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THE πNN VERTEX FUNCTION AND OFF-MASS-SHELL πN SCATTERING.

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Abstract:

We have extended the dispersion relation calculation of the πNN vertex function to include off shell terms in the πN scattering matrix. These off shell terms are constrained by the current algebra and PCAC results, and contribute to the $N\bar{N} \to \pi\pi$ s-wave. As such, they add to the kinematic off shell effects which Durso, Jackson and VerWest found in the p-wave terms. The off shell terms increase the calculated Goldberger-Treiman discrepancy from 0.02 to 0.03, bringing it into agreement with the field theory value of Jones and Scadron (0.035). The calculated discrepancy remains smaller than the experimental value of 0.06 ± 0.01.

1. INTRODUCTION

It is well known that the elementary one pion exchange potential behaves badly asymptotically, in the sense that the amplitude does not vanish in the limit as the momentum transfer approaches infinity. One way of correcting this divergence problem is to introduce a "form-factor" or vertex function, F(t), which in the first Born approximation for the exchange of one pion, has the effect of changing the πNN coupling G to GF(t).

Physically F(t) may be probed in a variety of processes involving pion exchange, but it is often difficult to separate F(t) from other effects. Some success in this program has been achieved in the analysis of pn → np and $\bar{p}p \to \bar{n}n$ reactions[2]. However the most readily accessible test of F(t) is the Goldberger-Treiman discrepancy.[1] Consider the divergence of the axial current between two nucleon states (a neutron and a proton say)

$$\langle N'|\partial_\mu A_\mu^+|N\rangle = \sqrt{\frac{m_n m_p}{EE'}}\,[(m_p+m_n)G_A(t)+tG_p(t)]\; i\bar{u}'\gamma_5\frac{\tau_\alpha}{2}u \tag{1}$$

Then writing a dispersion relation for $(m_p+m_n)G_A(t) + tG_p(t)$ and assuming pion-pole dominance

$$(m_p+m_n)G_A(t)+tG_p(t) = \frac{R_{\pi\text{-pole}}}{\mu^2-t} + \frac{1}{\pi}\int_{(3\mu)^2}^{\infty} dt'\,\frac{\mathrm{Im}D(t')}{t'-t} \tag{2}$$

where $R_{\pi\text{-pole}}$ is the residue of the pion pole evaluated at $t = \mu^2$:

$$R_{\pi\text{-pole}} = \sqrt{2}Gf_\pi\mu^2 \tag{3}$$

and G is the πNN coupling constant with $G = G(\mu^2)$.

At $t = 0$, equation (2) becomes

$$(m_p+m_n)G_A(0) = \sqrt{2}Gf_\pi + \frac{1}{\pi}\int_{(3\mu)^2}^{\infty} dt' \frac{\mathrm{Im}D(t')}{t'} \tag{4}$$

and if we assume the cut contribution to be negligible, this gives the Goldberger-Treiman[1] relation

$$(m_p+m_n)G_A(0) = \sqrt{2}Gf_\pi \tag{5}$$

The present experimental value of Δ_{GTR} is[3]

$$\Delta_{GTR}^{exp} = 1-\frac{(m_p+m_n)G_A(0)}{\sqrt{2}Gf_\pi} = 0.06 \pm 0.01$$

this difference from the theoretical value being the so-called Goldberger-Treiman "discrepancy", usually taken to be a measure of chiral symmetry breaking. If however, one replaces the coupling G by the full vertex function GF(t), then equation (4) becomes

$$(m_p+m_n)G_A(0) = \sqrt{2}G\, F(0)\, f_\pi \tag{6}$$

and thus

$$\Delta_{GTR} = 1 - F(0) \tag{7}$$

from which one can see that the Goldberger-Treiman discrepancy may also be interpreted as arising from the dependence of the coupling on the momentum transfer.

Assuming then, that the t-dependence of the πNN vertex function arises solely from the cut contribution, the pion Born term may be re-written as

$$\frac{\sqrt{2}G(t)f_\pi\mu^2}{\mu^2 - t} = \frac{\sqrt{2}G(\mu^2)f_\pi\mu^2}{\mu^2 - t} + \frac{1}{\pi}\int_{(3\mu)^2}^{\infty} dt' \frac{\mathrm{Im}D(t')}{t'-t} \tag{8}$$

and substituting for $G(t) = G(\mu^2)F(t)$ with $F(\mu^2) = 1$

$$F(t) = 1 + \frac{(\mu^2 - t)}{\sqrt{2} f_\pi \mu^2} \frac{1}{\pi} \int_{(3\mu)^2}^{\infty} dt' \frac{\text{Im}D(t')}{t'-t} \qquad (9)$$

This can be identified as a dispersion relation for F(t), once subtracted at $t = \mu^2$ and written as

$$F(t) = F(\mu^2) + \frac{(t-\mu^2)}{\pi} \int_{(3\mu)^2}^{\infty} dt' \frac{\text{Im}F(t')}{(t'-t)(t'-\mu^2)} \qquad (10)$$

where the identification

$$\text{Im}F(t') = \frac{(\mu^2-t') \, \text{Im}D(t')}{\sqrt{2} f_\pi \mu^2} \qquad (11)$$

has been made.

Two different ways of evaluating Im F(t) have been used in the literature. One uses field theory to evaluate the unitarity diagrams of figure 1. This was pioneered by Goldberger and Treiman[1] and for the latest results we refer to Jones and Scadron [4] who obtain

$$\Delta_{GTR}^{JS} = 0.035 \qquad (12)$$

The alternative, which we follow here, is to introduce the $\pi N \to \pi N$ scattering amplitude, to approximate the three pion cut, as in figure 2. This approach was introduced by Nutt and Loiseau [5]. Durso, Jackson and VerWest [6] pointed out that it was essential to include off shell effects in the πN amplitude in order to reproduce the field theoretic results for the $\rho\pi$ exchange terms, calculated in detail by Braathen [7]. Durso, Jackson and VerWest obtained

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$$\Delta^{DJV}_{GTR} = 0.02 \qquad (13)$$

In this paper we point out that the off mass shell effects are also important in the $\pi\pi \to N\bar{N}$ s-wave channel. While most off mass shell variation is of order q^2/m^2 where m is a typical hadronic mass (about 1 GeV), in this channel rapid off mass shell behaviour of order q^2/μ^2 is required to reproduce the current algebra and PCAC constraints. Thus the off shell corrections may be expected to be important, and we find that they give an additional contribution of 0.01 to Δ_{GTR}. Our final result is

$$\Delta^{theo.}_{GTR} = 0.03 \qquad (14)$$

which is still only half the experimental value, but is in good agreement with the field theoretic value of Jones and Scadron.

2. IM F(T) IN TERMS OF πN SCATTERING AMPLITUDES

Applying the Feynman rules to the diagrams of figure 2, and using pseudoscalar coupling for the pion, the method of Nutt and Loiseau gives

$$\bar{u}(p')\gamma_5\tau_\alpha u(p)\,\mathrm{Im}F(t) \qquad (15)$$

$$= -2\sum_\beta \mathrm{Im}\int \frac{d^4k}{(2\pi)^4}\,\bar{u}(p')\gamma_5\tau_\alpha G(p'-k)T_{\beta\alpha}(p'-k,k;p',p'-p)u(p)D(k)$$

where G and D are the free nucleon and pion propagators, and $\bar{u}(p_2)T_{\beta\alpha}(p_2,q_2;p_1,q_1)u(p_1)$ is the πN scattering amplitude for $\pi(q_1) + N(p_1) \to \pi(q_2) + N(p_2)$ (see figure 3a). The factor of two omitted in reference 5, arises because there are two Feynman diagrams in figure 2, which contribute equally to Im F(t).

Using the standard CGLN notation for the πN amplitudes [8] and the kinematics of figure 3b, equation (15) may be reduced to

$$\bar{u}(p')\gamma_5\tau_\alpha u(p)\,\mathrm{Im}F(t) \tag{16}$$

$$= 2\,\mathrm{Im}\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+\mu^2}\,\bar{u}(p')\,\frac{\gamma_5\tau_\alpha i\gamma.k}{(p'-k)^2+m^2}\left\{T^{(+)} + 2T^{(-)}\right\}u(p)$$

which defines $\mathrm{Im}\,F^{(\pm)}(t)$.

In figure 2, interpreted as a unitarity diagram for the discontinuity of F, it is clear that the internal nucleon and the external pion are both off mass shell. Thus as Durso, Jackson and VerWest showed, one must generalise the usual CGLN amplitude

$$T^{(\pm)}(p_1,q_1;p_2,q_2) = -A^{(\pm)} + i\tfrac{1}{2}\gamma.(q_1+q_2)B^{(\pm)} \tag{17}$$

to an off mass form, which they choose to be

$$T^{(\pm)} = -A^{(\pm)} + \tfrac{1}{2}i\left\{(\not{q}_1+\not{q}_2) - (\not{q}_1-\not{q}_2)\frac{(q_1-q_2).(q_1+q_2)}{(q_1-q_2)^2}\right\}C^{(\pm)} \tag{17}$$

$$+ \frac{1}{4m}\left\{\not{p}_2(\not{q}_1+\not{q}_2)+(\not{q}_2+\not{q}_1)\not{p}_1\right\}D^{(\pm)} \tag{18}$$

Equation (18) reduces to equation (17) on nucleon shell, with $B^{(\pm)} = C^{(\pm)} + D^{(\pm)}$. They then make the approximation of retaining only s- and p-wave terms when A,C and D are expanded in terms of the $N\bar{N} \to \pi\pi$ helicity amplitudes. Using the Cutkosky rules and co-ordinate frame $r = (\bar{0},\sqrt{\tau})$ one obtains

$$\mathrm{Im}\,F = \mathrm{Im}\,F^{(+)} + \mathrm{Im}\,F^{(-)} \tag{19}$$

$$\mathrm{Im}\,F^{(+)}(t) = \frac{1}{2\pi}\int_{4\mu^2}^{\infty} d\tau\;\Theta(t-(\sqrt{\tau}+\mu)^2)\;\mathrm{Im}f_0^+(\tau)\;\frac{m(t+\mu^2-\tau)}{t^{3/2}(m^2-\tfrac{1}{4}\tau)}\;A(t,\tau) \tag{20}$$

and

$$\mathrm{Im}\, F^{(-)}(t) = \int_{4\mu^2}^{\infty} d\tau\, \theta(t-(\sqrt{\tau}+\mu)^2) \frac{6}{2\pi\sqrt{t}(m^2-\frac{1}{4}\tau)} \left\{ \frac{Q(t-\mu^2-\tau)}{\tau} - \mu^2 A(t,\tau) \right\}$$

$$\times \left\{ m\, \mathrm{Im} f_1^{+}(\tau) - \frac{\tau}{4\sqrt{2}}\, \mathrm{Im} f_1^{-}(\tau) \right\}$$

$$+ \int_{4\mu^2}^{\infty} d\tau\, \theta(t-(\sqrt{\tau}+\mu)^2) \frac{6mQ^2}{2\pi(m^2-\frac{1}{4}\tau)\sqrt{t}} A(t,\tau) \left\{ \sqrt{\tfrac{1}{2}}\, m\, \mathrm{Im} f_1^{-}(\tau) - \mathrm{Im} f_1^{+}(\tau) \right\} \qquad (21)$$

where

$$A(t,\tau) = \frac{1}{\kappa} \tan^{-1} \left(\frac{4Q\kappa}{t-\mu^2-\tau} \right) \text{ with } \kappa = \sqrt{m^2-\tfrac{1}{4}t} \qquad (22)$$

and

$$Q = \left(\frac{1}{4t} \left\{ (t-\tau-\mu^2)^2 - 4\tau\mu^2 \right\} \right)^{\frac{1}{2}} \qquad (23)$$

Substituting (19), (20) and (21) into the once subtracted dispersion relation (10), and with suitable cut-offs for the integrals Durso, Jackson and VerWest obtain $F(t=0) = 0.98$ leading to a $\Delta^{DJW}_{GTR} = 0.02$ as quoted in the introduction.

In Fig. 4., we show the result of evaluating ImF(t) and F(t) in this way, using the helicity amplitude data of Höhler and Pietarienen [11]. The vector component of $\mathrm{Im}F^{-}(t)$ (given by the first term in equation (21)) is cut off at $t = 75\mu^2$, to simulate the cancellation of linear divergences which occurs between the vector coupled vertex term and the $\omega\pi$ propagator term. [6] The 'data' points shown in Fig. 4. are the result of our earlier, phenomenological analysis of F(t), where a Veneziano form for the form - factor $F_n(t)$ was used to reproduce the $d\sigma/dt$ data for averaged $np \to pn$ and $\bar{p}p \to \bar{n}n$ scattering. (Here the value of 13/2 has been used for the variable n, which gives a reasonable fit to the scattering data at intermediate energies and a Δ_{GTR} of .05).

Note that because the helicity amplitude data are given for $4\mu^2 \lesssim \tau \lesssim 50\mu^2$, though ImF(t) can be evaluated up to $t = 4m^2$ ($\sim 180\mu^2$) some of

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the integral will be missed. Further because of the almost linear behaviour of ImF(t), a once subtracted dispersion relation still gives a cut-off dependent Re F(t).

One way of estimating this uncertainty, is to subtract Re F(t) again, thus making the tail end of the integrand less important and ensuring that the major part of the contribution does come from the near t region. The only other reasonable subtraction point is at t = 0, with

$$\mathrm{Re}\ F(t) = \frac{t}{\mu^2} + \frac{1}{\mu^2}(\mu^2 - t)\ F(0) + \frac{t(t-\mu^2)}{\pi}\ P\int_{(3\mu^2)}^{\infty} \frac{\mathrm{Im}\ F(t')dt'}{t'(t'-\mu^2)(t'-t)} \qquad (24)$$

To apply equation (24) we must of course now use Δ_{GTR} (or F(0)) as an input to the calculation rather than as a test of its validity. While this second subtraction destroys one of the nice features of this essentially parameter free calculation, it does indicate to what extent the tail of Im F(t) and the omitted parts of the integral, contribute to Re F(t). As can be seen from Fig. 4(b), the form factor evaluated in this way is highly sensitive to the value of F(0) used in the second subtraction. The self consistency of the one subtracted curve giving F(0) = 0.980, and the twice subtracted curve where F(0) = 0.980 is used as an input, indicates that in fact no significant contribution is lost by having to cut off the integrations at $4\mu^2$.

The off shell modifications introduced by Durso, Jackson and VerWest were kinematic in nature. They were designed to ensure that terms corresponding to the spin-one projection of the ρ propagator,

$(g_{\mu\nu}+k_\mu k_\nu m^{-2})$, were retained in the dispersion approach, and that sufficient freedom was retained to accommodate γ_μ and $\sigma_{\mu\nu}$ couplings.

In the next section we examine some dynamical off shell corrections to the πN amplitude, which will introduce corrections to equation (12) for Im $F^{(+)}(t)$.

3. OFF MASS SHELL TERMS IN THE πN AMPLITUDE

While the kinematics associated with f_0^+, or σ meson exchange, in equation (12) do not vary in going off mass shell, the amplitude itself varies rapidly in order to satisfy the PCAC and current algebra constraints. This off shell behaviour has been studied by Coon et al. [9] who showed that, to order q^2, for the two pions off shell and the nucleons close to their mass shell, the non-nucleon pole contributions are given by

$$\bar{G}^{(+)}(\nu,t;q_1^2,q_2^2) = \bar{G}^{(+)}(\nu,t;-\mu^2,-\mu^2) - (q_1^2+q_2^2+2\mu^2)(g_1^+\mu^{-2}+g_2^+) \tag{25}$$

where the bars indicate that the nucleon pole term is subtracted out,

$$G^{(\pm)} = A^{\pm} + \nu B^{\pm}, \tag{26}$$

and ν is the usual CGLN parameter. $G^{(-)}$ and $B^{(\pm)}$ do not require extrapolation to this order since they are kinematically of order q^2 already. The parameters g_1^+ and g_1^+ are defined by the expansion in ν and t of the on shell arplitude

$$\bar{G}^{(+)}(\nu,t) = \bar{G}^{(+)}(\nu,t;-\mu^2,-\mu^2) = g_1^+ + g_2^+ t + g_3^+\nu^2 + g_4^+\nu^2 t + \ldots \tag{27}$$

In the evaluation of Im $F^{(+)}$, the Cutkosky rules require that the internal pion is on shell, and the resulting factor $\theta(t-(m+\mu)^2)$ restricts the nucleon line to about $5\mu^2$ off mass shell. Since we expect the off-nucleon mass shell terms to scale as $(p^2+m^2)M^{-2}$ with M a typical hadronic mass of order 1 GeV $\sim 50\mu^2$, neglecting the off mass shell terms for the nucleons should be an excellent approximation. Coon et al. estimated that (16) should be good at least up to $q^2 \sim 12\mu^2$.

In order to utilise the formalism developed in the previous section we need to develop the analogue of equations(16) appropriate to the helicity amplitude expansion. Noting that G depends on q_1^2 and q_2^2 implicitly through $t = -(q_1 - q_2)^2$ as well as explicitly, we can Taylor expand $\bar{G}^{(+)}(\nu,t;q_1^2,q_2^2)$ about the on mass shell point as

$$\bar{G}^{(+)}(\nu,t;q_1^2,q_2^2) = \bar{G}^{(+)}(\nu,t) + (q_1^2+q_2^2+2\mu^2)\left(\frac{-t}{2\mu^2}\right)\frac{\partial\bar{G}^{(+)}}{\partial t}(\nu,t) + (q_1^2+q_2^2+2\mu^2)c^{(+)}(\nu,t) \qquad (28)$$

where we have used $\left.\frac{\partial t}{\partial q_1^2}\right|_{q_1^2=q_2^2=-\mu^2} = \frac{-t}{2\mu^2}$

and the whole of the explicit dependence on $\bar{G}^{(+)}$ on q^2 has been included in the as yet undetermined function $c^{(+)}(\nu,t)$.

PCAC and current algebra supply two constraints on the off mass shell extrapolation

i) the Adler consistency condition

$$\bar{G}^{(+)}(0,\mu^2;0,-\mu^2) = \bar{G}^{(+)}(0,\mu^2;-\mu^2,0) = 0 \qquad (29)$$

which gives
$$\bar{G}^{(+)}(0,\mu^2) - \frac{\mu^2}{2}\left.\frac{\partial\bar{G}^{(+)}}{\partial t}\right|_{\substack{\nu=0\\ t=\mu^2}} + \mu^2 c^{(+)}(0,\mu^2) = 0 \qquad (30)$$

ii) the double soft pion limit

$$\bar{G}^{(+)}(0,0;0,0) = -\sigma f_\pi^{-2} \tag{31}$$

where σ is the πNN σ term, which gives

$$\bar{G}^{(+)}(0,0) + 2\mu^2 c^{(+)}(0,0) = -\sigma f_\pi^{-2} \tag{32}$$

Cheng and Dashen [10] have shown that

$$\bar{G}^{(+)}(0,2\mu^2) = \sigma f_\pi^{-2} \tag{33}$$

which with the approximation

$$\left.\frac{\partial\bar{G}^{(+)}(\nu,t)}{\partial t}\right|_{t=2\mu^2} = \frac{\partial\bar{G}^{(+)}(\nu,t)}{\partial t} \tag{34}$$

which may be justified in the small ν,t region from the expansion of equation (27), allows us to write

$$c^{(+)}(\nu,t) = \frac{-1}{\mu^2}\sigma f_\pi^{-2} + (1 + \frac{t}{2\mu^2})\frac{\partial\bar{G}^{(+)}}{\partial t}(\nu,t) \tag{35}$$

as a solution of equations (21) and (23).

Our final result is

$$\bar{G}^{(+)}(\nu,t;q_1^2,q_2^2) = \bar{G}^{(+)}(\nu,t) - (q_1^2+q_2^2+2\mu^2)\left\{\frac{1}{\mu^2}\sigma f_\pi^{-2} - \frac{\partial\bar{G}^{(+)}}{\partial t}(\nu,t)\right\} \tag{36}$$

which reduces to equation (25) when the small ν,t expansion of equation (27) is used.

Note that terms of order $q^2 t \frac{\partial F}{\partial t}$ cancel so that only terms of first order in q^2 remain.

One could alternatively try to derive an expression for $c^+(\nu,t)$ using the small t expansion of (27). However, this leads to confusion as to whether the f_2^+ appearing in the final result is really a constant or $\frac{\partial \bar{F}^{(+)}}{\partial t}$. The advantage of expanding the on-shell amplitude around the Cheng-Dashen point, is that we know that the σ term σf_π^{-2} is a constant independent of the momentum transfer for all t, and in this way, we have included any t-dependent parts in $\partial F/\partial t$. This is important, as we show later that terms in the scattering matrix which are constant in t give a zero contribution to ImF(t).

To summarize, the extrapolated relation (36) for $\bar{G}^{(+)}(\nu,t;q_1^2,q_2^2)$ satisfies to order q^2

i) The on-shell condition

$$\bar{G}^{(+)}(\nu,t;q_1^2 = -\mu^2,\ q_2^2 = -\mu^2) = \bar{G}^{(+)}(\nu,t)$$

ii) the Adler consistency condition

$$\bar{G}^{(+)}(0,\mu^2;\ 0,\ -\mu^2) = \bar{G}^{(+)}(0,\mu^2;\ -\mu^2,0) = 0$$

iii) the double soft pion limit :

$$\bar{G}^{(+)}(0,0;0,0) = -\sigma f_\pi^{-2}$$

4. THE OFF MASS SHELL CONTRIBUTION TO IM $F^{(+)}$(T)

Since to the approximation we are working at, $\bar{B}^{(+)} = 0$ and $\bar{A}^{(+)}_{Born} = 0$, we may identify $A^{(+)}(\nu,t;q_1^2,q_2^2)$ with $\bar{G}^{(+)}(\nu,t,q_1^2,q_2^2)$, and setting $q_1^2 = k^2$, $q_2^2 = -t$ we have

$$A^{(+)}(\nu,t';k^2,-t) = A^{(+)}(\nu,t') + \Delta A^{(+)} \tag{37}$$

where $\Delta A^{(+)}$ is the off mass shell correction term

$$\Delta A^{(+)} = (k^2-t+2\mu^2)\left\{-\mu^{-2}\sigma f_\pi^{-2} + \frac{\partial A^{(+)}}{\partial t'}(\nu,t')\right\} \tag{38}$$

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This contributes to Im $F^+(t)$ a term

$$\mathrm{Im}\Delta F^+(t) = \frac{4m}{t}\,\mathrm{Im}\int \frac{d^4k}{(2\pi)^4}\, k.r\, \Delta A^{(+)}(t')\, \frac{1}{k^2+\mu^2}\, \frac{1}{k^2-2k.p'} \qquad (39)$$

The constant term in (38) does not contribute to $\mathrm{Im}\Delta F^+(t)$, since it corresponds to the point interaction of figure 5a which has vanishing phase space when the intermediate meson is placed on mass shell by the Cutkosky prescription. This can be checked by expanding σf_π^{-2} as a Hilbert transform

$$\sigma f_\pi^{-2} = \lim_{a\to\infty}\left\{-\frac{1}{\pi^2}\,\sigma f_\pi^{-2}\, P\int_{-\infty}^{\infty}\frac{d\tau}{\tau-t'}\,\log\left|\frac{a+\tau}{a-\tau}\right|\right\} \qquad (40)$$

$$= \lim_{a\to\infty}\left\{-\frac{1}{\pi^2}\,\sigma f_\pi^{-2}\, P\int_{0}^{\infty}\frac{d\tau}{\tau-t'}\,\log\left|1+f_a(\tau,t')\right|\right\} \qquad (41)$$

where $$f_a(\tau,t') = \frac{4a(\tau-t')}{(a-\tau)(a-\tau+2t')} \qquad (42)$$

On applying the Cutkosky prescription to $\left(\dfrac{\sigma f_\pi^{-2}}{\tau+(r-k)^2}\right)$ and using $t' = -(r-k)^2$ this gives a term

$$\log\left|1+f_a(\tau,-(r-k)^2)\right|\delta^+(\tau+(r-k)^2)$$

$$= \log\left|1+f_a(\tau,\tau)\right|\delta^+(\tau+(r-k)^2)$$

$$= \log 1\;\delta^+(\tau+(r-k)^2) = 0 \qquad (43)$$

so that any term constant with respect to t' gives a zero contribution to ImF(t) (Figure 5).

To evaluate the remaining term, the dispersion relation for $A^{(+)}$, retaining only the s-wave helicity amplitude

$$A^{(+)}(t') = \frac{16}{4m^2-t'}\int_{4\mu^2}^{\infty} d\tau\,\frac{\mathrm{Im}f_0^+(\tau)}{\tau-t'} \qquad (44)$$

gives

$$\Delta A^{(+)}(t') = (k^2-t+2\mu^2)\left[\frac{16}{(4m^2-t')^2}\int_{4\mu^2}^{\infty} d\tau\, \frac{\mathrm{Im} f_o^+(\tau)}{\tau-t'}\right.$$

$$\left. + \frac{16}{4m^2-t'}\int_{4\mu^2}^{\infty} d\tau\, \mathrm{Im} f_o^+(\tau)\,\frac{\partial}{\partial t'}\left(\frac{1}{\tau-t'}\right)\right] \tag{45}$$

where the term in σf_π^{-2} has been omitted as it does not contribute to the final answer. We can then write

$$\mathrm{Im}\, F^{(+)}(t) = \mathrm{Im}\, F_o^{(+)}(t) + \mathrm{Im}\, \Delta F^{(+)}(t) \tag{46}$$

where $\mathrm{Im}\, F_o^{(+)}(t)$ is the on mass shell term of equation (20) and $\mathrm{Im}\, \Delta F^{(+)}(t)$ is the off mass shell correction term given by equation (39).

$\Delta A^{(+)}(t')$ itself can be separated into the two terms of equation (45)

$$\Delta A^{(+)}(t') = \Delta A_1^{(+)}(t') + \Delta A_2^{(+)}(t') \tag{47}$$

The contribution due to $\Delta A_1^{(+)}(t')$ (the first term) is straight forward to compute and gives an added term to Im F(t) of

$$\mathrm{Im}\, \Delta F_1^{(+)}(t) = \frac{1}{2\pi}\int_{4\mu^2}^{\infty} d\tau\, \Theta\,(t-(\sqrt{\tau}+m)^2)\,\frac{\mathrm{Im}\, f_+^o(\tau)(\mu^2-t)(t+\mu^2-\tau)}{t^{3/2}\quad 4\quad (m^2-\frac{1}{4}\tau)^2}\,\Lambda(t,\tau) \tag{48}$$

The contribution due to $\Delta A_2^{(+)}(t')$ is more difficult to evaluate since it involves the derivative of a delta function, and we now calculate this in more detail.

We wish to calculate

$$\mathrm{Im}\, \Delta F_2^+(t) = \frac{4m}{t}\,\mathrm{Im}\int_{\infty}\frac{d^4k}{(2\pi)^4}\, k.r\,(k^2-t+2\mu^2)\,\frac{16}{4m^2-t'}\, \times$$

$$\times \int_{4\mu^2} d\tau\, \mathrm{Im}\, f_+^o(\tau)\frac{\partial}{\partial t'}\left(\frac{1}{\tau-t'}\right)\frac{1}{k^2+\mu^2}\,\frac{1}{k^2-2k.p'} \tag{49}$$

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[In the following derivation, to keep expressions as brief as possible, we use the notation

$$f_x = \frac{\partial f}{\partial x} \text{ and } f_x(x_o) = \left.\frac{\partial f}{\partial x}\right|_{x = x_o}$$]

In order to utilize the Cutkosky rules as before, we note that

$$\frac{\partial}{\partial t'}\left(\frac{1}{\tau - t'}\right) = \frac{\partial}{\partial t'}\left(\frac{P}{\tau - t'}\right) \pm i\pi\delta_{t'}(\tau - t') \tag{50}$$

and as we are only interested in the discontinuity across the cut, we use the replacement

$$k^2 + \mu^2 \rightarrow 2\pi i\delta^+(k^2+\mu^2) \text{ and } \frac{\partial}{\partial t'}\left(\frac{1}{\tau - t'}\right) \rightarrow 2\pi i\delta^+_{t'}(\tau - t')$$

the derivative of a delta function being defined by

$$\int_{-\infty}^{\infty} \delta_x(x)\ f(x)dx = -f_x(0) \tag{51}$$

Changing the order of integration and applying the Cutkosky rules using the above prescription gives

$$\text{Im } \Delta F_2^+(t) = \frac{32m}{t}\int d\tau \text{ Im } f_+^o(\tau)\ \mathcal{F}_{c2}(\tau, t) \tag{52}$$

where

$$\mathcal{F}_{c2}(\tau,t) = \frac{1}{(2\pi)^2}\int d^4k\ \frac{k.r(k^2-t+2\mu^2)\delta^+_{t'}(\tau-t')\delta^+(k^2+\mu^2)}{(4m^2-t')(k^2-2k.p')}$$

and using $t' = -(r-k)^2$

$$= \frac{(t-\mu^2)}{(2\pi)^2}\int d^4k\ k.r\ \frac{\delta^+_{t'}(\tau+(r-k)^2)\delta^+(k^2+\mu^2)}{(4m^2+(r-k)^2)(-\mu^2-2k.p')} \tag{53}$$

We now do a number of changes of variable

$$x = 2k_o\sqrt{t}$$

$$\phi = 2\sqrt{t}\ \sqrt{k^2+\mu^2}$$

where $d^4k = dk_o k^2 dk d\Omega_k$

and note that the derivative of the delta function has to be handled with some care for

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$$\delta_x(x-y) = - \delta_y(x-y)$$

so that

$$\delta^+_{t'}(\tau+(r-k)^2)\delta^+(k^2+\mu^2) = - \delta^+_\alpha(\tau-t+\alpha-\mu^2)\delta^+(k^2+\mu^2) \tag{54}$$

Using the same co-ordinate frame as in section 2. and noting that

$$\delta^+ (k^2+\mu^2) = 2t\ \delta^+(\alpha - \phi) \tag{55}$$

we have

$$\mathcal{F}_{c2}\ (\tau,t)$$

$$= \frac{-(t-\mu^2)}{(2\pi)^2 8t} \int d\Omega_k d\phi\sqrt{\phi^2-4t\mu^2} \int d\alpha\ \frac{\alpha\delta^+_\alpha(\alpha-(\mu^2+t-\tau))\delta^+(\alpha-\phi)}{(\alpha+4m^2-t-\mu^2)(2\mu^2-\alpha+\frac{2}{\sqrt{t}}\sqrt{\phi^2-4t\mu^2}\ \hat{k}.\bar{p})} \tag{56}$$

This integral (except for the terms outside the integration sign and the integration over $d\Omega_k$) is now of the form

$$\mathcal{G}(x_o) = \int dy\ h(y) \int dx g(x) f(x,y)\ \delta_x(x-x_o)\delta(x-y) \tag{57}$$

with

$$y \leftrightarrow \phi$$

$$x \leftrightarrow \alpha$$

$$h(y) \leftrightarrow \sqrt{\phi^2-4t\mu^2}$$

$$g(x) \leftrightarrow \frac{\alpha}{2+4m^2-t-\mu^2}$$

$$f(x,y) \leftrightarrow \frac{1}{(2\mu^2-\alpha+\frac{2}{\sqrt{t}}\sqrt{\phi^2-4t\mu^2}\ \hat{k}.\bar{p})} \tag{58}$$

It is easy to calculate (59) in this general case using integration by parts and we find that

$$\mathcal{G}(x_o) = g(x_o)\ [h_y(x_o)f(x_o,x_o) + h(x_o)f_y(x_o,x_o)]$$

$$-h(x_o)\ [g_x(x_o)f(x_o,x_o) + g(x_o)f_x(x_o,x_o)] \tag{59}$$

Using this result and doing the appropriate substitutions we have

$$\mathcal{F}_{c2}(\tau,t)$$

$$= \frac{-(t-\mu^2)}{(2\pi)^2 16 t^{3/2} Q(4m^2-\tau)} \int d\Omega \{ (\mu^2+t-\tau)^2 \; B(t,\tau) - 4tQ^2 \; C(t,\tau) \} \qquad (60)$$

where $B(t,\tau) = \dfrac{1}{\mu^2-t+\tau+4Q\hat{k}.\bar{p}} \left[1 - \dfrac{4Q\hat{k}.\bar{p}'}{\mu^2-t+\tau+4Q\hat{k}.\bar{p}'} \right]$

and $C(t,\tau) = \dfrac{1}{\mu^2-t+\tau+4Q\hat{k}.\bar{p}} \left[\dfrac{(\mu^2+t-\tau)}{(\mu^2-t+\tau)+4Q\hat{k}.\bar{p}} + \dfrac{4m^2-t-\mu^2}{(4m^2-\tau)} \right]$ (61)

Doing the angular integration $\int d\Omega_k = 2\pi \int dz$ and using $\hat{k}.\bar{p} = ikz$ gives when we substitute into equation (32) for $\mathcal{F}_{c2}(\tau,t)$

$$\mathrm{Im}\,\Delta F^{+}(t) = \mathrm{Im}\,\Delta F_1^{+}(t) + \mathrm{Im}\,\Delta F_2^{+}(t) \qquad (62)$$

$$= \frac{1}{2\pi} \int d\tau \; \mathrm{Im}\, f_{+}^{o}(\tau) \; \Theta\,(t-(\sqrt{\tau}+\mu)^2) \; \frac{2(\mu^2-t)\; m}{t^{5/2}\; Q(4m^2-\tau)}$$

$$\left\{ \frac{(\mu^2+t-\tau)}{2(4m^2Q^2+\tau\mu^2)} \; (P(t,\tau)-4tQ^2) + 2tQA(t,\tau) \right\} \qquad (63)$$

where $P(t,\tau) = (\mu^2+t-\tau)(\mu^2-t+\tau)$.

Note that a term arises in $\mathrm{Im}\,\Delta F_2^{+}(t)$ which exactly cancels the whole of the contribution $\mathrm{Im}\,\Delta F_1^{+}(t)$ which is a product of the t' dependence of the purely kinematical term $\dfrac{1}{4m^2-t'}$ in equation (44).

Figure 6 shows the results of evaluating $\mathrm{Im}\,F^{+}(t)$ and ReF(t) using equations (20), (23), (46) and (63), and the helicity amplitudes of reference 11. The increased contribution by the dynamic off shell parts of the πN scattering matrix thus increases the value of $\Delta_{GTR}^{theo.}$ to .029 with .015 coming from the $\sigma\pi$ vertex and .014 from the $\rho\pi$ vertex. The contribution from the $\sigma\pi$ vertex particularly, is consistent with the results of Jones and Scadron and brings the value of Δ_{GTR} calculated from a dispersion relation approach into better agreement with that calculated from a field theory approach, showing that, provided off-shell effects are handled correctly the two methods are equivalent.

5. CONCLUSION

We have found that the inclusion of the off pion mass shell terms in the $\pi N \to \pi N$ scattering amplitudes alters the dispersion relation calculation of the πNN form factor significantly. Expressed in terms of the Goldberger - Treiman discrepancy, our results change the dispersion theoretic value from 0.02 to 0.03, bringing it into agreement with the field theory value of Jones and Scadron.

We would also draw attention to the enhancement we find in Im F(t) at $t - 42\mu^2$. Because of the long distance from on-shell of the virtual pion (and hence once of the pions in the πN scattering amplitudes) it is not clear what the origin of this enhancement is, (whether physical or pathological in nature) but it may repay further study.

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CAPTIONS:

Fig. 1.

t-channel unitarity diagrams for F(t).

Fig. 2.

Contribution to the three pion cut term represented by the $N\bar{N} \to \pi\pi$ annihilation process.

Fig. 3.

Kinematics of $\pi N \to \pi N$ scattering for the

a) general case

b) diagram contributing to ImF(t). Note that $s' + u' + t' \neq 2m^2 + 2\mu^2$ since three of the external lines are off shell.

Fig. 4.

a) Plot of ImF(t) evaluated using equations (20) to (23) and the data of Höhler and Pietarienen [11]. The vector part of ImF(t) is truncated at $t = 75\mu^2$ to simulate cancellation with the $\rho\pi$ propagator term.

b) Solid line: Re F(t) evaluated using the once subtracted dispersion relation of equation (10). This gives F(0) = 0.980 or $\Delta_{GTR} = 0.020$.

Broken lines: Re F(t) evaluated using the twice subtracted dispersion relations of equation (24), with

i) F(0) = 0.990 Δ_{GTR} = .010

ii) F(0) = 0.980 Δ_{GTR} = .020

iii) F(0) = 0.970 Δ_{GTR} = .030

Fig. 5.

Contribution to ImF(t) due to terms in $T_{\beta\alpha}$

a) independent of t' with zero imaginary part

b) which can be expressed as dispersion integrals over t'.

CAPTIONS: (Contd).

Fig. 6.

a) Plot of $ImF^+(t) = ImF_o^+(t) + Im\,\Delta F_o^+(t)$ evaluated using equations (20) and (63) and one data of Höhler and Pietarienen [11]. The inclusion of dynamic off-shell effects in the s-wave scattering has a sizeable effect on $ImF^+(t)$.

b) Plot of Re F(t) evaluated using a once subtracted dispersion relation with all off-shell effects included in ImF(t). The value of Δ_{GTR} is increased to .029.

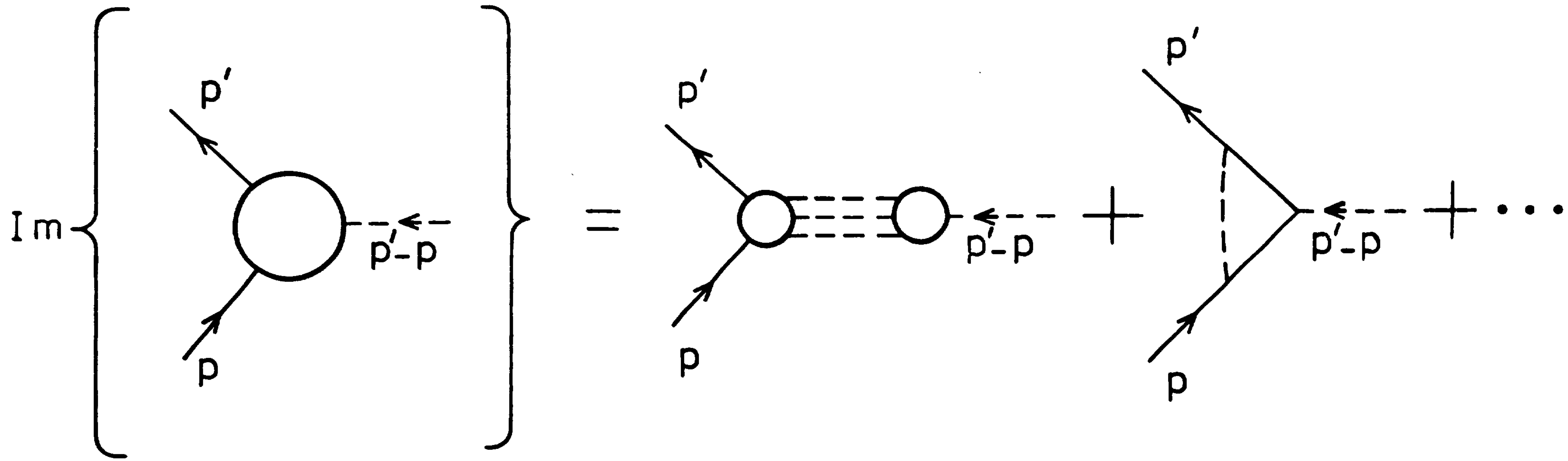


FIG. 1

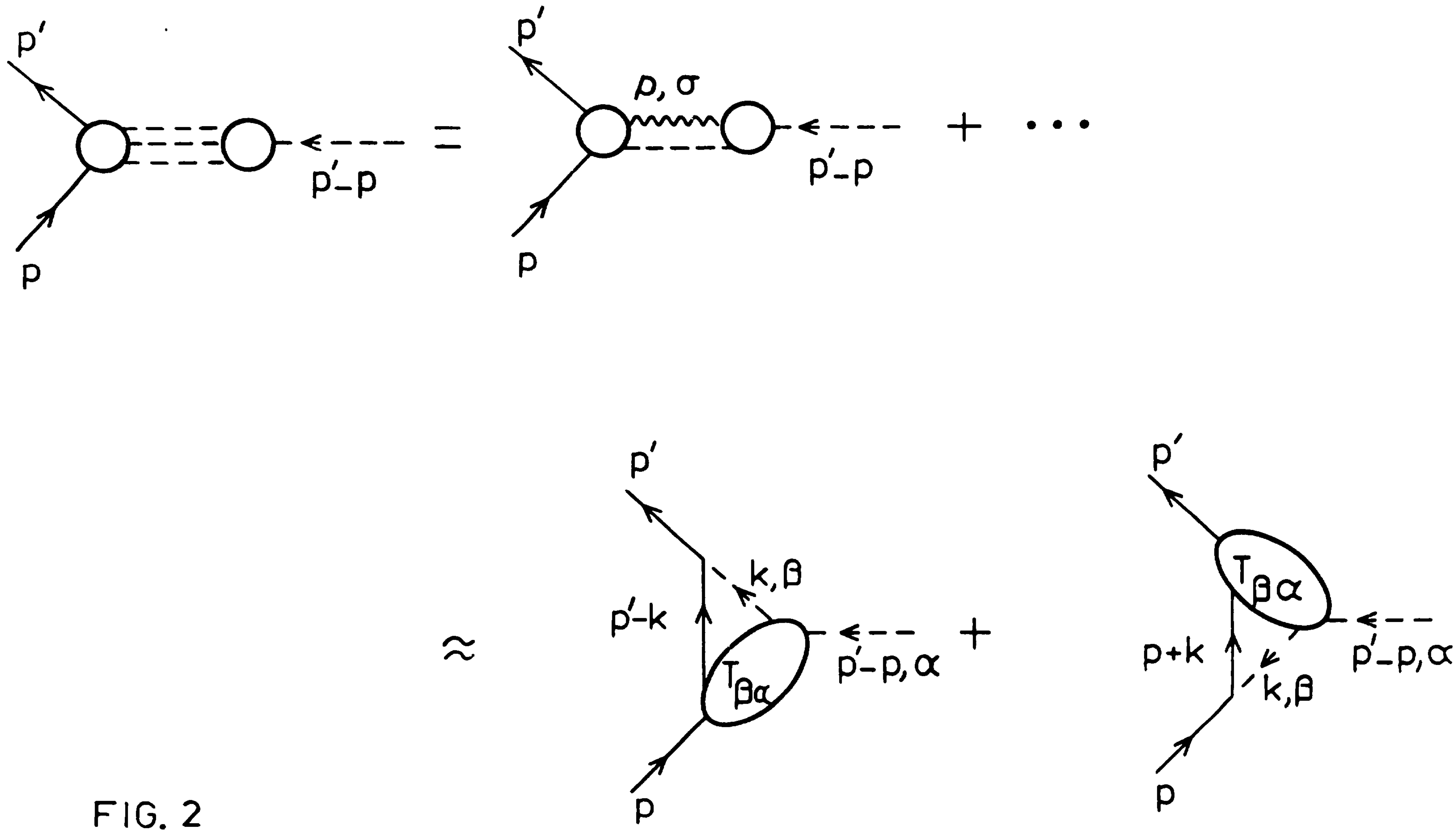
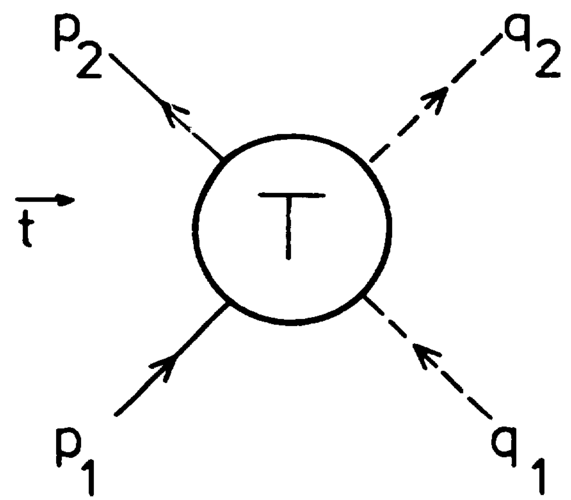


FIG. 2

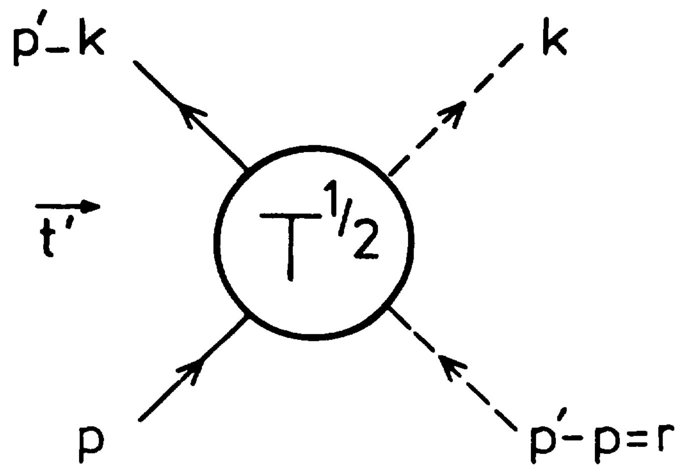


a) $s = -(p_1 + q_1)^2$

$t = -(p_1 - p_2)^2$

$u = -(p_1 - q_2)^2$

$\nu = \frac{s-u}{4m}$



b) $s' = m^2$

$t' = t + 2k \cdot r - k^2$

$u' = m^2 + 2k \cdot p - k^2$

$\nu' = \frac{1}{4m}(k^2 - 2k \cdot p)$

FIG. 3

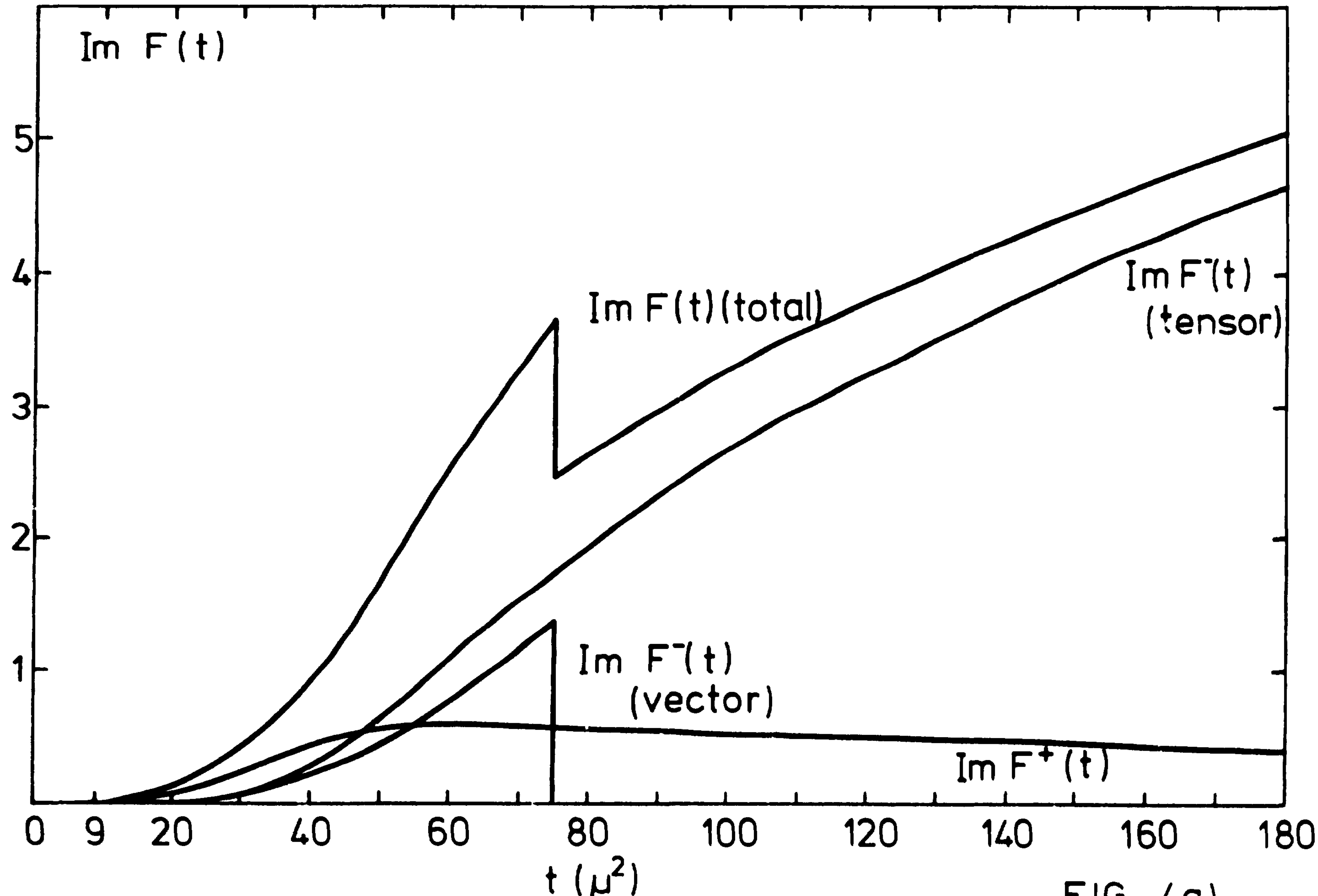

Im F(t)
5
4
3
2
1
Im F(t)(total)
Im F⁻(t)
(tensor)
Im F⁻(t)
(vector)
Im F⁺(t)
0
9
20
40
60
80
100
120
140
160
180
t (μ²)

FIG 4a)

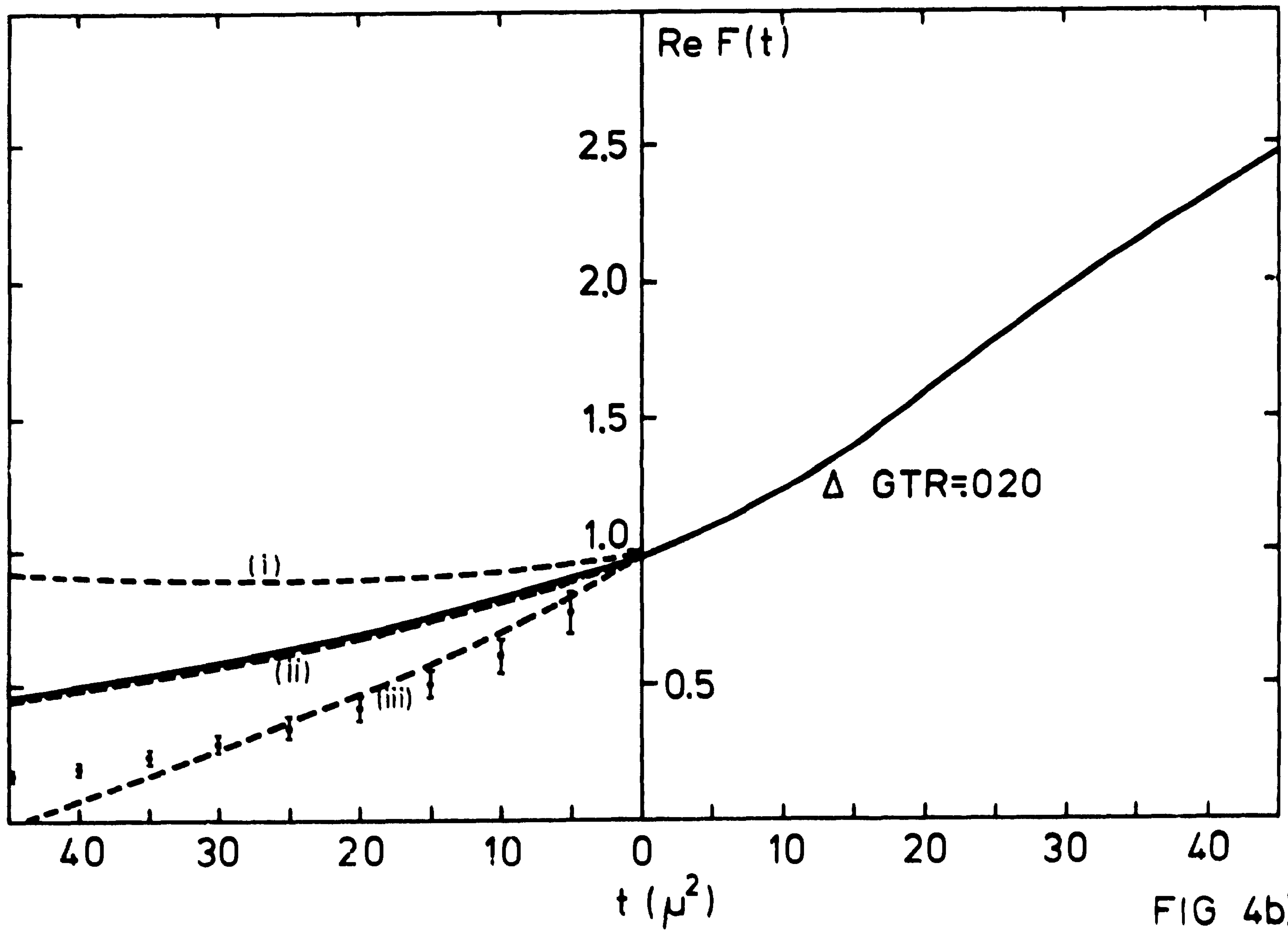


FIG 4b)

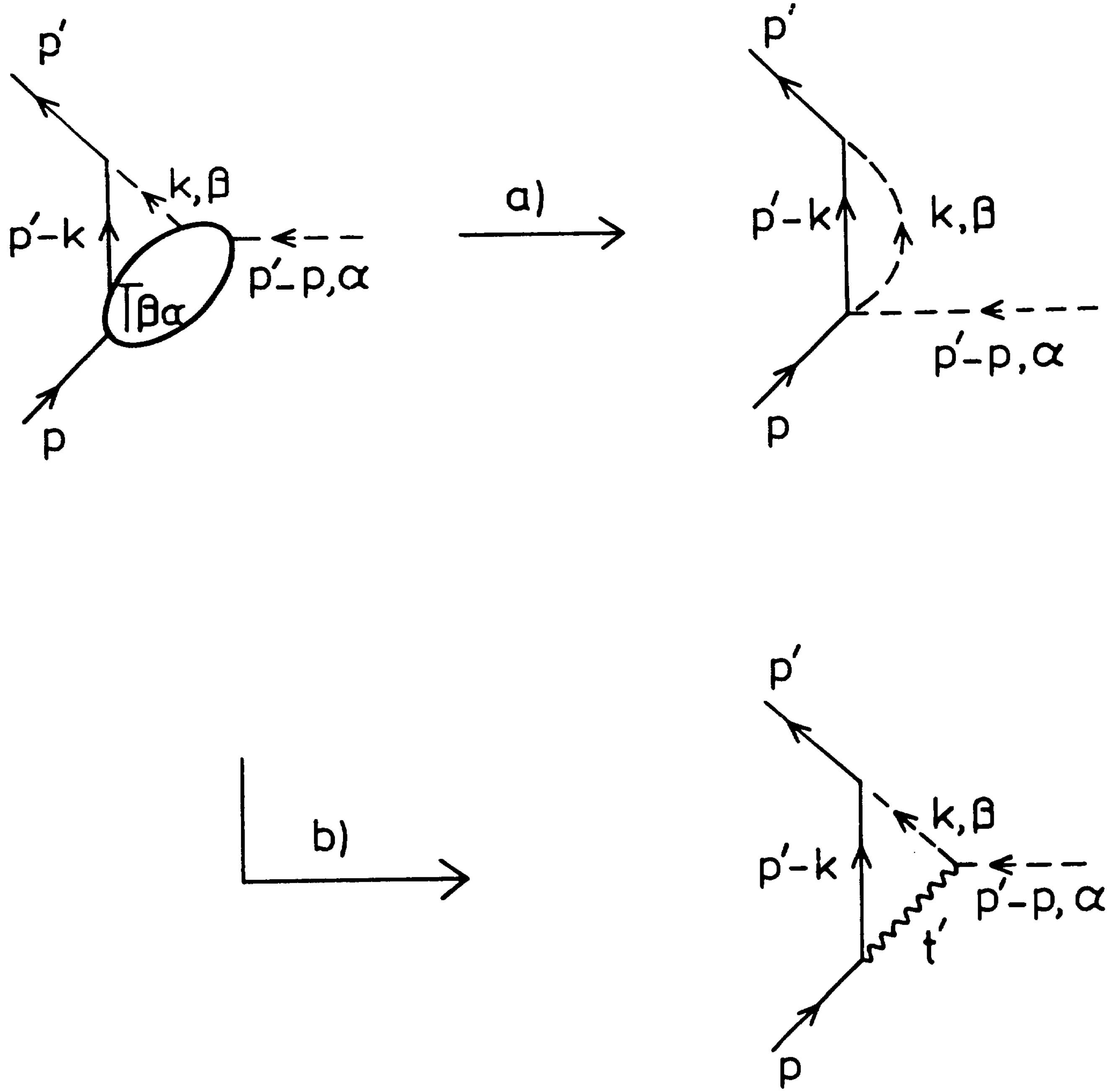


FIG 5.

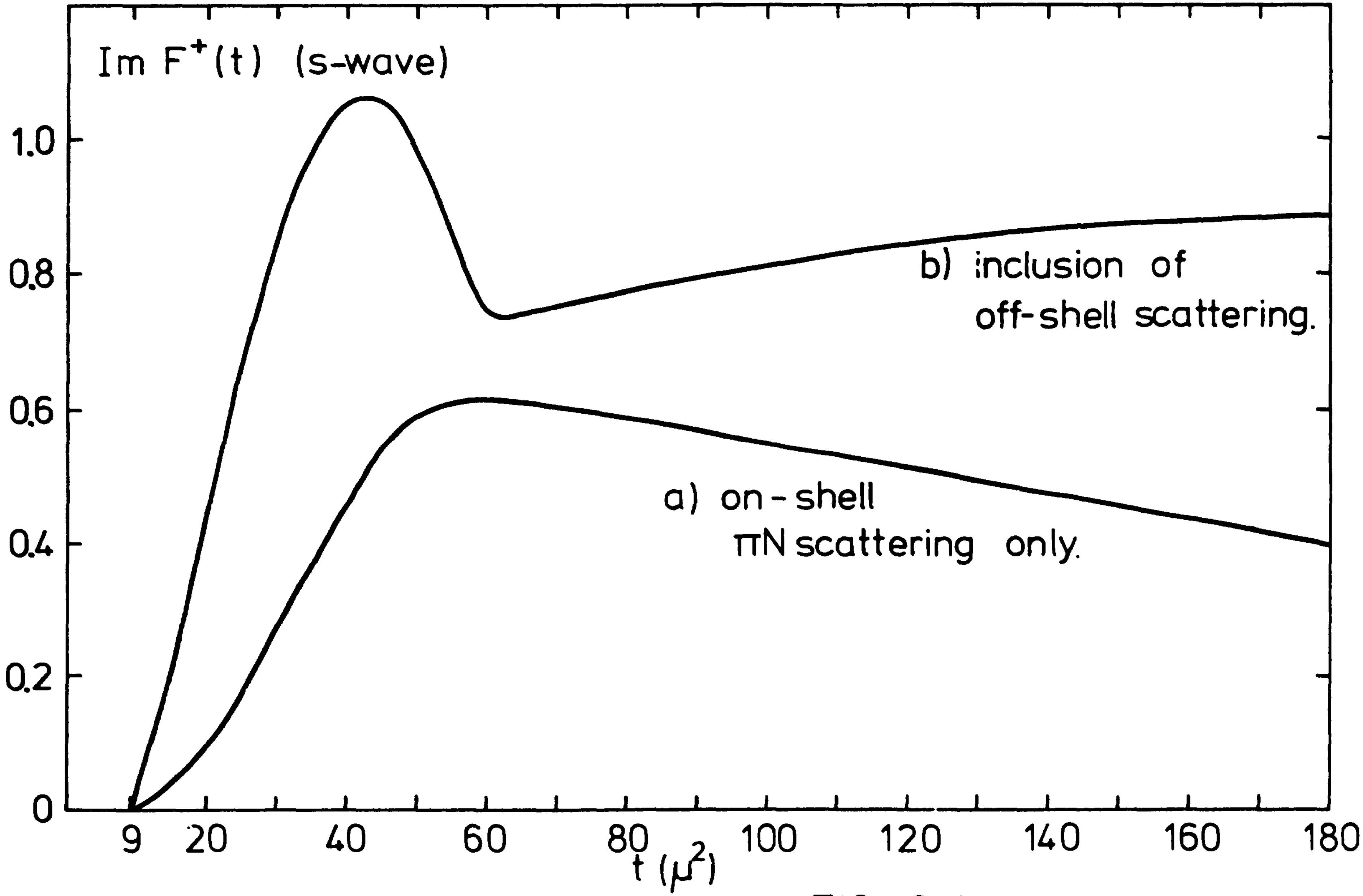

Im F+(t) (s-wave)
1.0
0.8
0.6
0.4
0.2
0
b) inclusion of
off-shell scattering.
a) on-shell
πN scattering only.
9
20
40
60
80
100
120
140
160
180
t (μ²)

FIG 6a)

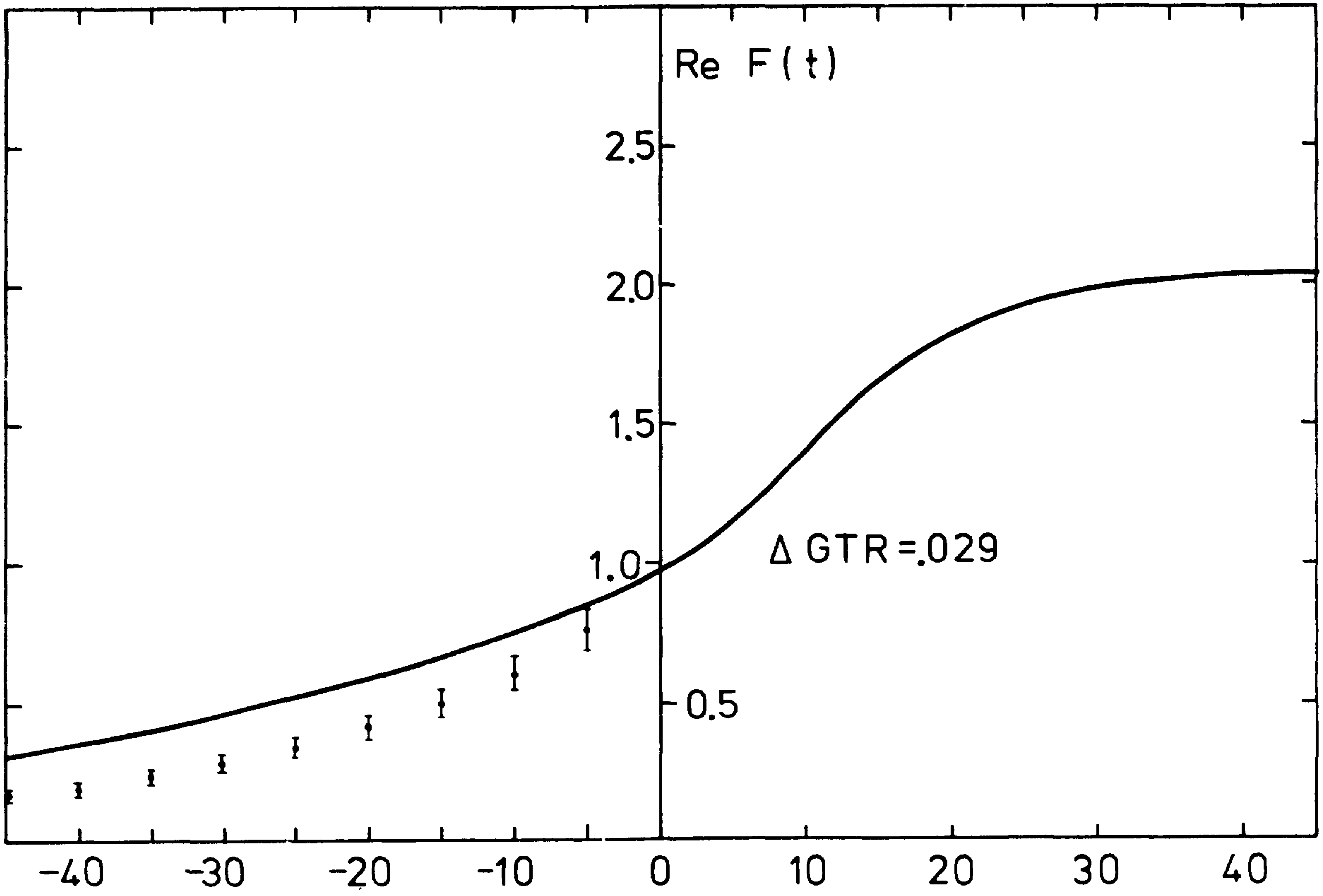


FIG (6b)