FR 8002009

CLASSICAL ACTION, THE WU-YANG PHASE FACTOR AND PREQUANTIZATION

Péter A. HORVÁTHY *
Université d'Aix-Marseille I
et
Centre de Physique Théorique, CNRS Marseille

ABSTRACT : For local variational systems (like a charged particle in the field of a Dirac monopole) a quantum mechanically well-defined action (Q.M.W.D.A.) can be introduced if the system is prequantizable in the Kostant-Souriau sense. If the configuration space is multiply connected (as in the Bohm-Aharonov experiment), different expressions for the classical action may emerge; they are quantum mechanically equivalent (Q.M.E.) if the corresponding prequantizations are equivalent. In both cases the situation depends on the behaviour of the non integrable phase factor of Wu and Yang.

CNRS-CPT--80/P.1182

* On leave from Veszprém University of Chemical Engineering Veszprém, (Hungary).

To be published in the Proceedings :

POSTAL ADDRESS : C.N.R.S. - LUMINY - CASE 907
Centre de Physique Théorique
F-13288 MARSEILLE CEDEX 2
France

[redacted] INTRODUCTION

The importance of classical action in quantum mechanics emerges the clearest way from Feynman's path integral approach [1]. To a path γ in spacetime between x and x' is associated the amplitude

$$\exp\left[\frac{i}{\hbar} S(\gamma)\right] \tag{1}$$

where $S(\gamma)$ is the classical action along γ ; the propagator is expressed as

$$K(x',x) = \int_{\mathcal{P}} \exp\left[\frac{i}{\hbar} S(\gamma)\right] \mathcal{D}\gamma \tag{2}$$

$\mathcal{P}$ being the "infinite dimensional manifold" of paths joining x to x'.

We are not concerned here with the tremendous problem of defining and computing this integral ; we shall accept its intuitive meaning and focus our attention to the amplitude (1).

The point is that in some interesting situations, as in the Bohm-Aharonov experiment [2] [3] the expression of classical action may be ambiguous [4]; in other cases, as for the motion of a charged particle in the field of a Dirac monopole [6], it may be even ill-defined [5].

Motivated by ordinary gauge transformation, we introduce the notion of quantummechanically well-defined action (Q.M.W.D.A.) and the idea of equivalent (Q.M.E.) actions.

The requirement of having a Q.M.W.D.A. leads to quantum conditions (like quantization of the monopole's strength) ; the equivalence of actions provides us with a classification scheme and with a simple proof of the C. DeWitt-Laidlaw theorem [7] [8] [9] on propagators.

These results can be reexpressed in a rather elegant geometric form : a Q.M.W.D.A. exists iff the system is prequantizable in the Kostant-Souriau (K-S) sense [10] [11] [13]. The classification scheme turns out to be just that of inequivalent prequantum bundles.

Our approach shows some similarities to that of Wu and Yang [15] who describe gauge fields in terms of a "non integrable phase factor". The relation is explained in the U(1) (electromagnetic) case.

1. LOCAL VARIATIONAL SYSTEMS [15][16]

Let Q be the manifold of all possible configurations of a classical system. If we are given a Lagrangian function $L: TQ \times R \longrightarrow R$, the variational problem can be translated to symplectic terms [11], [24], [25] : from L we can derive a 1-form Θ such, that the Euler-Lagrange equations have the geometric form

$$\dot{\gamma} \in \mathrm{Ker}\, d\Theta \tag{3}$$

The curves γ satisfying (3) - the lifts to $TQ\times R$ of the classical motions- are the extremals of the variational problem. $\sigma = d\Theta$ is a presymplectic form on the manifold $E = TQ\times R$ ("evolution space").

Souriau proposed [11] to enlarge classical mechanics by describing systems with such a pair (E, σ), without bothering about Lagrangians. The existence of a Lagrangian function is, however, a basic requirement in mechanics [23]. Also, as it will appear from the discussion which follows, (Sections 3, 4,5) in order to have a <u>meaningful quantization procedure</u>, we need some additional condition which rules out the velocity-dependence of potentials.

The exact relations between symplectic and variational description are the best established using the homogeneous formalism [17], [11], [15], [16] which we review here briefly.

Write $X = Q\times R$ for (configuration) space-time, denote $\pi : TX \to E$ ($E = TQ\times R$) the projection given locally as $\pi(x,\dot{x}) = (q, \frac{\dot{q}}{\dot{t}}, t)$, where $x = (q,t)$, $\dot{x} = (\dot{q},\dot{t})$; suppose $\dot{t} > 0$. The homogenized Lagrangian reads $\mathcal{L}(x,\dot{x}) = \dot{t}\, L\circ\pi$.
We have then a unique 1-form Λ on TX such that for any curve $\gamma \subset TX$

$$\int \mathcal{L}\circ\gamma(\tau)\, d\tau = \int_{\gamma} \Lambda \tag{4}$$

where $\tau \to \gamma(\tau) = (\gamma_q(\tau), \gamma_t(\tau), \gamma_{\dot{q}}(\tau), \gamma_{\dot{t}}(\tau))$ is any parametrization with $d\gamma_t/d\tau > 0$.

Explicitely, Λ is the <u>fiber derivative</u> of $\mathcal{L}$ [18],

$$\Lambda = \dot{d}\mathcal{L} \tag{5}$$

(recall the definition of $\dot{d}$:

For a function $f: TX \to R$ $\dot{d}f = (\partial f/\partial \dot{x}^\alpha) dx^\alpha$; the extension to forms is made by the requirements

$$\dot{d}(dx^\alpha) = \dot{d}(d\dot{x}^\alpha) = 0$$
$$\dot{d}(\omega \wedge \beta) = \dot{d}\omega \wedge \beta + (-1)^{\deg \omega} \omega \wedge \dot{d}\beta$$

this Λ is

- semibasic,

$$\Lambda_{(x,\dot{x})} = a_\alpha(x,\dot{x})\, dx^\alpha \tag{6a}$$

- homogeneous of order 0 in $\dot{x}$, for $0 \neq c \in \mathbb{R}$.

$$a_\alpha(x,\dot{x}) = a_\alpha(x, c\dot{x}) \tag{6b}$$

- of the form

$$\Lambda = \pi^* \Theta \tag{6c}$$

with a 1-form Θ on E (this is just the usual Cartan form [11], used in (3)).

Conversely, if we are given a Λ with these properties (6), we can always reconstruct a Lagrangian function

$$L(q,v,t) = \sum_{\alpha=1}^{n} v^\alpha a_\alpha(q,v,t) + a_{n+1} \tag{7}$$

Thus it is justified to call 1-forms on TX satisfying (6) <u>global variational 1-forms</u>, <u>(TX, Λ) is a global variational system.</u>

Denote $\Sigma = d\Lambda$; then L is regular (i.e. $\partial^2 L/\partial v^\alpha \partial v^\beta$ is a regular nxn matrix) iff

$$\dim \operatorname{Ker} \Sigma = 2 \tag{8}$$

If (8) holds then the smooth distribution $(x,\dot{x}) \longrightarrow \operatorname{Ker} \Sigma_{(x,\dot{x})}$ is integrable : the characteristic leaves [13] , [18] (which are in 1-1 correspondence with the curves in E satisfying (3)) are 2-dimensional submanifolds in TX . They project to the world lines in X , and thus it is justified to consider these leaves as the generalized solutions of the variational problem.

Σ satisfies [14]

$$d\Sigma = 0 \tag{9a}$$

$$\Sigma = \pi^* \sigma \quad , \quad \sigma, \qquad \text{presymplectique form on E} \tag{9b}$$

$$\dot{d}\Sigma = 0 \tag{9c}$$

In our case $\sigma = d\Theta$.

This is just this condition (9c) which singles out variational system among (pre)symplectic ones.

Unfortunately, global variational systems do not exhaust all the physically interesting situations : for a charged particle moving in the field of a Dirac monopole (see example [1] below) for instance, no global Λ exists. Conditions (9) are however satisfied.

On the other hand, Klein has shown [17] that (9) assures the existence of a local variational description at least.

Theorem, Definition 1.1

Let Σ be a 2-form on TX satisfying (9). Then, in a neighbourhood of any point at least, the equations

$$\Sigma = d\Lambda \qquad \text{or} \qquad \sigma = d\Theta \tag{10}$$

admit solutions such that Λ (or Θ) satisfy (6). Such 1-forms will be called <u>local variational</u> or <u>action forms</u>, (TX, Σ) or (E, σ) being a local <u>variational system</u>.

It is well-known (e.g. [19]) that the possibility of extending a local solution depends on the topology : if $H^2(TX, \mathbb{R}) = 0$ every local solution of (10) extends to the entire TX (or E).

<u>Proposition 1.2</u>

Let Λ and Λ' (or Θ, Θ') be local variational solutions of (10), then in the intersection of their domain

$$\alpha = \Lambda' - \Lambda = \Theta' - \Theta = A(q,t)\,dq + V(q,t)\,dt \tag{11}$$

is a <u>closed 1-form</u> on X , $d\alpha = 0$.

If this intersection is <u>simply connected</u> then α is <u>exact</u>.

<u>Proof</u> : α is obviously closed ; a closed semibasic 1-form can not depend on $\dot{x}$.

<u>Theorem 1.3</u> [15], [16]

If (E, σ) is a regular local variational system, $\operatorname{Ker} \pi^* \sigma$ defines a foliation of TX by 2-dimensional leaves. These leaves-considered as generalized solutions of the variational problem-project onto <u>curves</u> in X .

Thus, at a <u>purely classical level</u>, these systems admit a <u>completely satisfactory variational description</u>.

<u>Remark 1.4</u>

If we replace (8) by $\dim \operatorname{Ker} \Sigma = 2K$, $K > 1$, the whole formalism keeps on working ; this allows for including spin [15]. We study here, however, only spinless systems.

In what follows, we shall use the (E, σ) setting, (8) and (9) supposed being satisfied.

2. THE CLASSICAL ACTION

Consider first a <u>global</u> system with action form Θ. For $\gamma \subset E$ set

$$S(\gamma) = \int_{\gamma} \Theta \tag{12}$$

and call it <u>classical action</u> along γ. (If $\gamma \subset X$ is a curve, lift it to E : call the lift again γ to save characters) ; by (4), (12) reduces then to the usual expression).

Note however, that his definition is <u>ambiguous</u> : we are always allowed to change Θ to Θ' which also satisfies $d\Theta' = \sigma$; the requirements (6) imply (Prop. 1.2) that $\Theta' = \Theta + \alpha$ with a 1-form α on X . This has the effect of changing (12) by an additional term $\int_{\gamma} \alpha$.

If the configuration space is <u>simply connected,</u> then α is exact: $\alpha = d\xi$ with $\xi : X \to \mathbb{R}$; thus the additional term is just a constant $\{\xi(x') - \xi(x)\}$, which changes the amplitude (1) and thus the propagator (2) only by an overall phase factor

$$c\,(x', x) = \exp \frac{i}{\hbar} \left\{ \xi(x') - \xi(x) \right\} \tag{13}$$

which is physically <u>unobservable</u>.

However, if the underlying space is <u>multiply connected</u> (as in the Bohm-Aharonov experiment, see example 2 below), this term will depend on γ . and will change essentially the physics at the quantum level.

For local systems the situation is even worse : an action from Θ_α exists only locally, over an open set $\mathcal{U}_\alpha$. Consequently, the corresponding classical action $S_\alpha(\gamma) = \int_\gamma \Theta_\alpha$ will be meaningful only for paths γ contained entirely in $\mathcal{U}_\alpha$.

But even for such paths, we have an essential ambiguity: if we change $(\mathcal{U}_\alpha, \Theta_\alpha)$ to $(\mathcal{U}_\beta, \Theta_\beta)$ with $\gamma \subset \mathcal{U}_\beta$, then the new $S_\beta(\gamma) = \int_\gamma \Theta_\beta$ will be, generally, completely different from $S_\alpha(\gamma)$ (see Example 1 below). This is due again to topology: $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ may be non-simply connected, and thus $\Theta_\alpha - \Theta_\beta$ may be not exact, and so

$$S_\alpha(\gamma) - S_\beta(\gamma) = \int_\gamma (\Theta_\alpha - \Theta_\beta) \tag{14}$$

will be path dependent. Consequently, for local systems, it is generally meaningless to speak of classical actions.

[redacted] A QUANTUMMECHANICALLY WELL-DEFINED ACTION

Fortunately, as it is clear from (2) it is the amplitude (1) rather than the action itself, which is important for quantum mechanics.

Consider a local system (E, σ).

Definition 3.1

The classical action is quantummechanically well-defined (Q.M.W.D.) if to any choice $(\mathcal{U}_\alpha, \Theta_\alpha)$, and any path γ whose end points x, x' belong to $\mathcal{U}_\alpha$, we can associate an expression

$$\text{"}\exp\left[\tfrac{i}{\hbar} S_\alpha(\gamma)\right]\text{"} \tag{15}$$

such that

a) a change $(\mathcal{U}_\alpha, \Theta_\alpha) \to (\mathcal{U}_\beta, \Theta_\beta)$ introduces merely a phase factor

$$\text{"}\exp\left[\tfrac{i}{\hbar} S_\alpha(\gamma)\right]\text{"} = C_{\alpha\beta}(x', x) \cdot \text{"}\exp\left[\tfrac{i}{\hbar} S_\beta(\gamma)\right]\text{"} \tag{16}$$

where $|C_{\alpha\beta}(x', x)| = 1$, $C_{\alpha\beta}(x', x)$ depends only on x, x' and not the particular path γ between them.

b) for $\gamma \subset \mathcal{U}_\alpha$ (15) reduces to $\exp\left[\tfrac{i}{\hbar} S(\gamma)\right]$ with $S_\alpha(\gamma) = \int_\gamma \Theta_\alpha$.

If a Q.M.W.D.A. exists, then a change $(\mathcal{U}_\alpha, \Theta_\alpha) \longrightarrow (\mathcal{U}_\beta, \Theta_\beta)$ will introduce only a phase factor in the propagator (2).

Study first paths in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. It is easier to use loops:

Proposition 3.2

If a Q.M.W.D.A. exists, then for a loop $\gamma \subset \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, we have

$$\exp\left[\tfrac{i}{\hbar} S_\alpha(\gamma)\right] = \exp\left[\tfrac{i}{\hbar} S_\beta(\gamma)\right] \tag{17}$$

<u>Proof</u> : In fact, split up γ to $\gamma_1 \circ \gamma_2$; apply (16) to γ_1 and γ_2^{-1} ; divide, noting that $\exp[\frac{i}{\hbar} S(\gamma)] = \left(\exp[\frac{i}{\hbar} S(\gamma^{-1})]\right)^{-1}$

In other form :

<u>Proposition 3.3</u>

A necessary condition for the existence of a Q.M.W.D.A. is that for a loop $\gamma \subset U_\alpha \cap U_\beta$ we have

$$\exp\left[\frac{i}{\hbar} \oint_\gamma (\Theta_\alpha - \Theta_\beta)\right] = 1 \qquad (18)$$

It may happen, that it is possible to pull "caps" s_α and s_β over γ in U_α resp. U_β , each cap being diffeomorphic to $\mathbb{R}^2$; $s = s_\alpha \cup s_\beta$ is then diffeomorphic to S^2 ; let's apply Stokes' theorem to s_α, resp. s_β ; we get :

<u>Proposition 3.4</u>

(17) is equivalent to

$$\frac{1}{2\pi\hbar} \int_s \sigma \in \mathbb{Z} \qquad (19)$$

In Section 5, we shall show that these conditions are in fact <u>sufficient</u>.

<u>Remark 3.5</u>

As $\Theta_\alpha - \Theta_\beta$ is in fact a 1-form over X ,(Prop. 1.2) (18) and (19) hold if they hold for loops, resp. 2-surfaces in X .

<u>EXAMPLE 1</u> (Charged particle moving in the field of Dirac's monopole)

Suppose we have a magnetic monopole of strength g fixed in the origin , an electron moving in its field has the symplectic description [12] $Q = \mathbb{R}^3 \setminus \{0\}$, $E = TQ \times \mathbb{R}$, $\sigma = \sigma_{free} + e\,\mathbb{B}$, i.e.

$$\sigma = d\left(mv\,dq - m\frac{v^2}{2}\,dt\right) + eg\left\langle \frac{q}{|q|^3}, dq \times dq\right\rangle \qquad (20)\,.$$

It is easy to see that no global Θ with $\sigma = d\Theta$ (and thus no global vector potential), exists : if σ was $d\Theta$, $\int \sigma$ would be 0 by Stokes' theorem ; however one computes at once that $\int_{S^2} \sigma = 4\pi e g$.

Nevertheless, local solutions of (10) can be found on any chart corresponding to $\mathcal{U}_{n} = \mathbb{R}^3 \setminus \{$a "string" in the direction of $n\}$ e.g.

$$\Theta_{n} = \left\{ m v\, dq - m \frac{v^2}{2} dt \right\} + e A^{(n)}(q)\, dq \tag{21}$$

with the local vector potential [12]

$$A^{(n)} = g \frac{n \times q}{q^2 + |q| \langle n, q \rangle} \tag{22}$$

The ambiguity in the classical action can be tested on $\mathcal{U}_\alpha = \mathcal{U}_{(0,0,1)}$, $V_\beta = \mathcal{U}_{(0,0,-1)}$, $\gamma(\varphi) = (\cos\varphi, \sin\varphi, 0)$ with $0 \leq \varphi < 2\pi$ (the equator)

$$S_\alpha(\gamma) - S_\beta(\gamma) = \oint_\gamma \left(A^{(n)} - A^{(-n)}\right) dq = 4\pi e g \tag{23}$$

Thus a Q.M.W.D.A. exists iff the monopole is quantized as

$$2eg = \hbar k \qquad k \in \mathbb{Z} \tag{24}$$

More generally, one shows that a Q.M.W.D.A. exists iff

$$\exp\left[\frac{ie}{\hbar} \oint A_\alpha \right] \tag{25}$$

has the same value for all α (with $\sigma|_{V_\alpha} = d\Theta_\alpha$). (25) is just the phase factor of Wu and Yang [14].

4. A CLASSIFICATION SCHEME. THE PROPAGATOR IN MULTIPLY CONNECTED SPACES [4]

Let's consider a <u>global</u> system with multiply connected configuration space. The general solution of (10) among variational 1-forms is by Prop. 1.2

$$\Theta = \Theta_0 + \alpha \tag{26}$$

with Θ_0 a particular solution ; as a consequence of (6a), (6b), α is a 1-form on X

$$\alpha = A(q,t)\,dq + V(q,t)\,dt \tag{27}$$

<u>Definition 4.1</u>

Let $\Theta_1 = \Theta_0 + \alpha_1$ and $\Theta_2 = \Theta_0 + \alpha_2$ two actions forms for a global system. Two expressions $S_1(\gamma) = \int_\gamma \Theta_1$ and $S_2(\gamma) = \int_\gamma \Theta_2$ are told to be <u>quantummechanically equivalent</u> (Q.M.E.), (denoted also $\Theta_1 \sim \Theta_2$) iff

$$\exp\left[\frac{i}{\hbar} S_1(\gamma)\right] = C(x',x)\cdot \exp\left[\frac{i}{\hbar} S_2(\gamma)\right] \tag{28}$$

with a phase factor $C(x',x)$ depending only on (the projection onto X of the) end points of γ; $|C(x',x)| = 1$

<u>Proposition 4.2</u>

$\Theta_1 \sim \Theta_2$ iff for any loop γ

$$\exp\left[\frac{i}{\hbar} \oint_\gamma (\Theta_1 - \Theta_2)\right] = \exp\left[\frac{i}{\hbar} \oint_\gamma (\alpha_1 - \alpha_2)\right] = 1 \tag{29}$$

or

$$\frac{1}{2\pi\hbar} \oint_\gamma (\Theta_1 - \Theta_2) = \frac{1}{2\pi\hbar} \oint_\gamma (\alpha_1 - \alpha_2) \in \mathbb{Z} \tag{30}$$

(As the space is not simply connected, Stokes' theorem does not apply, and thus we cannot transform this to integrals over 2-cycles). Again, by Prop. 1.2, we can limit ourselves to path in X .

$\exp\left[\frac{i}{\hbar}\oint_\gamma \alpha\right]$ is studied the easiest way if we climb to the universal covering $(\tilde{X}, \Pi, P)$ of X : $\tilde{X} = \tilde{Q}\times R$, where $\tilde{Q}$ is the universal covering of Q ; Π is the (first) homotopy group of Q (and X) ; $P : \tilde{X} \ni (\tilde{q},t) \mapsto (q,t) \in X$ projection.

Set $\tilde{\alpha} = P^*\alpha$. As $\tilde{X}$ is already simply connected, $\tilde{\alpha} = d\tilde{f}$, with $\tilde{f} : \tilde{X} \longrightarrow R$.
Let $\gamma \subset X$ be any path, $x \in \gamma$, $\tilde{x} \in P^{-1}(x)$; γ has a unique lift $\tilde{\gamma}$ to $\tilde{X}$ through $\tilde{x}$. Evidently, $\int_\gamma \alpha = \int_{\tilde{\gamma}} \tilde{\alpha}$

In particular, if γ is a closed loop $\tilde{\gamma}$ will end at $g\tilde{x}$, where $g = [\gamma]$ is the homotopy class of γ. Consequently

$$\oint_\gamma \alpha = \tilde{f}(g\tilde{x}) - \tilde{f}(\tilde{x}) \tag{31}$$

Note that (31) depends only on g . Thus

Proposition 4.3

$$\chi(g) = \exp\left[\frac{i}{\hbar}\oint_\gamma \alpha\right] \tag{32}$$

is well-defined, and is in fact, a character of the homotopy group Π . In this way we get the following classification theorem:

Theorem 4.4

$\Theta_1 \sim \Theta_2$ iff for any loop γ

$$\chi_1(g) := \exp\left[\frac{i}{\hbar}\oint_\gamma \alpha_1\right] = \exp\left[\frac{i}{\hbar}\oint_\gamma \alpha_2\right]$$

where $g = [\gamma]$.

<u>The different situations are thus labelled by the characters of the homotopy group.</u>

Now we can prove an interesting theorem,[4] first stated explicitely by C. de Witt and Laidlaw [7] (see also [8],[9]).
Consider $x, x' \in X$, let $\mathcal{P}$ be the set of paths between them , choose any $\varrho \in \mathcal{P}$; any $\gamma \in \mathcal{P}$ can be written -1 to homotopy- as $\gamma = \varrho \circ \beta$, where β is a loop through x . γ and γ' are homotopic if β and β' are. The classical action is

$$S(\gamma) = S_o(\gamma) + \int_\varrho \alpha + \oint_\beta \alpha \tag{34}$$

where $S_o(\gamma) = \int_\gamma \Theta_o$.

Note that

- $\int_\varrho \alpha$ is independent of γ, denote

$$\exp\left[\frac{i}{\hbar} \int_\varrho \alpha\right] =: c \tag{35}$$

- $\exp\left[\frac{i}{\hbar} \oint_\beta \alpha\right] = \chi(g)$, with $g = [\beta]$, is constant on a homotopy class,

- define the partial amplitude

$$K_g(x', x) := \int_{\mathcal{P}_g} \exp\left[\frac{i}{\hbar} S_o(\gamma)\right] \mathcal{D}\gamma \tag{36}$$

where $\mathcal{P}_g \subset \mathcal{P}$ is the class of paths in $\mathcal{P}$ labelled by the same g ; as $\mathcal{P} = \bigcup_{g \in \Pi} \mathcal{P}_g$, the additivity of the path integral gives

<u>Theorem 4.5</u>

$$K(x', x) = c \sum_{g \in \Pi} \chi(g)\, K_g(x', x) \tag{37}$$

(a different choice in ϱ , the map $\gamma \longmapsto \beta$, or in Θ_o introduces only an unobservable phase factor.)

EXAMPLE 2 (Bohm-Aharonov Experiment) [2][3][4][14]

As the electron is classically excluded from the interior of the solenoid, the configuration space is $\mathbb{R}^2 \setminus \{\text{a disk}\}$; the presymplectic form is just that of a free particle restricted to $TQ\times R$: $\sigma = \sigma_{free}\big|_{TQ\times\mathbb{R}}$. It is of course exact, $\sigma = d\Theta_0$ with

$$\Theta_0 = \Theta_{free} = mv\,dq - m\frac{v^2}{2}\,dt$$

But, as we have pointed out, we can add any 1-form

$$\alpha = e\left(A(q,t)\,dq + V(q,t)\,dt\right)$$

with $d\alpha = 0$. However, as Q is not simply connected $\alpha \neq df$.

Now, as far as we take seriously geometry and do not look into the solenoid, there is no reason to call A, resp. V, vector, resp. scalar, potential ; in order to identify them, we have to consider our system to be the part of a larger one, consisting of the electron and the solenoid and the magnetic field. [3], [8].

The homotopy group is here $\mathbb{Z}$, thus the characters (32) are written as

$$\chi(n) = \left(\exp\left[\frac{i}{\hbar}\oint_{\gamma_0} A\right]\right)^n = \left(\exp\left[\frac{ie\Phi}{\hbar}\right]\right)^n$$

γ_0 being a loop going once around the solenoid ; Φ is the enclosed magnetic flux.

Here we recognize again the n-th power of the non-integrable phase factor of Wu and Yang [14].

Theorem 4.6

Two expressions of the classical action are Q.M.E. iff the corresponding Wu-Yang factors are the same : $\Theta_1 \sim \Theta_2$ iff $\chi_1(1) = \chi_2(1)$ iff

$$\Phi_1 - \Phi_2 = \frac{h}{e} k \quad , \quad k \in \mathbb{Z} \tag{38}$$

confirming the conclusions of Bohm and Aharonov. $(h = 2\pi\hbar)$.

5. PREQUANTIZATION

The fact that conditions (18), (19) are <u>sufficient</u> to the existence of a Q.M.W.D.A., will follow from noting the relation to prequantization.

<u>Theorem 5.1</u> [21]

A Q.M.W.D.A. exists iff the system is <u>prequantizable</u> in the K-S (Kostant-Souriau) sense [10],[11],[13].

<u>Proof</u> : By Weil's theorem, the system is prequantizable iff (18) or (19) holds. They ensure the possibility of constructing a U(1) principal bundle Y over E with connection form ω whose curvature form is $i\pi^*\sigma$, $\pi : Y \to E$ being the projection.

On the other hand, if the system is prequantizable, then, for any path γ in E , and $Y \ni \xi \in \pi^{-1}(y)$ with $y \in \gamma$, we have a unique horizontal lift $\hat{\gamma}$ through ξ .

If $\gamma \subset U_\alpha$, where Y has the local trivialization $\pi^{-1}(U_\alpha) \simeq U_\alpha \times U(1)$ $\hat{\gamma}$ is written here as $\hat{\gamma} = (\gamma , z^\alpha)$; furthermore, [12] [13]

$$\frac{z^\alpha(0)}{z^\alpha(1)} = \exp\left[\frac{i}{\hbar} S_\alpha(\gamma)\right] \tag{39}$$

(γ being parametrized by $t \in [0,1]$; $\xi = (y, z^\alpha(0))$).

Thus, the classical action can be recovered by dividing the "heights" above a U_α of the horizontal lift of γ .

Now if we change our local trivialization, the new expression will be related to the old one as

$$\frac{z^\beta(0)}{z^\beta(1)} = \frac{z_{\alpha\beta}(y)}{z_{\alpha\beta}(y')} \cdot \frac{z^\alpha(0)}{z^\alpha(1)} \tag{40}$$

with $y' = \gamma(1)$; the $z_{\alpha\beta}$'s are here the <u>transition functions</u> of the U(1) bundle Y .

For local variational systems the transition functions depend only of x, the projection of y onto X; this is the consequence of fact, that in (18) we could restrict ourselves to paths in X by (11). Thus

$$C_{\alpha\beta}(x',x) = \frac{Z_{\alpha\beta}(y)}{Z_{\alpha\beta}(y')} \tag{41}$$

will be a phase factor required in Definition 3.1.

On the other hand, as the horizontal lift of curve is well-defined independently of any local trivialization, $Z^{\alpha}(\gamma)$ and $Z^{\alpha}(1)$ will have a meaning as soon as y and y' are contained in U_{α}, even if γ zigzags out and back from U_{α}. Let's define " $\exp[\frac{i}{\hbar}S_{\alpha}(\gamma)]$ then just by (39), this will give a Q.M.W.D.A., as required (cf. [5]).

Q.E.D.

Consider a global system (E,σ). It is always prequantizable: Define $Y = E \times U(1)$, $\pi : Y \ni (y,z) \longrightarrow y \in E$; if Θ is a 1-form on E with $d\Theta = \sigma$, then

$$\omega = \pi^{*}\sigma + \hbar \cdot \frac{dz}{iz} \tag{42}$$

is a connection form, and any solution is written like this.

Two such constructions are told to be equivalent, if there exists a diffeomorphic map $\hat{F} : Y \to Y$ projecting onto E as identity, intertwining the actions of U(1) and carrying one connection form to the others. ($\hat{F}$ is then necessarily of the form $\hat{F}(y,z) = (y, F(y)\cdot z)$ with $|F(y)| = 1$).

As it is well-known, if the underlying space is not simply connected, we may have different inequivalent prequantizations for the same classical system. We propose to rederive this theorem ([10], [11], [13]) by establishing

<u>Theorem 5.2</u> [4]

Let Θ_1, Θ_2 be action forms for a global variational system (E, σ). Then Θ_1 and Θ_2 are Q.M.E. (Def. 4.1) iff the corresponding prequantizations are equivalent in the prequantum bundle sense.

<u>Proof</u> : If $\hat{F} : (Y, \omega_1, \pi) \to (Y, \omega_2, \pi)$ is a map establishing the equivalence between the prequantizations, then $\hat{F}^* \omega_2 = \omega_1$ implies that $\Theta_1 - \Theta_2 = \hbar \frac{dF}{iF}$ and thus, for any loop γ, we have

$$\exp\left[\frac{i}{\hbar} \oint_\gamma \Theta_1\right] = \exp\left[\frac{i}{\hbar} \oint_\gamma \Theta_2\right], \text{ i.e. } \Theta_1 \sim \Theta_2$$

On the other hand, write $d\tilde{\zeta} = \tilde{\Theta}_1 - \tilde{\Theta}_2$ (notation of Section 4) ; then $\Theta_1 \sim \Theta_2$ implies that $F(y) = \exp\left[\frac{i}{\hbar} \tilde{\zeta}(\tilde{y})\right]$ is well-defined $(\tilde{y} \in p^{-1}(y))$, $\hat{F}(y,z) := (y, F(y) \cdot z)$ establishes the equivalence.

5. RELATION TO THE WU-YANG APPROACH TO GAUGE THEORY

Primarily interested in describing the quantized motion of a system, we investigated the conditions under which the amplitude (1) is well-defined and is unique. In geometric terms these properties could be expressed using the prequantum bundle (Y, ω, π).

On the other hand, in their approach to gauge theory, Wu and Yang [14] proposed to study an expression slightly similar to the amplitude (1), namely (in U(1) case)

$$\exp\left[\frac{ie}{\hbar}\int_{\gamma} A_{\alpha}\, dx^{\alpha}\right] \qquad (43)$$

In our examples (monopole and Bohm-Aharonov experiment) also our conditions turned out to depend on this expression. It is not difficult to understand that this happens quite generally, at least for electromagnetic interactions.

Electromagnetism can, in fact, be conceived as constructing a U(1) principal bundle P with connection form $\mathcal{E}$ above space-time [21], [22]. The curvature form of this bundle is just then the electromagnetic 2-form F. $\mathcal{E}$ has the local expression

$$\mathcal{E} = \alpha + \hbar\,\frac{dz}{iz} \qquad (44)$$

where the local 1-form α is written as

$$\alpha = e\,A_{\alpha}\,dx^{\alpha} \quad \left(= e\,A_j\,dq^j - V\,dt\right) \qquad (45)$$

The existence and uniqueness of this construction depends on the "non integrable phase factor" (43).

Now, to see the connection between these two theories, imagine that a particle with charge e and mass m moves in the electromagnetic field. In describing the interacting quantum system electron + field the latter can be studied by the W.K.B. approximation. Then, the electron wave equation factors out from that of the composite system ; the effect of the field is retained by an interaction term ("minimal coupling") in the Schrödinger equation [3].

Minimal coupling has the following geometric expression [21], [22]: Let σ_0 denote the symplectif form of a free particle. Then the prequantum bundle for the electron + (passive) field is constructed as

$$Y = pr_X^{*} P \tag{46}$$

where $pr_X : TQ \times R \ni (q, v, t) \longrightarrow (q, t) = x \in X$; the connection form itself is written (with a slight ambiguity) as

$$\omega = \Theta_0 + \mathcal{E} = \text{locally} = (\Theta_0 + \alpha) + \hbar \frac{dz}{iz} \tag{47}$$

where Θ_0 is the global action form $(m v dq - m \frac{v^2}{2} dt)$ for σ_0

Now, as we have shown, the properties of the prequantum bundle depend on the amplitude

$$\exp\left[\frac{i}{\hbar} \int_\gamma (\Theta_0 + \alpha)\right] \tag{48}$$

$(\gamma \subset TQ \times \mathbb{R})$

but $\int_\gamma \Theta_0$ exists always and is unique. Thus all the problems of existence and uniqueness come from the factor $\exp\left[\frac{i}{\hbar} \int_\gamma \alpha\right]$. Finally, as α projects to X by construction, we can always use curves γ lying in X. Thus, we can explain the role of the Wu-Yang factor ; it is just the factor which determines whether a test particle moving in the exterior classical field has a meaningful quantum description.

A C K N O W L E D G E M E N T S

I am indebted to Jean-Marie SOURIAU for hospitality, help and encouragement at Marseille. Discussions with Christian DUVAL and David SIMMS are also gratefully acknowledged.

REFERENCES

[1] R.P. FEYNMAN and A.R. HIBBS, Quantum Mechanics and Path Integrals, Mc. Graw-Hill Book Co., (1965).

[2] Y. AHARONOV and D. BOHM, Phys. Rev. 115, 3, 145 (1959).

[3] Y. AHARONOV and D. BOHM, Phys. Rev. 123, 4, 1511 (1961).

[4] P.A. HORVÁTHY, Phys. Letters A101 (1980), (to appear).

[5] T.T. WU and C.N. Yang, Phys. Rev. D14, 2, 437 (1976).

[6] P.A.M. DIRAC, Phys. Rev. 74, 817 (1948).

[7] C. DEWITT and M.G.G. LAIDLAW, Phys. Rev. D3, 6, 1375 (1971).

[8] L. SCHULMAN, J. Math. Phys. 12, 2, 304 (1971).

L. SCHULMAN, in "Functional Integration and its Applications", Proc. Intern. Conf. London 1974, Clarendon Press (1975) A.M. Arthurs Ed.

[9] J.S. DOWKER, J. Phys. A (Gen. Phys.) 5, 936 (1972).

[10] B. KOSTANT, in Lecture Notes in Math. 170, Springer (1970) Taam Ed.

[11] J.M. SOURIAU, Structure des systèmes dynamiques, Dunod (1970).

[12] J.M. SOURIAU, Structure of Dynamical Systems, to appear at North-Holland.

[13] N.M.J. WOODHOUSE and D.J. SIMMS, Lecture Notes in Physics <u>53</u>, Springer (1976).

[14] T.T. WU and C.N. YANG, Phys. Rev. <u>D12</u>, 3845 (1975).

[15] P.A. HORVÁTHY, J. Math. Phys. <u>20</u>, 1,49 (1979).

[16] P.A. HORVÁTHY, Ph.D. Thesis, (in Hungarian) (1978).

[17] J. KLEIN, Ann. Inst. Fourier (Grenoble) <u>12</u>, 1-124 (1962).

J. KLEIN, Ann. Inst. Fourier (Grenoble) <u>13</u>, 191 (1963).

[18] C. GODBILLON, "Géométrie différentielle et mécanique analytique" Hermann, Paris (1969).

[19] SULANKE and WINTGEN, Differentialgeometrie und Faserbündel, VEB Deutscher Verlag der Wissenschaften, Berlin (1972).

[20] P.A. HORVÁTHY, Feynman Integral for Spin, Preprint CPT Marseille 79/P.1099 (1979) (unpublished).

[21] S. STERNBERG, in Lecture Notes in Math. <u>676</u>, 1-80, Bleuler et al. Eds., Springer (1978).

[22] Ch. DUVAL, Sur les mouvements classiques dans un champ de Yang-Mills, CPT Preprint Marseille 78/P.1056 (1978) (unpublished).

[23] L. LANDAU and E. LIFCHIFTZ, Mécanique, Mir (1965).

[24] R. ABRAHAM and J. MARSDEN, Foundations of Mechanics, Benjamin, (1978).

[25] P.A. HORVÁTHY and L. ÚRY, Acta Physica Hungarica <u>42</u>, 3 (1977).