HU8005071

KFKI-1980-67

1980 OKT 6

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MONTE CARLO STUDY OF SU(2) GAUGE THEORY AT FINITE TEMPERATURE

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BUDAPEST

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Submitted to Physics Letters

HU ISSN 0368 5330
ISBN 963 371 703 6

ABSTRACT

We find numerical evidence for the phase transition between the confinement phase and free Coulomb phase of SU(2)Yang-Mills theory with lattice cut-off.

АННОТАЦИЯ

Численным расчетом найдена точка фазового перехода на решетке между удерживающей и кулоновской фазой SU(2) теории Янга и Миллса.

KIVONAT

Numerikusan megtaláljuk a fázisátmeneti pontot rácson az SU(2)Yang-Mills elmélet bezáró fázisa és szabad Coulomb-fázisa között.

ABSTRACT

We find numerical evidence for the phase transition between the confinement phase and free Coulomb phase of SU(2) Yang-Mills theory with lattice cut-off. The search for the critical temperature is based on a Monte Carlo study of the string tension between a heavy $Q\bar{Q}$-pair in heat bath. The arbitrary normalisation 0.2 GeV^2 is used for the string tension at zero temperature when a smooth extrapolation of the lattice theory to the continuum limit is carried out. Our numerical estimate for the critical temperature is $T_c \sim 160 \pm 30$ MeV in the absence of quark degrees of freedom. It is suggestive that the phase transition is of second-order.

A few weeks ago we announced [1] the first numerical evidence for the existence of a phase transition between the confinement phase and free Coulomb phase of the SU(2) Yang-Mills theory with lattice cut-off.

There has been a long-standing conjecture that a phase transition must take place between the high temperature and low temperature phases of a non-Abelian gauge field theory [2]. Polyakov [3] and Susskind [4] gave convincing arguments for the presence of this phase transition in the strong coupling limit of the lattice model.

Our Monte Carlo calculation confirms the existence of a phase transition between confined and liberated phases in the strong coupling limit of SU(2) lattice gauge theory. Besides, we find the two phases and the critical point in the region of intermediate coupling where a smooth extrapolation of the lattice model to its continuum limit exists.

It did not escape our attention that the results presented here may be useful for the early universe, for quark matter search in heavy ion collisions, and for a broader view and better understanding of the confinement problem in Quantum Chromodynamics. It is remarkable that an environment can be simulated inside the computer which corresponds to thermal quark liberation at a few hundred MeV temperature.

The physical properties of a quantum field theory at finite temperature can be calculated in terms of the partition function,

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right), \tag{1}$$

and the thermal averages of physical observables,

$$\langle O \rangle = \frac{1}{Z}\mathrm{Tr}\left(Oe^{-\beta H}\right), \tag{2}$$

where $\beta = 1/T$ is the inverse temperature with $k_B = 1$.

The partition function of the SU(2) gauge theory in the absence of quarks may be written as a Euclidean path integral

$$Z = \int DA_\mu \exp\left[-\frac{1}{2g^2}\int_0^\beta dt\int d^3x \; \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}\right] \tag{3}$$

over periodic gauge fields,

$$A_\mu(\beta, \vec{x}) = A_\mu(0, \vec{x}) \ , \tag{4}$$

with period β in the fictive imaginary time direction. The standard notation $A_\mu = A_\mu^a \sigma^a/2$ is used throughout the paper. The index a runs from 1 to 3 in SU(2) and σ^a denotes the standard Pauli matrices. The trace tr operates on SU(2) matrices.

In order to study the string tension in a heat bath we have to introduce an external color source Q at location R and a color sink $\bar{Q}$ at the origin. The free energy $V(\beta, R)$ of this heavy $Q\bar{Q}$-pair is related to the correlation function of thermal Wilson loops by the formula [5]

$$\langle \mathrm{tr}\, W(0) \;\; \mathrm{tr}\, W^+(\vec{R}) \rangle = e^{-\beta V(\beta, R)} \ . \tag{5}$$

The thermal Wilson loop $W(\vec{R})$ is defined as a closed path in the fictive imaginary time direction,

$$W(\vec{R}) = P \exp\left[i \int_0^\beta dt \; A_0(t, \vec{R}) \right] , \tag{6}$$

where P denotes path ordering in the standard fashion.

The free energy $V(\beta, R)$ is a measure of the potential energy between the heavy $Q\bar{Q}$-pair at finite temperature. In the confinement phase we expect the behavior

$$\langle \mathrm{tr}\, W(0) \;\; \mathrm{tr}\, W^+(\vec{R}) \rangle \sim \mathrm{const}\; e^{-\beta \sigma(\beta) R} \tag{7}$$

at large distances. The free Coulomb phase is characterised by a screened Coulomb potential, and accordingly,

$$\langle \mathrm{tr}\, W(0) \;\; \mathrm{tr}\, W^+(\vec{R}) \rangle \underset{R \to \infty}{\sim} \mathrm{const}\left(1 + \beta \frac{3 g^2}{16\pi R} e^{-\kappa R}\right) . \tag{8}$$

The effective string tension $\sigma(\beta)$ at finite temperature is defined by Eq.(7). The Debye screening length κ^{-1} in Eq.(8) is known to be $\kappa^2 = \frac{2}{3} g^2 T^2$ in the lowest order of perturbation theory.

The two phases are clearly distinguished by the different behaviors of the correlation function in Eqs.(7) and (8). The trace of the gauge invariant thermal Wilson loop may be regarded as an order parameter in SU(2) gauge theory at finite temperature. The thermal average of $\mathrm{tr}\, W(\vec{R})$ measures the free energy of an isolated quark with respect to the vacuum. It vanishes in the confinement phase, since the infinite free energy of the isolated quark is in the exponent of Eq.(5). The order parameter $\mathrm{tr} W(\vec{R})$ is some non-vanishing constant in the Coulomb phase where it is related to the finite self-energy of an isolated free quark on the lattice.

For the numerical evaluation of the functional integral on the left-hand side of Eq.(5) a lattice cut-off is introduced in the model following the standard procedure [6]. The periodic boundary condition in the imaginary time direction is required by the finite temperature of the heat bath. To eliminate surface effects in the three-dimensional physical space we also impose periodic boundary conditions in the three spatial directions.

The Lagrangian formulation of the continuum theory requires a symmetric lattice with equal lattice spacing in the spatial and imaginary time directions. The Hamiltonian method starts directly from Eq.(1) and operates with the transfer matrix [9]. A dense slicing is required then operationally in the imaginary time direction for fixed spatial cut-off a [1]. The two methods yield compatible results for the phase transition. First, we study the symmetric lattice.

The inverse temperature β is given in lattice spacing units a by the relation $\beta = n_t \cdot a$ where n_t is the number of lattice sites in the imaginary time direction. The spatial volume $n_s^3 \cdot a^3$ must be reasonably large for the calculation of thermodynamical quantities. There is no other restriction on the spatial size and the number of sites, n_s, in the three spatial directions is limited only by the performance of the computer.

The partition function

$$Z = \int \prod_{\{i,j\}} dU_{ij} \, e^{-\frac{4}{g^2} S(U)} \qquad (9)$$

defines now the thermodynamics of SU(2) gauge theory. The finite dimensional integral in Eq.(9) includes all independent link variables U_{ij} and dU_{ij} designates the invariant group measure of SU(2) . The action S is a sum over all elementary plaquettes,

$$S = \sum_{\text{plaquettes}} \left(1 - \frac{1}{2} \, \mathrm{tr}\left(U_{ij} U_{jk} U_{kl} U_{li}\right)\right) ,$$

where i, j, k and l represent the labeling of the sites around a plaquette. The connection between the link variable U_{ij} and the exponentiated gauge field variable $\exp(i \cdot a \cdot A_\mu)$ is well-known [6].

The Monte Carlo method [7,8] was applied for the calculation of the order parameter tr W(R) and for the evaluation of the correlation function $\mathrm{tr}\, W(0) \cdot \mathrm{tr}\, W(R)$. In our program the heat bath method of Creutz [8] was implemented for sweeping through all lattice sites in each step of the iteration towards thermal equilibrium.

The order parameter is shown in Fig.1a for the inverse temperature $\beta = 3a$ at different values of the coupling constant g^2 . At $4/g^2 = 1.9$ the order parameter drops to zero and a phase transition occurs in the system. The same behavior of the order parameter is seen in Fig.1b for a fixed value of the coupling constant at $4/g^2 = 2.5$ as the inverse temperature β varies. The thermal average of tr W drops to zero at about $\beta = 10a$.

It is easy to show that the action S is invariant under the global symmetry transformation $U \to -U$ in a selected spacelike hyperplane on each link in the imaginary time direction. The transformation flips the order parameter $\mathrm{tr}\, W \to -\mathrm{tr}\, W$.

In the symmetric disordered phase $\langle \mathrm{tr}\, W \rangle$ vanishes and the free energy of an isolated quark is infinite. Therefore, this phase confines quarks and the asymptotic form $\langle \mathrm{tr}W(0) \cdot \mathrm{tr}W(R) \rangle \underset{R \to \infty}{\longrightarrow}$ $\to \mathrm{const} \cdot \exp(-\beta\, \sigma(\beta) \cdot R)$ is observed. In the free Coulomb phase a spontaneous symmetry breaking occurs and the behavior $\langle trW(0)\, trW(R)\rangle \underset{R\to\infty}{\longrightarrow}$ $\langle trW\rangle^2 \neq 0$ is expected. This implies finite free energy for isolated free quarks on the lattice.

The observation of symmetry breaking in the Coulomb phase is influenced by the finite size of the spatial lattice. The order parameter develops a constant expectation value for long time periods and the probability that it flips sign gradually decreases with growing n_s .

In Fig.1b only cold starts are shown where all U's are set to the same constant matrix at the beginning of the iteration. The run with $\beta = 6a$ actually starts at the positive value $\langle trW\rangle = +1$ and we had to change the sign of the points on the plot throughout this particular run for convenient comparison with other values of β . Similarly, in Fig.1a we changed the sign of $\langle trW\rangle$ for the run at $4/g^2 = 3.5$ with hot start. The U variables are set to random matrices at hot starts and the runs begin with $\langle trW\rangle = 0$. The runs at $4/g^2 = 3.5$ with hot and cold starts reach the same thermodynamical limit within one hundred iteration steps.

The behavior of the correlation function is shown in Fig.2a at $\beta = 4a$ and $g^2 = 2$ in the confinement phase. We find the numerical value $\sigma(\beta)\cdot a^2 = 0.54$ for the tension as extracted from the exponential shape of the correlation function. Creutz measures $\sigma(0)\cdot a^2 = 0.6$ at $g^2 = 2$. His result corresponds to zero temperature within some technical limitations.

Fig.2b depicts the correlation function in the free Coulomb phase at $\beta = 4a$ and $4/g^2 = 2.3$. The arrow marks the value of $\langle trW\rangle^2$ which is the asymptotic limit of the correlation function.

In our search for the phase transition point the order parameter never exhibits a discontinuous jump at T_c and a second-order transition is suggested. It is also supported by the observation of large fluctuations near the critical point.

The critical coupling constant is shown in Fig.3 at various temperatures. The interpretation of these results requires a smooth extrapolation to the continuum limit of the theory. The scale is set on the lattice by the lattice Λ parameter in lattice spacing units as

$$\Lambda = \lim_{a\to 0} \frac{1}{a} \left(\gamma_0 g^2(a)\right)^{-\frac{\gamma_1}{2\gamma_0^2}} \exp\left(-\frac{1}{2\gamma_0 g^2(a)}\right) \qquad (10)$$

in the continuum limit. The coupling constant $g(a)$ is used throughout the Monte Carlo calculations. For SU(N) gauge groups the coefficients in Eq.(10) are $\gamma_0 = \frac{11}{3}\frac{N}{16\pi^2}$ and $\gamma_1 = \frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2$.

The lattice Λ parameter is related to Λ^{MOM} in the continuum limit theory by a recent calculation of Hasenfratz and Hasenfratz [10] . Creutz calculated in his Monte Carlo program the tension at zero temperature in terms of the Λ parameter in the SU(2) gauge theory. He finds [11]

$$\Lambda = (1.3 \pm .2) \times 10^{-2}\ \sigma^{1/2}(0) \quad . \qquad (11)$$

The arbitrary normalization 0.2 GeV^2 is used for the string tension at zero temperature in our numerical estimate of the critical point in the continuum limit. This would correspond to $\Lambda^{MOM} = 330$ MeV in continuum SU(2) gauge theory [10] .

Our calculated Monte Carlo points in Fig.3 follow the renormalization group relation

$$T_c a = \text{const}\ \left(g^2(a)\right)^{-\frac{51}{121}} \cdot \exp\left(-\frac{12\pi^2}{11 g^2(a)}\right) \qquad (12)$$

for $4/g^2 \geq 2$. The otherwise arbitrary constant in Eq.(12) is determined by the Monte Carlo points. The best estimate of the critical temperature with the presented extrapolation to the continuum limit is

$$T_c = (0.35 \pm .05)\, \sigma(0)^{1/2}, \text{ or } T_c = 160 \pm 30 \text{ MeV} . \qquad (13)$$

The error bar on the critical point in Eq.(13) is two-fold.

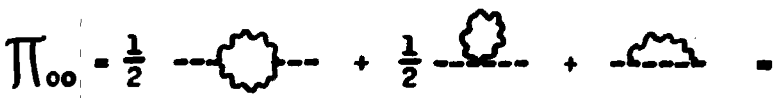
There is a statistical error in Creutz's relation of Eq.(11). Our statistical inaccuracy is represented in Fig.3 by the horizontal error bars of the critical coupling for a given value of $T \cdot a$.

There are finite temperature corrections to the relation between the lattice spacing a and coupling constant g which we calculated in Coulomb gauge on the one-loop level. These corrections are small for $4/g^2 > 2$ where $T_c \ll \frac{1}{a}$.

Our estimate for the renormalization of the coupling at finite temperature is as follows. We calculate the Coulomb force on the scale a at temperature T on a superfine lattice a_o as

$$\left.\frac{g^2(a,T)}{\vec{p}^2}\right|_{\vec{p}^2=\frac{1}{a^2}} = \frac{g^2(a_o)}{\vec{p}^2 - \Pi_{oo}(p_o=o,\ \vec{p}^2,\ g^2(a_o),\ T)} \qquad (14)$$

where the right-hand side is also taken at $\vec{p}^2 = 1/a^2$. The polarization tensor Π_{oo} is given by



$$= \Pi_{oo}\left(p_o=o,\ \vec{p}^2,\ g^2(a_o),\ T=o\right) + g^2(a_o)\,\frac{N_c}{18}\,T^2 \qquad (15)$$

for SU(N) gauge theory, in the limit $\vec{p}^2 \gg T^2$.

Eq.(14) is obtained by the bubble summation of the one-loop diagrams in Coulomb gauge [5].

Eqs.(14) and (15) relate the coupling constant $g^2(a,T)$ on two different scales and different temperatures as

$$g^2(a_2,T_2) = \frac{g^2(a_1,T_1)}{1+g^2(a_1,T_1)\left[\frac{11}{24\pi^2}\,N_c \ln\frac{a_1}{a_2} + \frac{N_c}{18}\left(T_1^2 \cdot a_1^2 - T_2^2 \cdot a_2^2\right)\right]}\,.$$

The dashed line in Fig.3 includes this finite temperature correction.

The dotted line in Fig.3 is an estimate of the critical temperature [5] in the strong coupling limit:

$$T_c \simeq a \frac{\zeta(o)}{\ln 5} .$$

Its derivation is based on the observation that the partition function develops a singularity at the critical point by the condensation of chromo-electric vortices.

At about $4/g^2 = 1.9$ the Monte Carlo points break away from the corrected curve (dashed line in Fig.3) and gradually approach the strong coupling estimate.

It is interesting to search for the critical point of the phase transition in the Hamiltonian formulation of the theory. It also provides a consistency check of the overall picture.

Our starting point is Eq.(1) with the lattice Hamiltonian of Kogut and Susskind [12]. The partition function can be approximated through the transfer matrix [9] by

$$Z = \int \prod_{\{i,j\}} dU_{ij} \exp\Big[-\frac{4}{g_H^2}\frac{\tau}{a} \sum_{\substack{\text{spacelike}\\ \text{plaquettes}}} \Big(1-\frac{1}{2}\mathrm{tr}\big(U_{ij}U_{jk}U_{kl}U_{li}\big)\Big) - \frac{4}{g_H^2}\frac{a}{\tau} \sum_{\substack{\text{timelike}\\ \text{plaquettes}}} \Big(1-\frac{1}{2}\mathrm{tr}\big(U_{ij}U_{jk}U_{kl}U_{li}\big)\Big)\Big] \quad (16)$$

where τ is a single slice of the interval β. Eq.(16) becomes exact in the limit $\tau/a \to 0$ for fixed lattice spacing a of the three-dimensional space. The coupling constant $g_H(a)$ appears in the original lattice Hamiltonian [12]. The expectation values in Eq.(2) can be calculated in a similar way.

The phase transition is shown in Fig.4 for a fixed coupling $4/g_H^2 = 2$. The critical temperature was found at $T_c \cdot a = 0.32$. For smaller coupling $4/g_H^2 = 2.15$ we find $T_c \cdot a = 0.22$. The two points fit the continuum limit renormalization scheme with

$T_c \simeq 0.35 \sqrt{\sigma(0)}$ provided that the lattice scale parameter in the Hamiltonian formulation is approximately the same as in the Lagrangian method.

Our numerical calculation may be regarded as the first direct determination of the Hamiltonian scale parameter Λ_H in terms of the Lagrangian Λ parameter. If the applied transfer matrix approximation with our dense slicing of the finite time interval β is adequate, we predict $\Lambda_H \simeq \Lambda$.

After the work reported here was complete and presented [1], similar work [13] came to our attention.

ACKNOWLEDGEMENTS

We thank Ferenc Szabó for his generous help during the preparation of our work. We are also indebted to Magda Zimányi and the staff of the computer center for their effort to provide us with sufficient computational time. One of us (J. K.) appreciates discussions with Larry D. McLerran and Giorgio Parisi. He also appreciates the assistance of Julia Ember.

REFERENCES

[1] Talk presented by one of us (J. Kuti) at the XX^th International Conference on High Energy Physics, July 18, (1980), Madison ; J. Kuti, J. Polónyi and K. Szlachányi, to be published in the conference proceedings.

[2] N. Cabibbo and G. Parisi, Phys. Lett. 59B (1975) 67 ; J.C. Collins and M.J. Perry, Phys. Rev. Lett. 34 (1975) 1353 ; P.D. Morley and M.B. Kislinger, Phys. Rep. 51C (1979) 63 and references therein ; E.V. Shuryak, Phys. Rep. 61C (1980) 71 and references therein ; for a recent interesting work, see D.J. Gross, R.D. Pisarski and L.G. Yaffe, Princeton preprint (1980) .

[3] A.M. Polyakov, Phys. Lett. 72B (1978) 477 .

[4] L. Susskind, Phys. Rev. D20 (1979) 2610 .

[5] The details of our calculation will be published elsewhere.

[6] For a review, see, for example, J.M. Drouffe and C. Itzykson, Phys. Rep. 38C (1978) 133.

[7] K.G. Wilson, Cornell preprint (1979) .

[8] M. Creutz, Brookhaven preprint (1979) .

[9] M. Creutz, Phys. Rev. D15 (1977) 1128.

[10] A. Hasenfratz and P. Hasenfratz, Phys. Lett. 93B (1980) 165.

[11] M. Creutz, Proceedings of the Johns Hopkins workshop (1980) 85.

[12] J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395.

[13] L.D. McLerran and B. Svetitsky, SLAC-PUB-2572 (1980) .

FIGURE CAPTIONS

Fig. 1a The evolution of the order parameter $\langle tr\ W \rangle$ is shown at $\beta = 3a$ for various values of the coupling constant as the number of iteration steps increases. The value of the order parameter averaged over 10 iterations is plotted. Cold starts from $\langle tr\ W \rangle = 1$ are indicated and the corresponding runs are more densely populated for the first few iterations. For the black triangle points the plot is interrupted between iteration steps 50 and 80 for the clarity of the figure.

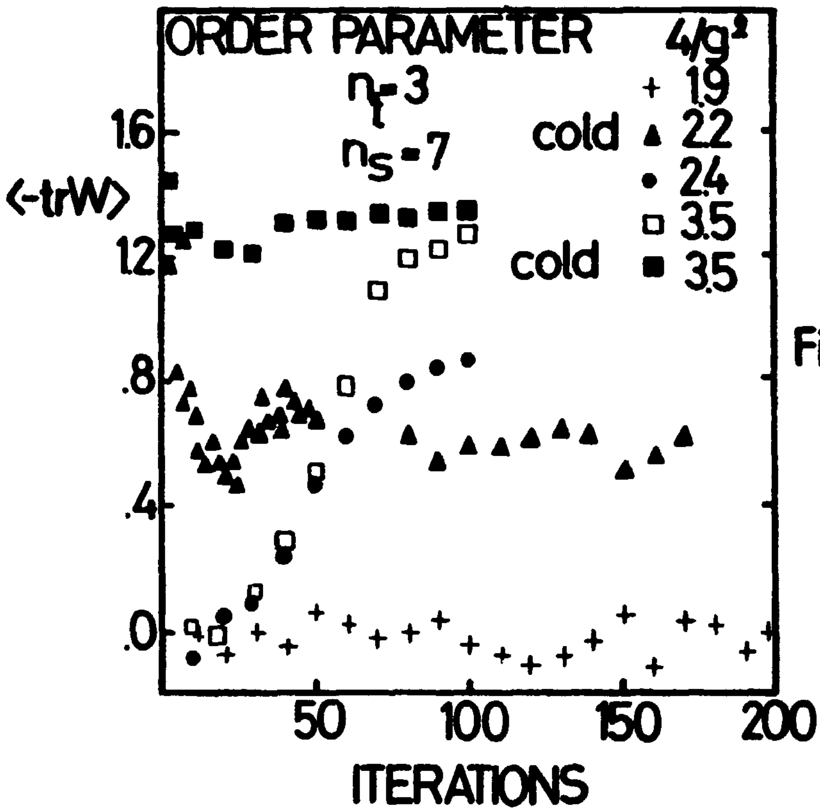
Fig. 1b Runs are shown for fixed coupling as the temperature varies. All runs are selected with cold starts. The order parameter disappears in the noise for $\beta > 10a$.

Fig. 2a The correlation function with exponential decay at $g^2 = 2$ determines the tension $\sigma(\beta)$ in the confinement phase at $\beta = 4a$ near the critical point. R is given in lattice spacing units.

Fig. 2b The correlation function is shown at the same temperature but for weaker coupling $4/g^2 = 2.3$. The points follow the Debye screened Coulomb law with screening length κ^{-1} determined by the temperature and coupling.

Fig. 3 Our Monte Carlo points for the critical temperature follow the renormalization group relation (solid line) in the intermediate coupling region. Points for the critical coupling are given at $\beta = a, 2a, 3a, 4a, 5a, 6a, 7a, 8a, 9a, 10a$.

Fig. 4 The order parameter is shown for runs at $g_H^2 = 2$ in the Hamiltonian formulation. Typically, twelve to sixteen time slices were adequate for inverse temperatures around $\beta = 3a$. The order parameter disappears in the noise at about $T \cdot a = 0.3$.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● 2.4
□ 3.5
cold ■ 3.5
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1.2
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.4
.0
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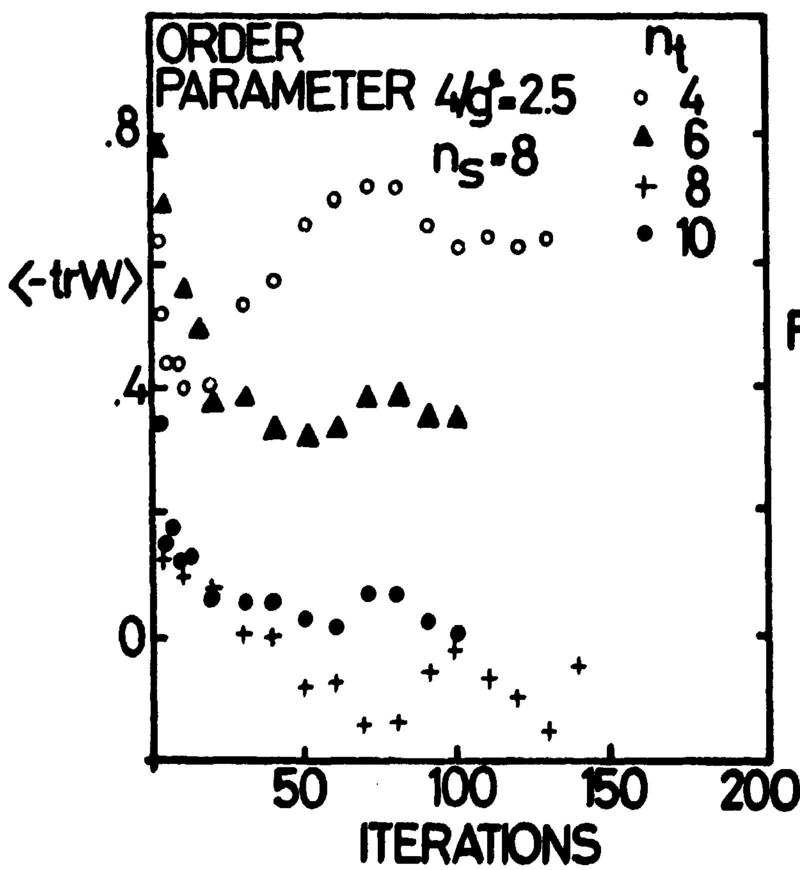
50
100
150
200
ITERATIONS

Fig.1a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n_t
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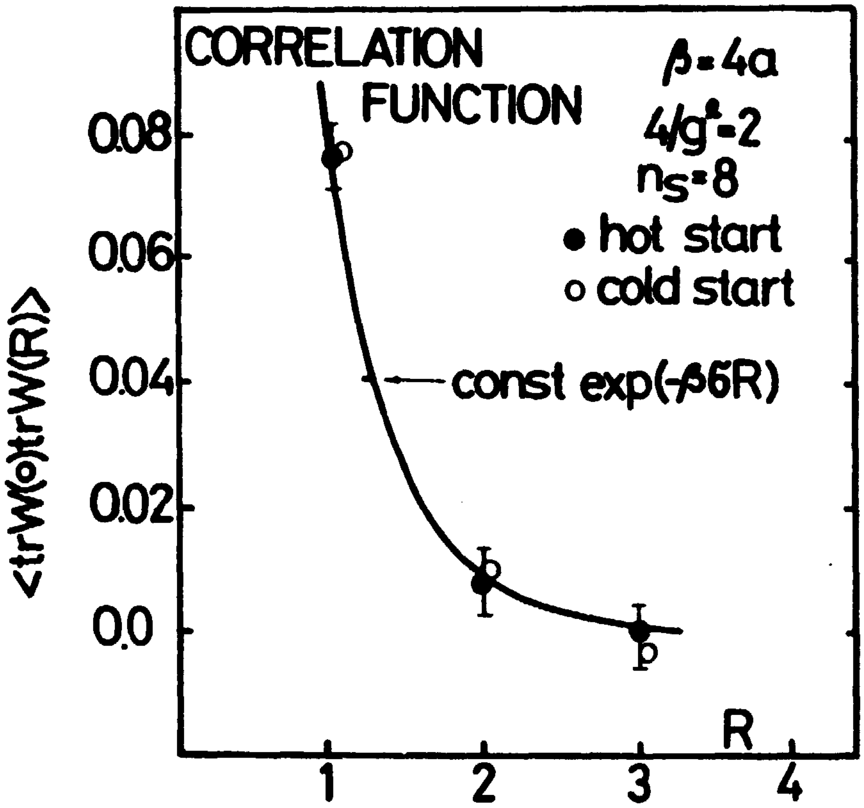
▲ 6
+ 8
● 10
.8
.4
0
<-trW>
50
100
150
200
ITERATIONS

Fig.1b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FUNCTION
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4/g²=2
n_S=8
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o cold start
const exp(-β6R)
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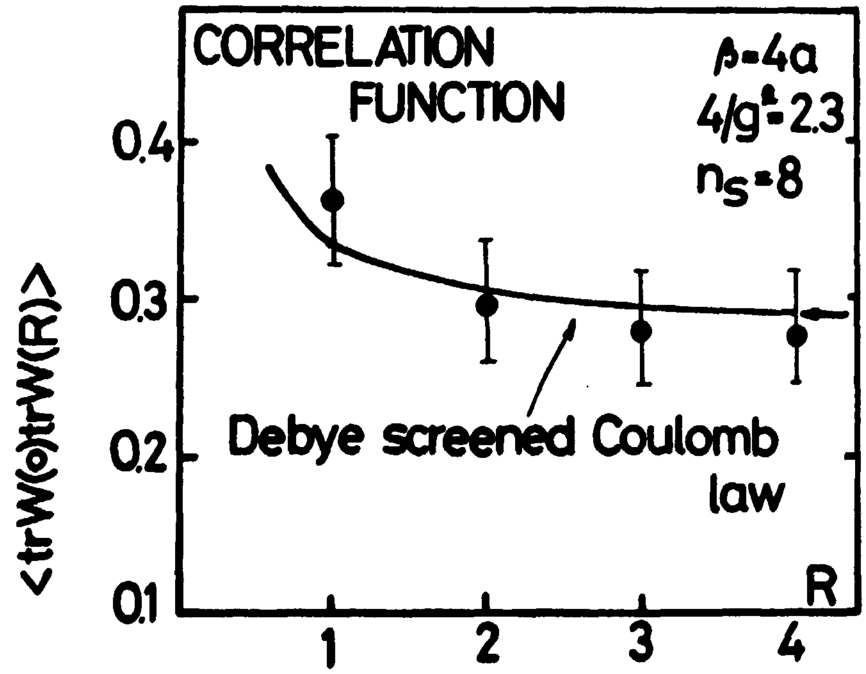
0.06
0.04
0.02
0.0
R
1
2
3
4

Fig.2a


CORRELATION
FUNCTION
β=4a
4/g²=2.3
n_S=8
Debye screened Coulomb
law
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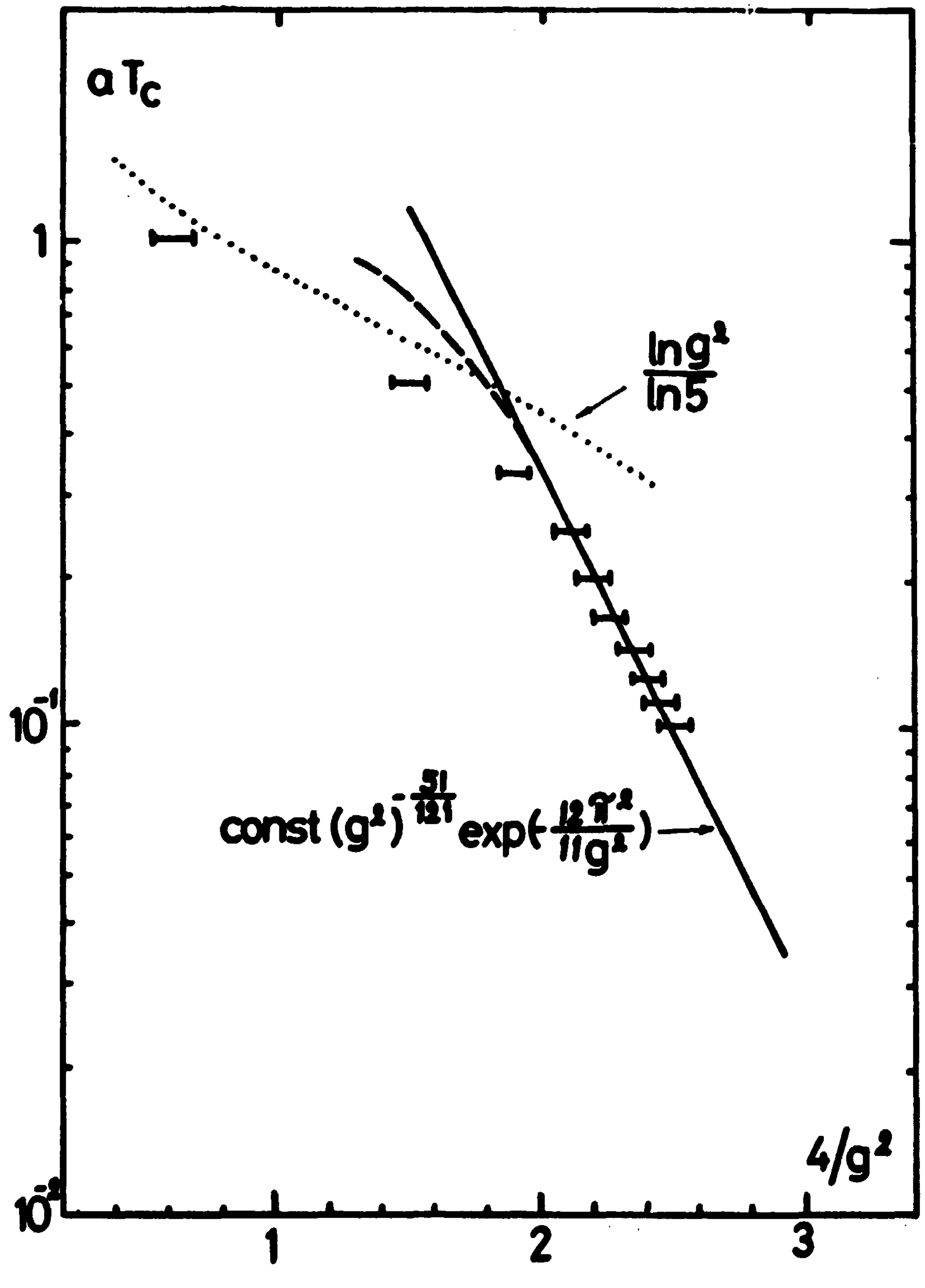
0.2
0.1
R
1
2
3
4

Fig.2b


a T_c
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10^{-1}
10^{-2}
$\frac{\ln g^2}{\ln 5}$
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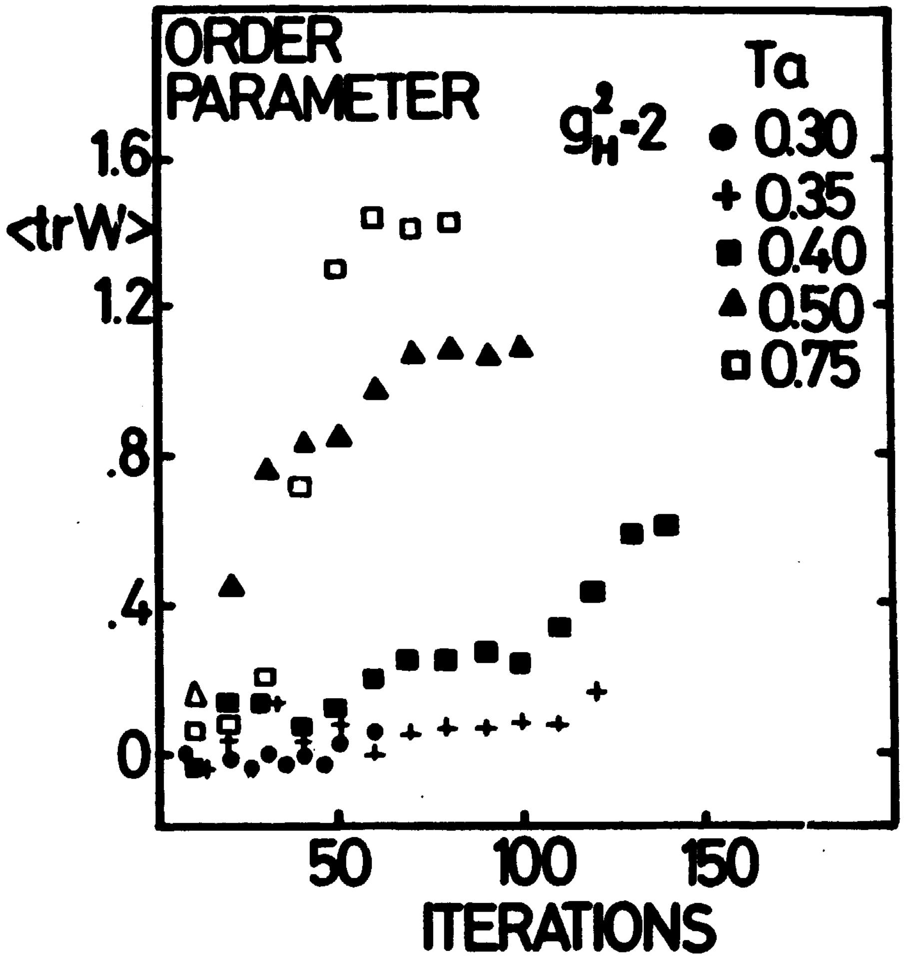
$4/g^2$
1
2
3

Fig. 3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1.2
.8
.4
0
50
100
150
ITERATIONS

Fig.4

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Szegő Károly
Szakmai lektor: Hraskó Péter
Nyelvi lektor: Forgács Péter
Példányszám: 490 Törzsszám: 80-535
Készült a KFKI sokszorosító üzemében
Budapest, 1980. szeptember hó