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Vortices and Phase Transitions in Two-Dimensional Non-Abelian Spin Models

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ABSTRACT

We modify O(n) models ($n>2$) in two dimensions so as to compare different theories with identical local properties and different global ones.

Our O'(3) model with a particular interaction has vortex-like configurations ($\pi_1(P_2)=Z_2$) though it is locally equivalent to an O(3) model ($\pi_1(S_2)=0$)

Our results have been obtained by means of strong coupling methods. The Pade extrapolants show a critical value $x_c = 11.8$.

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Phase diagrams of two dimensional lattice spin models (2DS) with SU(n) x SU(n) global symmetry (also named "O(n + 1)" models) were intensively studied in the last years . (for updated reference lists see[1-3]. The reason for the interest of the high energy theorists in this statistical mechanics problem is its common properties with the SU(n) four-dimensional gauge models (4 DG)[2-7].

The simplest examples of the two classes of the aforementioned models are the O(2) 2DS and the U(1) 4DG[8]. Neither of them are asymptotically free and both of them possess vortex-like configurations. The vortices are shown to induce in both of them, phase transitions[9-11]

For $n \geq 2$ the SU(n) 4DG and the O(n+1) 2DS are nonabelian and asymptotically free [12-15]. However there is a difference: The SU(n) 4DG have singular topological objects for any n[16] while the O(n) 2DS do no present vortices for $n > 2$[17]. To push the analogy further it will be useful to construct models that are locally isomorphic to O(n) but such that their global structure allows for the appearance of singular topological objects (vortices). We call these models O'(n).

In principle the appearance of topological objects in these systems could induce phase transitions analog to the one shown by the O(2) model. First we sketch the Kosterlitz - Thouless argument[9]. Each vortex has an energy

$$E_0 = \pi J \ln \frac{R}{a} \qquad (1)$$

where R is the linear dimension of the two-dimnsional world and a is the lattice spacing. The entropy is:

$$S_0 = K \ln\left(\frac{R}{a}\right)^2 \tag{2}$$

So the free energy of a vortex is

$$F_0 = (\pi J - 2KT) \ln\frac{R}{a} \tag{3}$$

Therefore for all temperatures greater than $T_c = \frac{\pi J}{2K}$ we expect vortices to mix into the ground state. One can therefore state: "T_c is the temparature of the phase transition towards vortex condensation".

In the case of (a slightly modified) O(2)2DS[18] the positions of the vortices describe completely the state of the system[10]. This allows rewriting the integral over configurations in terms of vortex variables and to obtain the logarithmic energy of the vortices rigourously. (For the analog 4DG work[11]). In the case of the O'(n) models (with n > 2) writing an effective action for the vortices requires an integration over all the other degrees of freedom. This is impossible to do exactly. However, this integration may lead to an effective action where Eo doesn't diverge logarithmically and consequently the vortices are always (at any finite temperature) mixed into the ground state. One says in such a case that the phase transition towards vortex condensation is renormalized to $T_c = 0$ by the (nontopological) fluctuations around the vortices."

Another way to argue for such a scenario is to relate it to the known weak coupling behaviour of the O(n) models[14-15]. Since an O'(n) model has the same local structure with O(n) (by construction) their low temperature properties are the same. But the O(n) is known to be even at low temperatures in its high temperatures phase of total disorder (exponential decay of the correlation function). Therefore the presence or the creation of vortices cannot induce any further disordering in the system[5].

A third observation that makes the appearance of phase transitions in the O'(n) models unlikely is the following. A phase transition is characterized by a zero of the β-function. Now the high temperature limit shows that $\beta(\infty)>0$. Asymptotic freedom means $\beta'(0)>0$, and $\beta(0)=0$ The conclusion is that β has an even number of zeros at finite temperature. Since it is difficult to find reasons for the existence of two or more phase transitions one is left with the conjecture of no phase transitions as a highly plausible one[5].

The possibility of a phase transition poses a potential difficulty to the program of extending the lattice results to a meaningful asymptotically free continuum theory[8]. In the light of the above analysis the results reported below suggesting a phase transition in an O'(3) model are both intriguing and troublesome.

First let us give a few recipes of constructing O'(n) models. In two dimensional spin systems the appearance of vortices is related to a nontrivial topological group π1 of the parameter space[17]. Since $\pi_1(G/H) = \pi_o(H)_G$ [19] the procedure for constructing an O'(n) model is to begin with a non abelian (asymptotically free) model with parameter space G and to choose an interaction that is invariant under a discrete group H.

For example one may begin with a 3 component unit vector $\vec{n}(\vec{s})$ on each lattice site $\vec{s}$ and to construct a lagrangian that is invariant under the following local (gauge) transformation:

$$T(\vec{S}_o): \begin{cases} \vec{n}(\vec{S}) \to \vec{n}(\vec{S}) & \text{if } \vec{S} \neq \vec{S}_o \\ \vec{n}(\vec{S}) \to -\vec{n}(\vec{S}) & \text{if } \vec{S} = \vec{S}_o \end{cases} \tag{4}$$

Such a Lagrangian is for instance:

$$L = \sum_{\vec{S} \in (\text{sites})} \; \sum_{\vec{i} \in \{\vec{u}_1, \vec{u}_2\}} (\vec{n}(\vec{S}) \; \vec{n}(\vec{S} + i))^2 \tag{5}$$

where $\vec{u}_1$, $\vec{u}_2$, are the unit vectors that generate the lattice. Another such lagrangian is obtained by the introduction of a 'gauge" auxiliary field $\sigma(\vec{S}, \vec{S} + \vec{i})$ defined on links:

$$L = \sum_{\vec{S}} \sum_{\vec{i}} \vec{n}(\vec{S}) \sigma(\vec{S}, \vec{S} + \vec{i}) \; \vec{n}(\vec{S} + \vec{i}) \tag{6}$$

This is equivalent to

$$L = \sum_{\vec{S}} \; \sum_{\vec{i}} \left| \vec{n}(\vec{S}) \; \vec{n}(\vec{S} + \vec{L}) \right| \tag{7}$$

Continuous models can be constructed using the group identity

$$U(N)/U(1) \equiv SU(N)/Zn \tag{8}$$

One takes an interacting model of unitary matrices and introduces a complex auxiliary field that compensates (rends unphysical) the overall phase degree of freedom:

$$L(x) = (\partial_\mu - iA_\mu)\, U_{K\ell}\, (\partial_\mu + iA\ )\, U^*_{K\ell} + i\, \lambda_{mn}\, (x)(U_{mj}\, U^*_{nj} - \ell_{mn}) \quad (9)$$

Also since $SO(3) \equiv SU(2)/Z_2$ (10)

a model with 3x3 orthorgonal matrices is an O'(4) model:

$$L(x) = \partial_\mu\, a_{ij}\, \partial_\mu\, a_{ij} + \lambda_{ij}(a_{iK}\, a_{jK} - \delta_{ij}) \quad (11)$$

where $i,j, \in \{1,2,3\}$

For the dilute gas of vortices of the above models the naive Kosterlitz-Thouless argument goes through with minor modifications required by the finiteness of $\pi 1$ associated with each of them.

Let us take a closer look at the model (5)*. The configurations of fig 1a (which the model shares with O(3)) are not stable since they can decay to the vacuum fig. 1.c (remember the spins are three dimensional). However the configurations of fig. 1b, which have in the O(3) model, a linearely divergent infrared energy, have in this model a logarithmic divergence exactly like the vortices of the O(2) model. They are also (relatively) stable against decay into the vacuum. (This means more precisely: There is no existing continuous family of configurations that have the following properties:

1) It interpolates continuously between 1b and 1c

2) The energy of each of the configurations in the family is at most logarithmically divergent).

* Study of the other models and other methods will appear in a longer paper.

In order to study the model (5) we used a transfer matrix formalism[20] that leads to a strong coupling expansion for the β- function (and other interesting quantities). It consists in compactifying one direction of the lattice. This reduces the problem to the study of the first excited state of an one dimensional lattice quantum system with hamiltonian:

$$H = \frac{g}{2a} \sum_{m} \left(J^2(m) - x \sum (-1)^i \sigma_{-i}(m)\sigma_i(m+1)\right) \tag{12}$$

where g is the coupling constant, a is the lattice constant, $x = \frac{2}{g^2}$, $\vec{J}$ is a three component angular momentum operator,

$$\sigma_i(m) = \sqrt{\frac{4\pi}{5}}\; Y_{2i}(\theta(m),\phi(m)) \tag{13}$$

m is the lattice site, Y_{2i} are the spherical harmonics associated with J^2 and J_z. The ground state is characterized by

$$J^2|0> = 0 \tag{14}$$

while a first excited state is characterized by

$$\begin{aligned} J^2|1> &= 6 \\ J_z|1> &= 2 \end{aligned} \tag{15}$$

We computed the difference $(\omega_1 - \omega_0)$ of the energies of the states $|1>$ and $|0>$ to the sixth order in x.

Since not all the selection rules that simplify the diagramatic calculations for 0(3) and 0(2) models hold here, the number of nonvanishing diagrams in this modified model is much larger. Consequently the programme that would produce a relevant number of terms in a limited amount of time is more cumbersome.

We have checked our own programme carefully by applying it to $O(2)$ and $O(3)$ models and then comparing the output with known results.*

For our model $O'(3)$ the Pade' approximants[21)] give very good convergence (fig 2,3). They indicate a critical coupling x_c = 118 (tables 2 and 3). In order to make sure that the phase transition is not an artifact of the high value of J^2(=6) we computed a few terms for the J = 2 state of the O(3) model. They have a very different structure (table 4) than the $|1>$ of $O'(3)$ and show no sign of phase transition. (no zeros in the Pade's). In fig. 4 we plot some of the Pade's of the $\omega_1 - \omega_0$ in the $O(2)$ model. The tendency to vanish is weaker than in $O'(3)$ model fig. 2,3.

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* As a byproduct we obtained the coeficient of x^7 for $\omega_1 - \omega_0$ in the O(3) model: it is - 0.18756×10^{-6}. Detailed discussion of following numerical results will appear in a forthcoming paper.

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TABLE 1

Coeffients for X^n in the O'(3) model

ω_o/N	$\omega_1 - \omega_o$	n
0	6	0
0	-0.4	1
0.16667×10^{-1}	-0.21714×10^{-1}	2
0.39683×10^{-3}	$+0.17752 \times 10^{-2}$	3
0.88183×10^{-6}	-0.85988×10^{-4}	4
0.27975×10^{-6}	$+0.18126 \times 10^{-5}$	5
0.2394×10^{-8}	$+0.82098 \times 10^{-6}$	6

TABLE 2

Closests zeroes of the M/N Padé's of O'(3) energy

N \ M	1	2	3	4	5
1	8.2	11.36	12.02	11.53	11.00
2	16.7	11.89	11.75	11.94	
3	8.63	11.69	11.87		
4	11.93	14.09			
5	13.97				

TABLE 3

Closests zeros of Pade's for $\frac{\beta(g)}{g}$ of O'(3)

N \ M	1	2	3	4	5
1	5.7	7.12	x	6.6	6.5
2	88	13.57	11.7	x	
3	1.5	10.6	12.7		
4	37	x			
5	x				

TABLE 4

Coefficients of x^n for the second excited state

energy of the O(3) model

n	$\omega_2 - \omega_0$	β/g
0	6.0	1
1	0	0
2	0.38333	2.
3	0	0
4	- 0.01754	- 0.865
5	0	0
6	0.00164	1.117

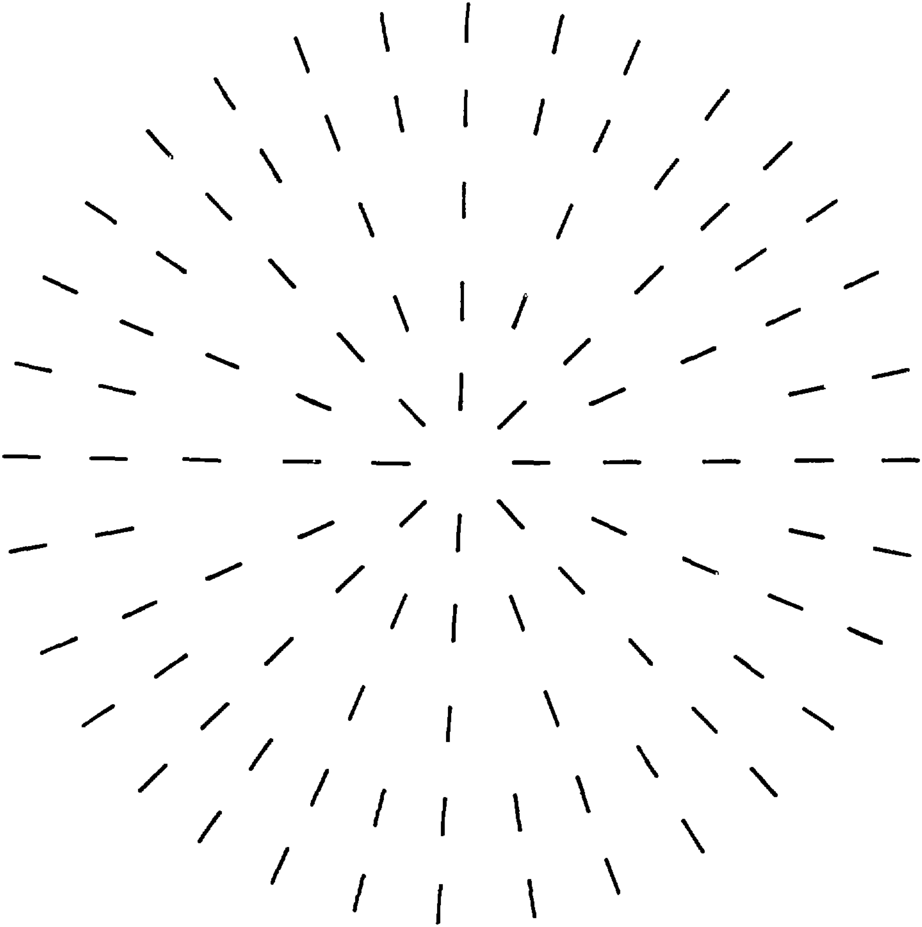

Fig. 1a

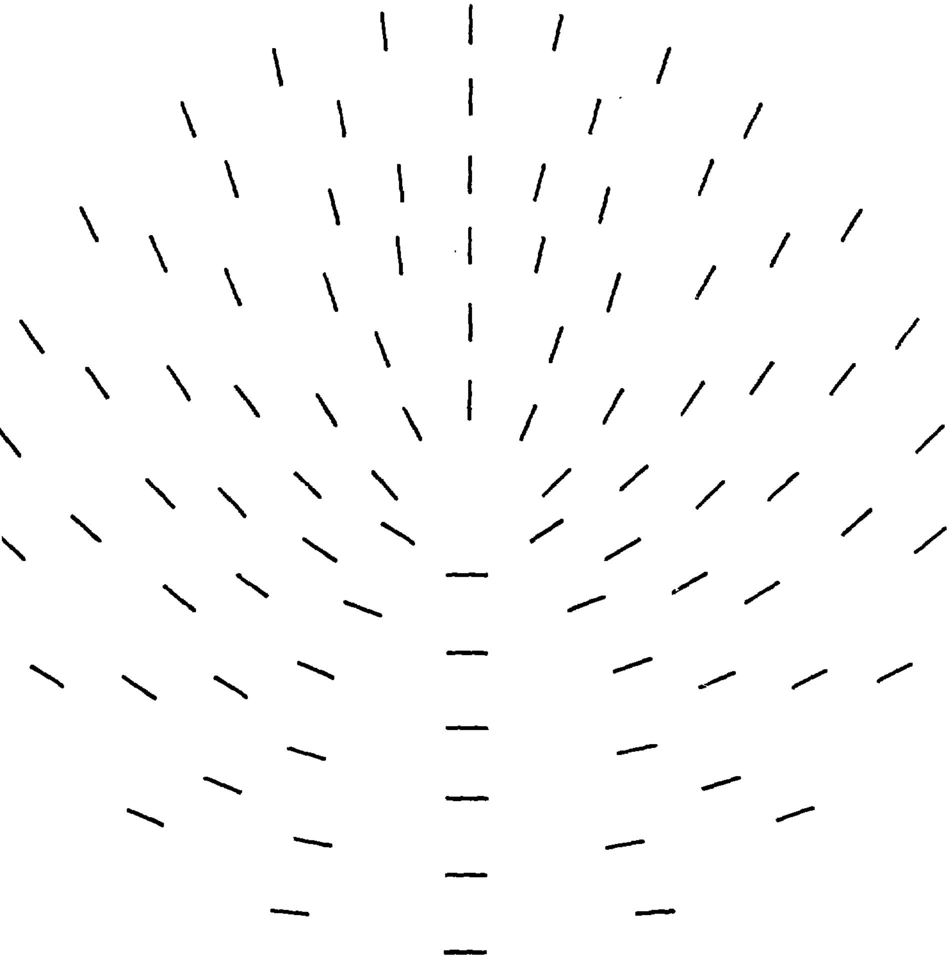

Fig. 1b

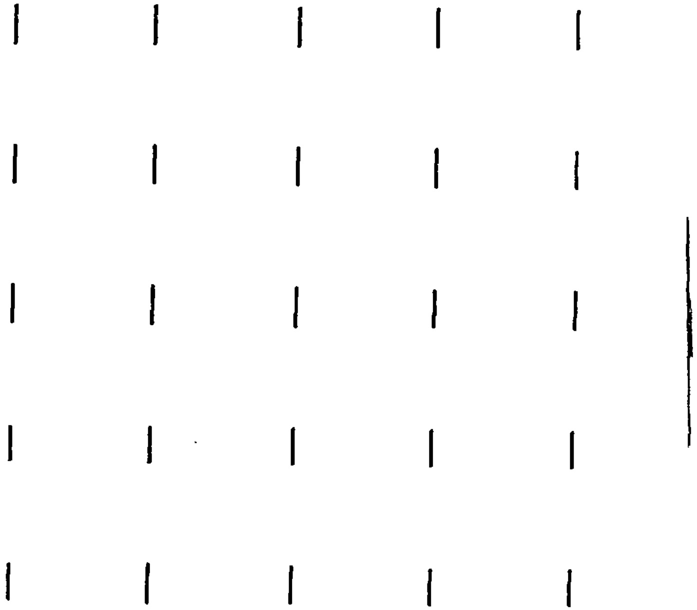

Fig. 1c

Padé approximants for $\omega_1 - \omega_0$ in the $O'(3)$ model.

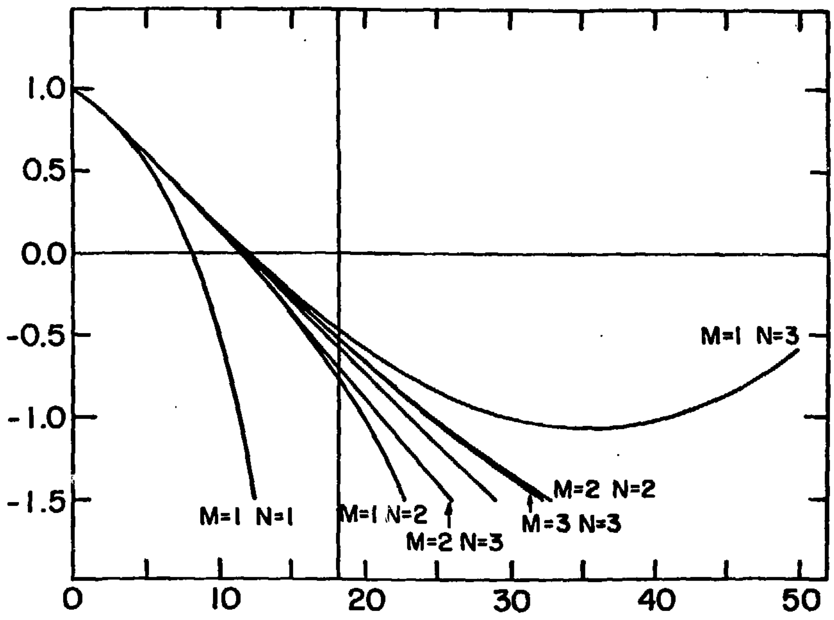

Fig. 2

Pade' approximants for the $\frac{\beta}{g}$-function of the O'(3) model

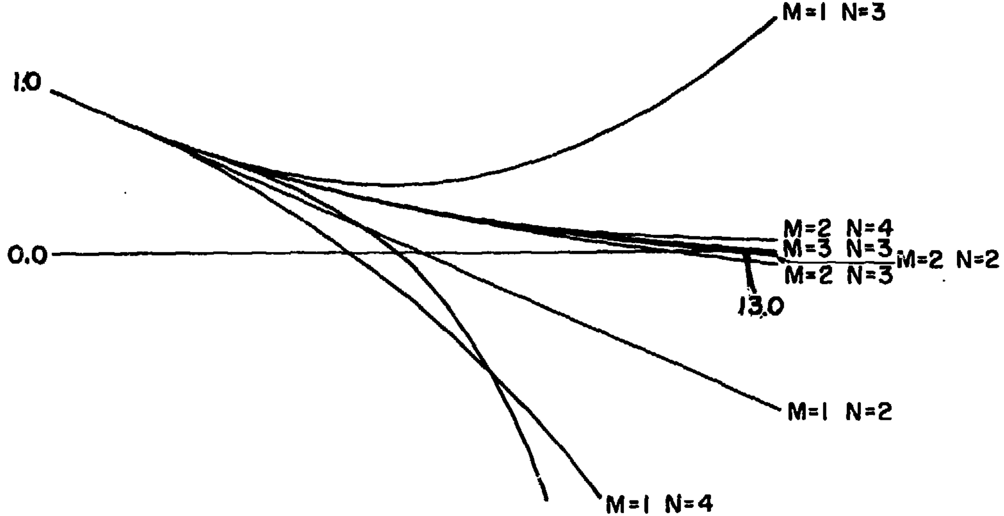

Fig. 3

Pade' approximants for $\omega_1 - \omega_0$ in the O (2) model.

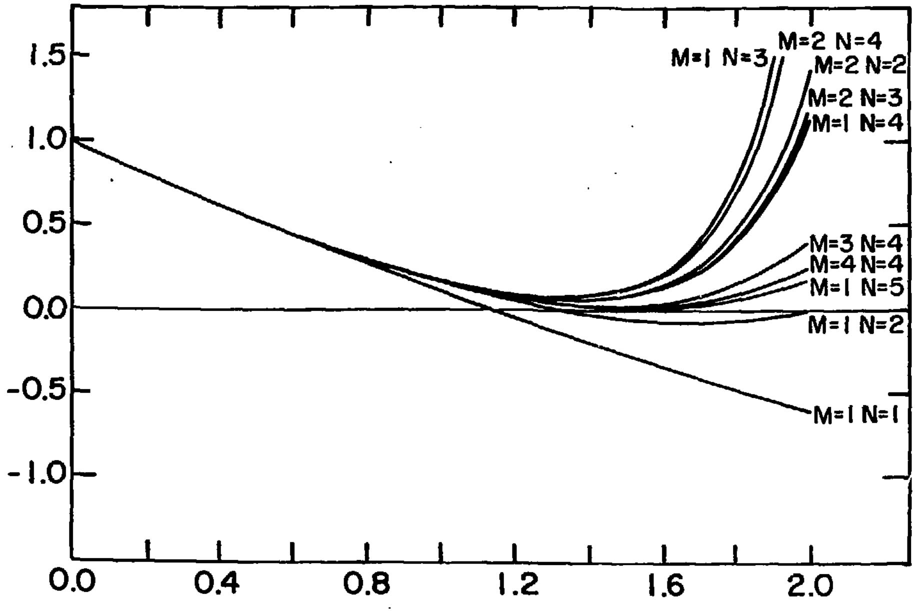

Fig. 4

Pade´ approximants for the $\frac{\beta}{g}$-function of the O(2) model

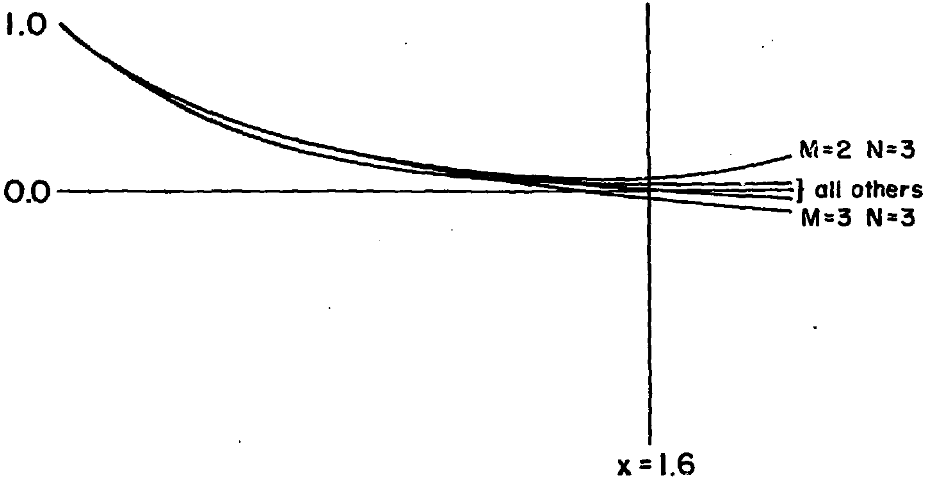

Fig. 5