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ON LIEB'S ENTROPY CONJECTURE

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Abstract. *The reformulation of Lieb's entropy conjecture, in the frame of the harmonic analysis on the SO(3) group, makes it evident that the exact value of the classical entropy of a pure quantum state, which belongs to the Hilbert space H_J of a (2J+1)-dimensional, unitary, irreducible representation U_J of the SO(3) group, depends only on the parameters which characterize the orbits of U_J in H_J. In the case $J = 1$ we give the exact analytic dependence of the classical entropy of a quantum state on the parameter which characterizes the orbits and as a consequence we obtain a verification of Lieb's entropy conjecture. We verify this conjecture also for any value of J for the states of the canonical basis of H_J. A natural generalization of Lieb's entropy conjecture, which is a new "phenomenon" in the harmonic analysis on SO(3), is discussed in the case $J = 1$.*

1. INTRODUCTION

The present paper is devoted to a verification of Lieb's entropy conjecture /1, section 3/ in some particular cases. From the beginning we point out the connection of this problem with the harmonic analysis on the group SO(3) of rotations in three dimensions. Let $U_J(g)$, ($g \in SO(3)$), be a unitary, irreducible representation of SO(3) in the (2J+1)-dimensional Hilbert space H_J, where $J=1/2,1,3/2,\ldots,$ and let $\{v_m\}$, $m=-J,-J+1,\ldots,0,\ldots,J-1,J$, be the canonical basis in H_J. We denote by $\|\cdot\|$ the norm in H_J and suppose that $\|v_m\| = 1$ for any value of m. The matrix elements of the representation U_J in this canonical basis are denoted by :

$$t^J_{mn}(g) = (v_m, U_J(g)v_n) = e^{-i(m\varphi + n\psi)}\, t^J_{mn}(\theta) \qquad (1.1)$$

where (φ,ϑ,ψ) are the Euler angles which define the rotation $g \in SO(3)$, and

$$t^J_{mn}(\theta) = P^J_{mn}(\theta) \qquad (1.2)$$

where

$$P^J_{mn}(\cos\theta) = i^{-m-n}\left(\frac{(J-m)!(J-n)!}{(J+m)!(J+n)!}\right)^{\frac{1}{2}} \left(\operatorname{ctg}\frac{\theta}{2}\right)^{m+n} \sum_{k=\max(m,n)}^{J} \frac{(-1)^k (1+k)!}{(1-k)!(k-m)!(k-n)!}\left(\sin\frac{\theta}{2}\right)^{2k} \qquad (1.3)$$

Because the unitary, irreducible representations of the compact groups are square integrable we have in the particular case of the SO(3) group, for any $u,v \in H_J$:

$$\frac{2J+1}{8\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi} |(u,U_J(g)v)|^2 \sin\theta \, d\theta d\varphi d\psi = \|u\|^2 \|v\|^2 \qquad (1.4)$$

and from this, for any $u \in H_J$, we obtain :

$$\frac{2J+1}{4\pi}\int_0^{2\pi}\int_0^{\pi} |(u,U_J(g)v_{\pm J})|^2 \sin\theta \, d\theta d\varphi = \|u\|^2 \qquad (1.5)$$

Lieb's entropy conjecture takes in these notations the following form :

$$-\frac{d}{dp}\left(\frac{2J+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left|\left(\frac{u}{\|u\|},U_J(g)v_{\pm J}\right)\right|^{2p}\sin\theta\, d\theta\, d\varphi\right)\Big|_{p=1} \geqslant \frac{2J}{2J+1} \quad (1.6)$$

where the equality is attained only for Bloch coherent states :

$$u = U_J(g)v_{\pm J} \quad (1.7)$$

for any $g \in SO(3)$. In fact this conjecture may be considered as a consequence of the following conjecture :

$$\frac{2J+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left|(u,U_J(g)v_{\pm J})\right|^{2p}\sin\theta\, d\theta\, d\varphi \leqslant \frac{2J+1}{2pJ+1}\|u\|^{2p} \quad (1.8)$$

where , when $p \geqslant 1$, the equality is attained only for Bloch coherent states (1.7), and when $p = 1$,for any $u \in H_J$. This last conjecture is in fact a sharp estimation of the $L^{2p}(S^2)$ - norm of the matrix coefficients $(u,U_J(g)v_{\pm J})$:

$$\left(\frac{2J+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left|(u,U_J(g)v_{\pm J})\right|^{2p}\sin\theta\, d\theta\, d\varphi\right)^{1/2p} \leqslant \left(\frac{2J+1}{2pJ+1}\right)^{1/2p}\|u\| \quad (1.9)$$

A result of this kind is unknown in the harmonic analysis on the SO(3) group . For the Heisenberg group such a result was proved in /3/. In section 2 we obtain the exact value of the classical entropy of a quantum state /1,4/ and as a consequence we verify (1.6) for $J = 1$. In section 3 we obtain the exact value of the left hand side of (1.8) and prove (1.6) and(1.8) for any value of J, for the states of the canonical basis : $u = v_m$, $m = -J,-J+1,\dots,0,\dots,J-1,J$ .In section 4 we discuss the conjecture (1.8) for $J = 1$.

2. THE EXACT VALUE OF THE CLASSICAL ENTROPY OF A QUANTUM STATE FOR J=1 .

The essential property of the integral :

$$I_p^J(O_u) = \frac{2J+1}{4\pi} \int_0^{2\pi}\int_0^{\pi} \left| \left(\frac{u}{\|u\|}, U_J(g)v_{\pm J} \right) \right|^{2p} \sin\theta \, d\theta d\varphi \qquad (2.1)$$

which we shall exploit in the following, is the fact that it depends only on the orbit O_u of U_J in H_J to which belongs the vector u. From this property it follows that it is sufficient to calculate the integral $I_p^J(O_u)$ only for one representant from each orbit O_u.; this representant may be chosen to be the most simple one. The space H_1 splits /5/ into three strata (union of orbits with the same stabilizer up to conjugacy) which are characterized by a real valued parameter $a \in [0,1]$ which is defined for any vector $u = c_{-1}v_{-1}+c_0v_0+c_1v_1 \in H_1$ in the following way:

$$a(u) = \left|c_0^2 - 2c_{-1}c_1\right| \,/\, \left(|c_{-1}|^2+|c_0|^2+|c_1|^2\right) \qquad (2.2)$$

A typical vector u of a stratum , which is characterized by a given value a of this parameter , is of the following form :

$$u = \|u\| \left(\sqrt{1-a}\; v_{-1} + \sqrt{a}\; v_0\right) \qquad (2.3)$$

The stratum for which a =o contains one two-dimensional orbit $O_0=O_{v_1}=O_{v_{-1}}$, which is the orbit of Bloch coherent states . The stratum with $a \in (0,1)$ is a continuous set of three-dimensional orbits O_a, one for each value of the parameter a . The stratum for which a = 1 contains one two-dimensional orbit $O_1 = O_{v_0}$. We shall obtain the classical entropy /1,4/ of a pure quantum state

$u/\|u\| = (\sqrt{1-a}\, v_{-1} + \sqrt{a}\, v_o)$, defined by :

$$S^{cl}\left(\frac{u}{\|u\|}\right) = -\frac{d}{dp} I^1_p(O_a) \Big|_{p=1} \qquad (2.4)$$

as a function of $a \in [o,1]$. With the notation $x = \cos\theta$ we have :

$$I^1_p(O_a) = \frac{3}{4\pi} \int_{-1}^{1} dx \int_{o}^{2\pi} d\varphi \Big[(1-a)\left(\frac{1-x}{2}\right)^2 + 2a\left(\frac{1-x}{2}\right)\left(\frac{1+x}{2}\right) + 2\left(2a(1-a)\left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right)^3\right)^{1/2} \cos\left(\varphi - \frac{\pi}{2}\right)\Big] \qquad (2.5)$$

For $a = o$ and $a = 1$ we obtain :

$$I^1_p(O_o) = \frac{3}{2p+1} \qquad (2.6)$$

and

$$I^1_p(O_1) = \frac{3}{2p+1} \frac{2^p \Gamma(p+1)^2}{\Gamma(2p+1)} \qquad (2.7)$$

respectively. For each $a \in (o,1)$ we split the integral with respect to x in two pieces : one from -1 to $(1-3a)/(1+a)$ and the other from $(1-3a)/(1+a)$ to 1. Further we change the variable in the first integral into $t = 2a(1+x)/(1-a)(1-x)$ and in the second integral into $t = (1-a)(1-x)/2a(1+x)$. Then after integration with respect to φ and use of the formula /6/ :

$$F(-p,-p;1;t) = \frac{1}{2\pi} \int_{o}^{2\pi} (1+2t\cos(\varphi+\alpha) + t^2)^p \, d\varphi \qquad (2.8)$$

we obtain

$$I^1_p(O_a) = \frac{3}{2}\Big[\frac{(1-a)^{p+1}}{2a} \int_{o}^{1} dt \left(\frac{1-a}{2a}t+1\right)^{-2(p+1)} F(-p,-p;1;t) + \frac{(2a)^{2p+1}}{(1-a)^{p+1}} \int_{o}^{1} dt \left(1+\frac{2a}{1-a}t\right)^{-2(p+1)} t^p F(-p,-p;1;t)\Big] \qquad (2.9)$$

From the fact that the integrands which appear in (2.9) are free of singularities for $t \in [0,1]$, $a \in (0,1)$ and $p \geq 1$, it follows that $I_p^1(O_a)$ is a differentiable function of a and p for $a \in (0,1)$ and $p \geq 1$. We shall calculate the classical entropy (2.4) using the representation (2.9) for $I_p^1(O_a)$. From the fact that

$$F(-p,-p;1;t) = 1 + p^2 t + \left(\frac{p(p-1)}{2!}\right)^2 t^2 + \left(\frac{p(p-1)(p-2)}{3!}\right)^2 t^3 + \ldots \tag{2.10}$$

and because this series is absolutely converging for all $t \in [0,1]$ we obtain :

$$\frac{d}{dp} F(-p,-p;1;t) \Big|_{p=1} = 2t \tag{2.11}$$

for all $t \in [0,1]$. After tedious calculations we obtain the following simple expression for the classical entropy of a pure quantum state $u/\|u\| \in O_a$:

$$S^{cl}(a) = \frac{2}{3} + (a - \ln(1 + a)) \tag{2.12}$$

for $a \in (0,1)$. When $a = 0$ or $a = 1$ we obtain directly from (2.6) and (2.7)

$$S^{cl}(0) = \frac{2}{3} \tag{2.13}$$

and

$$S^{cl}(1) = \frac{2}{3} + (1 - \ln 2) \tag{2.14}$$

respectively . We remark that (2.13) and (2.14) are particular cases of (2.12) for $a = 0$ and $a = 1$ respectively. Relation (2.12) is thus valid for all $a \in [0,1]$. Now Lieb's entropy conjecture for $J = 1$:

$$S^{cl}(a) \geq \frac{2}{3} \tag{2.15}$$

where the equality is attained only for Bloch coherent states $u \in O_o$, is a simple consequence of (2.12),(2.13) and of the well known inequality :

$$a - \ln(1 + a) \geq o \qquad (2.16)$$

valid for all $a \geq o$. From (2.12) it is obvious that the classical entropy attains its maximum value for $a = 1$ i.e. for $u/\|u\| \in O_{v_o}$.

3. THE VERIFICATION OF LIEB'S CONJECTURE AND OF ITS GENERALIZATION FOR ANY VALUE OF J FOR THE CANONICAL BASIS .

We shall calculate the exact value of the integrals $I_p^J(O_{v_m})$ for $m = -J,-J+1,\dots,o,\dots,J-1,J$ where $J = 1/2,1,3/2,2,\dots\dots$ and $p \geq 1$. In this case we have

$$I_p^J(O_{v_m}) = \frac{2J+1}{2}\int_{-1}^{1}\left|P^J_{m,-J}(x)\right|^{2p}dx = \frac{2J+1}{2}\int_{-1}^{1}\left|P^J_{m,J}(x)\right|^{2p}dx \qquad (3.1)$$

and obtain

$$I_p^J(O_{v_m}) = \frac{2J+1}{2pJ+1}\left(\frac{(2J)!}{(J+m)!(J-m)!}\right)^p \frac{\Gamma(p(J-m)+1)\Gamma(p(J+m)+1)}{\Gamma(2pJ+1)} \qquad (3.2)$$

From this formula it is obvious that $I_p^J(O_{v_m}) = I_p^J(O_{v_{-m}})$ for all values of m .The classical entropy of a pure quantum state v_m , $m = -J,-J+1,\dots,o,\dots,J-1,J$ is then given by :

$$S^{cl}(v_m) = (J+m)\left(\frac{1}{J+m+1} + \frac{1}{J+m+2} + \ldots + \frac{1}{2J}\right) + (J-m)\left(\frac{1}{J-m+1} + \frac{1}{J-m+2} + \ldots + \frac{1}{2J}\right) - \ln\left(\frac{(2J)!}{(J+m)!\,(J-m)!}\right) + \frac{2J}{2J+1} \qquad (3.3)$$

where

$$S^{cl}(v_{-J}) = S^{cl}(v_J) = \frac{2J}{2J+1} \qquad (3.4)$$

Lieb's entropy conjecture is then equivalent with the following inequality in which we have used the notations $k = J+m$, $j = J-m$:

$$k\left(\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{k+j}\right) + j\left(\frac{1}{j+1} + \frac{1}{j+2} + \ldots + \frac{1}{j+k}\right) \geq \ln\left(\frac{(k+j)!}{k!\,j!}\right) \qquad (3.5)$$

For $k = 1$ we obtain a known /7/ inequality

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{j} \geq \ln(j+1) \qquad (3.6)$$

valid for any nonnegative integer j. This inequality [illegible] proved by induction , using the well known inequality

$$\frac{1}{k+1} > \ln\left(\frac{k+2}{k+1}\right) \qquad (3.7)$$

valid for any nonnegative integer k. We [illegible] prove also the inequality (3.5) by induction , first with respect to k and finally with respect to j using (3.7). In this way we have proved Lieb's conjecture for any value of J and for all states v_m , $m = -J, -J+1, \ldots, 0, \ldots, J-1, J$. We remark that with the use of the inequality (3.7) we may prove that $S^{cl}(v_m)$ attain it maximum value for $m = 0$. In the following we shall show that the generalized conjecture :

$$I_p^J(\rho_{v_m}) \leq \frac{2J+1}{2pJ+1} \qquad (3.8)$$

is valid for any value of J and for p = 1. From (3.2) it follows that this inequality is equivalent with the following inequality for the Γ-function :

$$\frac{\Gamma(kp+1)\Gamma(jp+1)}{\Gamma((k+j)p+1)} = \left(\frac{\Gamma(k+1)\Gamma(j+1)}{\Gamma(k+j+1)}\right)^p \qquad (3.9)$$

which is unknown. This inequality may be written as an inequality for the B-function :

$$((k+j)p+1)B(kp+1,jp+1) = ((k+j+1)B(k+1,j+1))^p \qquad (3.10)$$

for any nonnegative integers k, j and any p ≥ 1. We shall consider the more general inequality :

$$((a+b)p+1)B(ap+1,bp+1) = ((a+b+1)B(a+1,b+1))^p \qquad (3.11)$$

for any real nonnegative numbers a, b and p ≥ 1. We obtain this inequality from an integral representation for the B-function /8,§1,1.6.3/ :

$$\frac{1}{(a+b+1)B(a+1,b+1)} = 2^{a+b}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(a-b)\varphi}(\cos\varphi)^{a+b}\,\frac{d\varphi}{\pi} \qquad (3.12)$$

and from Jensen's inequality /9,chap.[illegible],Th.3.3/ only for a=b :

$$\left(\frac{1}{(2b+1)B(b+1,b+1)}\right)^p = \left(2^{2b}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\varphi)^{2b}\,\frac{d\varphi}{\pi}\right)^p$$

$$\leq 2^{2bp}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\varphi)^{2bp}\,\frac{d\varphi}{\pi}$$

$$= \frac{1}{(2bp+1)B(pb+1,pb+1)} \qquad (3.13)$$

Hence the inequality (3.11) , for a ≠ b, remains a conjecture.

4. DISCUSSION OF THE GENERALIZED CONJECTURE IN THE CASE J = 1.

In this section we discuss , for J = 1 , the conjecture (1.8) which in this case becomes :

$$I_p^1(O_a) \leq \frac{3}{2p+1} \tag{4.1}$$

for any value of a $\in$ [o,1] and of p $\geq$ 1. In order to verify (4.1) we try to find the explicit form of the integral $I_p^1(O_a)$ as a funct. n of the parameter a. First we calculate this integral , in a straightforward manner, in the case in which p is a positive integer (p = n$\geq$1), and obtain :

$$I_n^1(a) = \frac{3}{2n+1} \sum_{s=o}^{[\frac{n}{2}]} \sum_{r=o}^{n-2s} \sum_{t=o}^{n-s-r} \frac{(-1)^t 2^{s+r}(2n-s-r)!(s+r)!\, n!\, a^{s+r+t}}{(s!)^2 r!(n-2s-r)!(2n)!(n-s-r-t)!t!} \tag{4.2}$$

After tedious calculations we obtain from this expression that the coefficients of a^{2k+1} are equal to zero for k=o,1 and that the coefficients of a^{2k} for k =1,2 are of the following form :

$$I_n^1(a) = \frac{3}{2n+1}\left(1 - \frac{n(n-1)}{2(2n-1)}\, a^2 + \frac{n(n-1)(n-2)(n-3)}{2^2 2(2n-1)(2n-3)}\, a^4 - \ldots\ldots\right) \tag{4.3}$$

The comparison of this expression with the functio[illegible]

$$\frac{2^n(n!)^2}{(2n)!}\, a^n P_n\left(\frac{1}{a}\right) = \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^k n(n-1)(n-2)\ldots(n-2k+1)}{2^k k!(2n-1)(2n-3)\ldots(2n-2k+1)}\, a^{2k} \tag{4.4}$$

where $P_n(.)$ are Legendre polynomials, suggests that :

$$I_n^1(a) = \frac{3}{2n+1}\, \frac{2^n(n!)^2}{(2n)!}\, a^n P_n\left(\frac{1}{a}\right) \tag{4.5}$$

If we assume that (4.5) is valid we obtain that :

$$I_n^1(O_a) \leq \frac{3}{2n+1} \qquad (4.6)$$

where the equality is attained only for a = o . Indeed, from the fact that the roots of the Legendre polynomials lie all in the interval (-1,1) and from the fact that if $P_n(b)$ =o it result that either b = o or $P_n(-b)$ = o , we obtain that :

$$P_n(x) \leq \frac{(2n)!}{2^n (n!)^2} x^n \qquad (4.7)$$

for arbitrary $x > 1$. From (4.7) we get

$$\frac{2^n (n!)^2}{(2n)!} a^n P_n(\tfrac{1}{a}) \leq 1 \qquad (4.8)$$

which together with (4.5) gives (4.6). Now we try to extend the formula (4.5) to all values of $p \geq 1$ using spherical functions $P_p(x)$ instead of Legendre polynomials. Then we make the hypothesis that :

$$I_p^1(O_a) = \frac{3}{2p+1} \; \frac{2^p \Gamma(p+1)^2}{\Gamma(2p+1)} \, a^p \, P_p(\tfrac{1}{a}) \qquad (4.9)$$

and from the fact that /10/ :

$$P_p(z) = (\tfrac{1+z}{2})^p F(-p,-p;1;\tfrac{z-1}{z+1}) \qquad (4.10)$$

for Re(z) > o , we obtain :

$$I_p^1(O_a) = \frac{3}{2p+1} \; \frac{2^p \Gamma(p+1)^2}{\Gamma(2p+1)} (\tfrac{1+a}{2})^p \; F(-p,-p;1;\ \tfrac{1-a}{1+a}) \qquad (4.11)$$

From this formula we obtain immediately that :

$$\left.\frac{dI_p^1(O_a)}{dp}\right|_{p=1} = \frac{2}{3} + (\, a - \ln(\, 1+a)\,) \qquad (4.12)$$

which coincides with the result proved in section 2.

Finally we remark that the inequality (4.4) is equivalent

$$P_p(x) \leq \frac{\Gamma(2p+1)}{2^p \Gamma(p+1)^2} x^p \qquad (4.13)$$

for all $x \geq 1$, or with the inequality :

$$F(-p,-p;1;t) \leq \frac{\Gamma(2p+1)}{2^p \Gamma(p+1)^2} (1+t)^p \qquad (4.14)$$

for all $t \in [0,1]$. We have not a proof of (4.13) or of (4.14) for noninteger values of p.

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