} = g_{\bar{y}y} = g_{z\bar{z}} = g_{\bar{z}z} = 1 \qquad /2/$$

then the condition of selfduality reduces to

$$F_{yz} = F_{\bar{y}\bar{z}} = 0, \qquad /3/$$

$$F_{y\bar{y}} + F_{z\bar{z}} = 0. \qquad /4/$$

In the Yang's R gauge defined by

$$R = \left[1 + \varphi + \rho(\sigma_1 - i\sigma_2) + (1-\varphi)\sigma_3\right]\frac{1}{2\sqrt{\varphi}} \qquad /5/$$

the potentials are given by:

$$\varphi\vec{A}_y = (i\rho_y, \rho_y, -i\varphi_y), \qquad \varphi\vec{A}_{\bar{y}} = (-i\bar{\rho}_{\bar{y}}, \bar{\rho}_{\bar{y}}, i\varphi_{\bar{y}}), \qquad /6/$$

$$\varphi\vec{A}_z = (i\rho_z, \rho_z, -i\rho_z), \qquad \varphi\vec{A}_{\bar{z}} = (-i\bar{\rho}_{\bar{z}}, \bar{\rho}_{\bar{z}}, i\varphi_{\bar{z}}),$$

where

$$\varphi = \text{real}, \qquad \bar{\rho} = \rho^* \qquad /7/$$

If we substitute these into /4/ the field equations will be:

$$\varphi[\varphi_{y\bar{y}} + \varphi_{z\bar{z}}] - \varphi_y\varphi_{\bar{y}} - \varphi_z\varphi_{\bar{z}} + \rho_y\bar{\rho}_{\bar{y}} + \rho_z\bar{\rho}_{\bar{z}} = 0, \qquad /8/$$

$$\varphi[\rho_{y\bar{y}} + \rho_{z\bar{z}}] - 2\rho_y\varphi_{\bar{y}} - 2\rho_z\varphi_{\bar{z}} = 0,$$

$$\varphi[\bar{\rho}_{y\bar{y}} + \bar{\rho}_{z\bar{z}}] - 2\bar{\rho}_{\bar{y}}\varphi_y - 2\bar{\rho}_{\bar{z}}\varphi_z = 0.$$

All self-dual fields in Yang's R gauge can be expressed by /6/ with $\rho, \bar{\rho}$ and f satisfying /7/ and /8/. Conversely /7/ and /8/ guarantee that the gauge potential given by /6/ is self-dual.

If we define f and ρ as follows:

$$f = r\phi(x_3, r) \;, \qquad r = \sqrt{x_1^2 + x_2^2} \qquad /9/$$

$$\rho = \sqrt{2}\,\bar{y} g(x_3, r), \qquad /10/$$

where ϕ is a real and g is a complex function, the equations /8/ takes the following form:

$$\begin{aligned} &\phi\left(\phi_{00} + \phi_{11} + \frac{1}{r}\phi_1\right) - \phi_0^2 - \phi_1^2 + g_0\bar{g}_0 + g_1\bar{g}_1 = 0, \\ &\phi\left(g_{00} + g_{11} + \frac{1}{r}g_1\right) - 2g_0\phi_0 - 2g_1\phi_1 = 0, \\ &\phi\left(\bar{g}_{00} + \bar{g}_{11} + \frac{1}{r}\bar{g}_1\right) - 2\bar{g}_0\phi_0 - 2\bar{g}_1\phi_1 = 0. \end{aligned} \qquad /11/$$

These equations are equivalent to the generalized form of the Ernst equations, known from the general relativity. If g is real these equations will reduce to the Ernst equations, which describe stationary, axially symmetric solutions of Einstein equations is vacuum.

It was proved in Ref. [8] too, that the Manton ansatz which corresponds to real g is equivalent to the Ernst equations

The field equations /11/ can be obtained from the

Lagrangian:

$$\mathcal{L} = \frac{\phi_0^2 + \phi_1^2 + g_0\bar{g}_0 + g_1\bar{g}_1}{\phi^2} \qquad /12/$$

This Lagrangian defines the model.

We can write the /11/ field equations in forms of divergences:

$$\partial_1\left(\frac{r g_1}{\phi^2}\right) + \partial_0\left(\frac{r g_0}{\phi^2}\right) = 0, \qquad /13/$$

$$\partial_1\left(r\frac{\phi_1}{\phi} + r\frac{g_1\bar{g}_1}{\phi^2}\right) + \partial_0\left(r\frac{\phi_0}{\phi} - r\frac{g_0\bar{g}_0}{\phi^2}\right) = 0. \qquad /14/$$

We can solve the equation /13/ in the following way:

$$\omega_1 = \frac{r}{\phi^2} g_0 \quad , \quad \omega_0 = -\frac{r}{\phi^2} g_1 . \qquad /15/$$

It is simple to show that the integrability conditions and the equation /14/ imply the following equations:

$$\phi\left(\phi_{00} + \phi_{11} + \frac{1}{r}\phi_1\right) - \phi_0^2 - \phi_1^2 + \frac{\omega_0\bar{\omega}_0\phi^4}{r^2} + \frac{\omega_1\bar{\omega}_1\phi^4}{r^2} = 0,$$

$$\phi\left(\omega_{00} + \omega_{11} - \frac{1}{r}\omega_1\right) + 2\omega_0\phi_0 + 2\omega_1\phi_1 = 0, \qquad /16/$$

$$\phi\left(\bar{\omega}_{00} + \bar{\omega}_{11} - \frac{1}{r}\bar{\omega}_1\right) + 2\bar{\omega}_0\phi_0 + 2\bar{\omega}_1\phi_1 = 0.$$

The advantage of introducing the new functions consists of the fact that defining

$$\phi' = \frac{r}{\phi} \quad , \quad g' = i\omega \quad , \quad \bar{g}' = i\bar{\omega}, \qquad /17/$$

the equations /11/ are also satisfied. It means that /17/ is also the solution of equations /11/. With the aid of this transformation it is possible to get new solutions. This is the transformation discovered by Corrigan, Fairlie, Yates and Goddard [9].

The transformation used in our case is more general than that of ref. [5] because we proved its existence also for the complex function. This transformation shall be denoted with I.

Besides this transformation, it is possible to find other ones. New solutions can also be generated in the following way:

$$\phi' = \frac{\phi}{\phi^2 + g\bar{g}} , \quad g' = \frac{-\bar{g}}{\phi^2 + g\bar{g}} , \quad \bar{g}' = \frac{-g}{\phi^2 + g\bar{g}} , \qquad /18/$$

as well as

$$\phi' = c\phi , \quad g' = ce^{i\alpha}g , \quad \bar{g}' = ce^{-i\alpha}\bar{g}, \qquad /19/$$

where c is a real constant, and

$$\phi' = \phi , \quad g' = g + d , \quad \bar{g}' = \bar{g} + \bar{d} \qquad /20/$$

with d being a complex constant.

These transformation are elements of a group G, which is an SL/2,C/ group. The SO/3,1/ Lorenz group is in 1 to 2 correspondence with this group.

In the next section we will show that it is possible to write these transformations in a more simple form.

Combining these two groups I and G we obtain a new transformation that can generate further solutions of equations /11/. If we perform the I transformation once we get such a solution where g' and $\bar{g}'$ are not the complex conjugates of each other and if we perform it twice we get the identity transformation. After the combined IhI /where $h \in G$/ transformation we obtain new solutions. The product IhI is not contained in G, thus the repeated applications of these two transformations generate an infinite parameter invariance group. This new group denoted by K is the generalized Geroch group in our case.

Because K is an infinite parameter group the situation is very complicated. Applying a finite number of the elements h and IhI usually generates singularities in an uncontrollable way. Therefore it is important to sum up in a suitable way an infinite number of similar infinitesimal transformations.

3. σ-model Representation of the Field Equations

In this section we will show that instead of the complicated SL/2,C/ transformation group we can introduce an SO/3,1/ group with which the solutions are obtainable in a much easier way.

If we perform the following transformation on the /12/ Lagrangian:

$$\phi = \frac{1}{\sigma_4 + \sigma_1} \quad , \quad g = \frac{\sigma_2 + i\sigma_3}{\sigma_4 + \sigma_1} \quad , \quad \bar{g} = \frac{\sigma_2 - i\sigma_3}{\sigma_4 + \sigma_1} \qquad /21/$$

and if σ satisfies the subsidiary condition:

$$\sigma_4^2 - \sigma_3^2 - \sigma_2^2 - \sigma_1^2 = 1, \qquad /22/$$

we get the following Lagrangian:

$$\mathcal{L} = (\nabla\sigma_4)^2 - (\nabla\sigma_3)^2 - (\nabla\sigma_2)^2 - (\nabla\sigma_1)^2. \qquad /23/$$

These σ -s can be interpreted as four-vectors:

$$\sigma_i = \begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{vmatrix} \qquad /24/$$

and if we perform an SO/3,1/ transformation on σ_i the /23/ Langrangian will remain invariant. This is the SO/3,1/ group which acts on the functions ϕ and g we mentioned in section 2. It is conceivable that the complicated transformation given by SL/2,C/ group can be performed in a more simpler way on the functions /21/.

The Lagrangian /23/ with the subsidiary condition /22/ defines a nonlinear SO/3,1/ σ-model. In σ-models we get the following field equations:

$$\Delta\sigma_i = \lambda(x_3, r)\sigma_i, \qquad /25/$$

where Δ is the Laplace operator defined in cylindrical co--ordinates and λ is the Lagrange multiplicator.

The two-dimensional nonlinear σ-model possesses many properties which can be found in four-dimensional, non-Abelian gauge theories. Both of them are scale invariant and asymptotically free and because of their rich topological structure the σ -models contain topologically nontrivial solutions /e.g. multiinstanton solutions/.

The two-dimensional model is much simpler than the four-dimensional Yang-Mills theories and therefore the model seems ideal for studying the effects of topologically nontrivial solutions of four-dimensional gauge theories.

The classical two-dimensional nonlinear σ -model has an infinite number of conservation laws. By using them the quantum field theory of some σ -models can be determined.

When proving the existence of the infinite number of conservation laws an important role is played by the fact that it was possible to integrate the classical field equations by the inverse scattering method. This method can be brought into connection with the existence of certain Bäcklund transformations.

In general the constraint /22/ for σ-models contains only positive signs, while in our case the signs are different according to the fact that σ-fields can be identified with the elements of the noncompact SL/2,C/ group.

We define an SL/2,C/ matrix with $g, \bar{g}$ and ϕ functions in the following way:

$$g = \begin{pmatrix} \frac{1}{\phi}, & \frac{\bar{g}}{\phi} \\ \frac{g}{\phi}, & \phi + \frac{g\bar{g}}{\phi} \end{pmatrix}, \qquad \text{where} \qquad \det g = 1 \qquad /26/$$

With the correspondence /21/ the g matrix can be expressed with the aid of components of the σ_i-fields:

$$g = \begin{vmatrix} \sigma_4 + \sigma_1, & \sigma_2 + i\sigma_3 \\ \sigma_2 - i\sigma_3, & \sigma_4 - \sigma_1 \end{vmatrix} \qquad /27/$$

where the condition det g=1 means the satisfaction of the subsidiary condition /22/ in a natural way.

By direct substitution we proved that the /11/ field equations in the language of the matrix g take the form:

$$(rg^{-1}g_{,0})_{,0} + (rg^{-1}g_{,1})_{,1} = 0. \qquad /28/$$

Because of the connection found with σ-models it is not suprising that the form of the self-duality equation corresponds to the form of field equations of σ-models.

The knowledge of this fact makes it possible to apply the above mentioned inverse scattering method to the self-duality equations of Yang-Mills theories /7/.

By means of ϕ, ω and $\bar{\omega}$ we define the following matrix:

$$f = \begin{pmatrix} \phi, & -\phi\omega \\ -\phi\omega, & -\frac{r^2}{\phi} + \phi\omega\bar{\omega} \end{pmatrix} \qquad /29/$$

which fulfils

$$\det f = -r.$$

It can be shown that the equations /16/ in terms of the matrix f take the form:

$$(r f^{-1} f_{,0})_{,0} + (r f^{-1} f_{,1})_{,1} = 0. \qquad /30/$$

If we define by means of the matrix f a matrix $\tilde{f}$ in the following way:

$$\tilde{f} = \frac{1}{r} f$$

then we get

$$\tilde{f} = \begin{pmatrix} \frac{\phi}{r}, & -\frac{\phi\omega}{r} \\ -\frac{\phi\omega}{r}, & -\frac{r^2}{\phi}+\phi\omega\bar{\omega} \end{pmatrix} \qquad /31/$$

the determinant of which is

$$\det \tilde{f} = -1.$$

With the aid of this matrix we get an equation similar to /30/:

$$(r\tilde{f}^{-1}f_0)_0 + (r\tilde{f}^{-1}f_1)_1 = 0. \qquad /32/$$

If the elements of f matrix are written as follows:

$$\tilde{f}' = \begin{pmatrix} \psi_4+\psi_1, & \psi_2+i\psi_3 \\ \psi_2-i\psi_3, & \psi_4-\psi_1 \end{pmatrix}, \qquad /33/$$

we get again a σ-model with ψ_i-s as field variables. The field equation on the basis of /32/ will be /25/.

Notice that if we perform the I transformation on the the elements of g matrix we will get the f matrix, which means that the I transformation also transforms the σ-models embedded in these matrices into each other.

As we mentioned before the σ-models are invariant under the SO/3,1/ symmetry group. Let us denote the SO/3,1/ group which has an effect on σ-fields with $SO/3,1/_{\sigma}$ and the group which has an effect on ψ-fields with $SO/3,1/_{\psi}$. Since the two σ-models are connected with the I transformation, a similar relation exists between $SO/3,1/_{\sigma}$ and $SO/3,1/_{\psi}$ groups, consequently these groups are dual.

$$I\ SO(3,1)_{\delta}\ I = SO(3,1)_{\varphi}. \qquad /34/$$

The analogy with the Geroch group is evident and therefore /34/ can be used for generating new solutions. We hope that among them the multimonopole solution can be found in BPS limit.

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