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Calculation of the Transfer Matrix T in Six Dimensions for an rf-Deflector Element

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CALCULATION OF THE TRANSFER MATRIX T
IN SIX DIMENSIONS FOR AN rf-DEFLECTOR ELEMENT

by

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ABSTRACT

One possible element for funneling two beams together is a deflector with a constant or time-varying electric-field strength. With such an element, arbitrary beams can be brought together and maintained on the axis, if the appropriate combination of deflector parameters is chosen. A parallel beam can be handled only with a time-varying voltage of the deflector. The six-dimensional transfer matrices are calculated for constant or time-varying fields; all the results are correct in first-order approximation.

I. INTRODUCTION

The idea of funneling two or more beams together is an important point for most heavy ion fusion scenarios.[1] The idea of funneling means that particle beams, coming from a low-frequency acceleration structure, are brought together in a second accelerator, which is operating at a higher frequency in such a way that every bucket of the high-frequency acceleration field is filled. Such an arrangement has two great advantages: the space-charge problem is not severe, because the number of particles in each radio-frequency (rf) bucket can be kept small, and filling all the buckets reduces the total length of the system and the operation costs. One example of such a funneling concept is the proposed arrangement of six different types of linear accelerators for a heavy ion fusion facility.[2] In this case, a X_e^{+1} beam of 800 mA and 10 GeV at the end is produced by starting with 32 individual beams of 25 mA. In general, the funneling idea allows handling a final high-current beam in an elegant and less expensive way.

One possible element for bringing the beams together is a deflector with a constant or time-varying electric field. Two beams are assumed to be symmetrically distributed around the longitudinal axis of the following accelerator. In this case, funneling means that at the end of the funneling section, the two beam centers, separated in time, are colinear for minimizing the longitudinal and transverse emittance growth. Therefore, for a symmetric arrangement of the two beams, the deflector's electric-field strength has to be changed from a positive to a negative value. This can be done either with a fast switcher and E = constant, or with a time-varying electric field with an appropriate frequency choice.

Each funneling section may increase the transverse and longitudinal beam emittance; therefore, the funneling line must be optimized. To minimize the emittance growth, the transfer matrix of the funneling section must be known, at least in first-order approximation.

In this report, we would like to present the linear transfer matrix T in six dimensions for a deflector element with a constant or time-varying field. All formulas are in first-order approximation, correct for an arbitrary movement of the beam centers inside the deflector. Space-charge effects are not included, but in a funneling line they are not so important because for low energies the current is low, and for high-current values the beam energy is high.

The deflector element is the most important part of our following proposed funneling line: two symmetrically located beams, produced by arbitrary, but identical accelerators, are first brought near the z-axis by similar bending magnets of opposite polarity. We define the z-axis to be the longitudinal axis of the following accelerator structure. Therefore the parameters of the two beam centers differ only in sign and we can consider one beam only.

After the bending magnet, the angle x' of the beam center is smaller than or equal to zero for a positive value of the displacement x. A zero angle can be achieved by using two separated bending magnets. The parameters of the following deflector element are chosen so that the beam, coming from the bending magnet, is transferred to the z-axis. For a parallel injection ($x' = 0$), we have to use a time-varying field, whereas for $x' < 0$ either constant or time-varying fields are possible. At the end of the deflector, the beam center is moving along the z-axis for every given initial displacement and angle, if the parameters of the bending magnet and the deflector element are compatible. In the deflector element, the longitudinal and transverse motions of the particles are coupled. Therefore, we are expecting some emittance growth, and a matching section has to be constructed for transferring this beam into another accelerator.

All above statements are correct for the second beam if its beam center, arriving at the deflector at some later time, is seeing the opposite electric field. A concrete design for such a whole funneling section will be done later.

This report is organized in the following way: in Sec. II we calculate the linear transfer matrix T of a deflector with a constant electric field for an arbitrary movement of the beam center. In Sec. III, the same is done for a time-varying field. In Sec. IV, the two six-dimensionals transfer matrices and the definitions of all the terms are listed.

The most important part of this report is the calculation of the phase difference. In the Appendix, we derive a general formula for the phase difference, which is valid for arbitrary movement of the beam center and the particles. The movement of the particles is described in its own specific curved coordinate system. The expressions used for the phase differences in Sec. II and Sec. III are based on this Appendix.

II. THE DEFLECTOR ELEMENT WITH A CONSTANT ELECTRIC FIELD

The deflector for funneling the beams together is shown in Fig. 1.

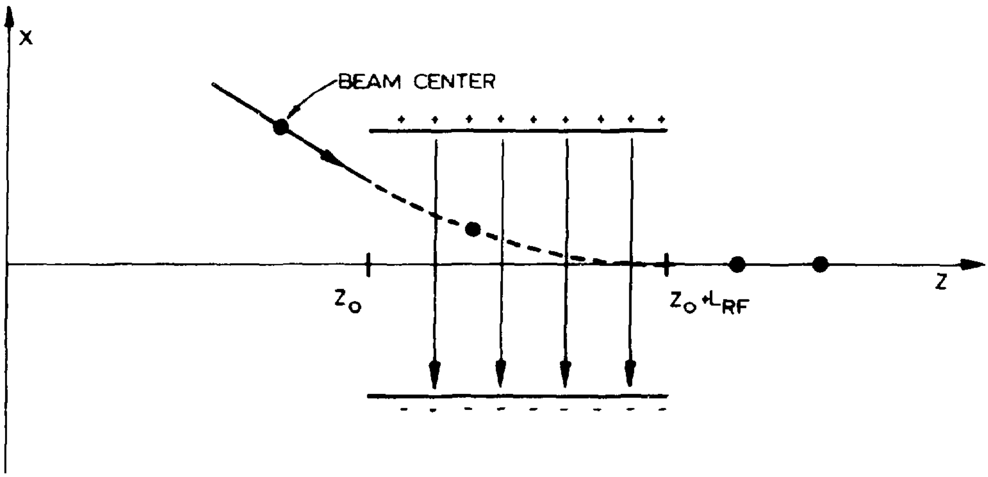

Fig. 1. Schematic Drawing of a Deflector Element

We are using this coordinate system: (x,y,z) are normal rectangular cartesian coordinates and z is the longitudinal axis of the following acceleration structure; (x,y) are the displacements of the particles from the z-axis.

The deflector element is a normal plate conductor of the length L_{RF}, parallel deposit around the z-axis, with a constant or time-varying field. The beam centers are moving along the lines $x_c(z)$ in the (x,z)-plane. The two beams are thought to enter symmetrically $\begin{pmatrix} x_c \\ x_c' \end{pmatrix}^1 = - \begin{pmatrix} x_c \\ x_c' \end{pmatrix}^2$ but separated in time in this arrangement, and we consider one beam only.

In this entire report, the subscript c refers to the beam center (the synchronous particle) and $x_c(z)$ always means the movement of the centers, called reference line. The quantities m and q are the particles rest mass and charge. Velocity-dependent parameters are carrying a subscript c for beam center and p for particles. The quantity v (γ) is the velocity's absolute value (the relativistic factor $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$) of either the beam center or the particles.

With a bending magnet, the particles of each beam are brought to the beginning of the deflector element; therefore, the displacement and the angles of the particles are determined by the magnet parameters. For a parallel entry of the beam ($x_c' = 0$) in such a magnet with a homogenous field strength B_0, we get

$$x_c(z_1 + a_1) = x_c(z_1) - \rho_1 + \rho_1 \sqrt{1 - (a/\rho)_1^2} \tag{1}$$

$$x_c'(z_1 + a_1) = \frac{-(a/\rho)_1}{\sqrt{1 - (a/\rho)_1^2}} \tag{2}$$

with $x_c'(z_1) = 0$ and $\left|\frac{a}{\rho}\right|_1 < 1$. Here $x_c(z_1)[x_c(z_1 + a_1)]$ is the displacement of the synchronous particle at the beginning (end) of the bending magnet and $\frac{1}{\rho_1}$ is the curvature radius of the magnet:

$$\frac{1}{\rho_1} = \frac{q(B_0)^1}{m\, v_c\, \gamma_c} \ . \tag{3}$$

The magnetic field is acting on the synchronous particle in the interval $z_1 \leqslant z \leqslant z_1 + a_1$.

A parallel entry of the beam into the deflector element is possible if we use a second bending magnet. The beam parameters are given by

$$x_c(z_2 + a_2) = x_c(z_2) + \rho_2 - \frac{(a/\rho)_2}{x_c'(z_2)} \quad \rho_2 \tag{4}$$

$$x_c'(z_2 + a_2) = 0 \tag{5}$$

with $\left|\frac{a}{\rho}\right|_2 < 1$ and $\left|\frac{(a/\rho)_2}{x_c'(z_2)}\right| < 1$. Note that $x_c(z_2)$ is the displacement of the beam center at the entrance of the second magnet.

For the deflector element, we get these equations of motion if we apply a constant voltage between the two plates

$$m \frac{d^2\vec{r}}{dt^2} = q\,\vec{E} \tag{6}$$

with $\vec{r} = (x,y,z)$ and $\vec{E} = (E_x,0,0)$. For a constant velocity $v_z = \frac{dz}{dt}$, the solutions of Eq. (6) in x-direction can be written as

$$\begin{aligned} x(L) &= x(0) + x'(0) + (a/2)L^2 \\ x'(L) &= x'(0) + a\,L \end{aligned} \tag{7}$$

with $L = z - z_0$ or $0 \leqslant L \leqslant L_{rf}$

and

$$a = \frac{qE_x}{mv_z^2\gamma} = \text{const} . \tag{8}$$

The quantity $x(L)$ is the particle displacement from the z-axis, and $x' = dx/dL$ is the angle between the particle trajectory and the z-axis.

For funneling the two beams together, we would like to have at the end of the deflector

$$\begin{pmatrix} x_c(L_{rf}) \\ x_c'(L_{rf}) \end{pmatrix}^{\text{beam 1}} = \begin{pmatrix} x_c(L_{rf}) \\ x_c'(L_{rf}) \end{pmatrix}^{\text{beam 2}} = 0 \ . \tag{9}$$

For one beam center, the reference line $x_c(L)$, fulfilling the restriction of Eq. (3), is given by

$$x_c(L) = \frac{a_c}{2} \ (L_{rf} - L)^2 \tag{10}$$

with $a_c = \dfrac{qE_x}{mv_c^2 \ \gamma_c}$ and $|v_z^c| \approx v_c = \text{const}$.

Therefore, for our symmetric arrangement of the two beams, Eq. (9) can only be fulfilled by changing the electric-field strength from $+E_x$ to $-E_x$ for the two beams.

Now $x_c(0)$ and $x_c'(0)$ are functions of the parameters a_c and L_{rf} and for a positive value of the electric-field strength E_x, we get $x_c(0) > 0$, $x_c'(0) < 0$. This sign combination of x_c and x_c' can be made by bending the particles near the z-axis. Thus, with a fast change of the constant electric-field strength E_x, both beam centers can be brought to the z-axis and maintained on the axis. Please notice that a parallel beam ($x_c' = 0$) cannot be handled with this approach.

Equation (7) is used for the synchronous particle, but not for the other particles in a bunch, for the following reason: Eq. (7) is only valid for a constant velocity $v_z(t)$, which is approximately correct for the beam center, because the synchronous particle has no motion in y-direction and x_c' can be made small enough with a bending magnet. For the other particles, both statements are incorrect; the differential equation (6) is solved in a curved coordinate system $(\tilde{x},\tilde{y},s)$.[3] (Appendix).

All the following formulas are correct for an arbitrary reference line $x_c(z)$ and not only for the function $x_c(z)$ given in Eq. (10). The results for $\tilde{x}$ and $\tilde{y}$ are independent of the form of $x_c(z)$; however, the phase difference $\Delta\phi$ can be calculated only if the function $x_c(z)$ is explicitly known. For the reference line $x_c(z)$ the differential equation (6) must be solved.

Assuming the existence of an arbitrary reference line $x_c(z)$, we obtain the following differential equations

$$\frac{d^2\tilde{y}}{ds^2} = 0$$
$$\frac{d^2\tilde{x}}{ds^2} = \frac{1}{v_c^2}\frac{d^2\tilde{x}}{dt^2} \quad , \tag{11}$$

where $(\tilde{x},\tilde{y})$ are the displacements of the particles from the reference line and the coordinate s is the path length of the beam center (Appendix). From Eq. (11) we get

$$\tilde{y}(s) = \tilde{y}(0) + s\tilde{y}'(0)$$
$$\tilde{y}'(s) = \tilde{y}'(0) \tag{12}$$

with $0 \leq s = \int_0^L \sqrt{1 + x_c'^2(L')}\; dL' \leq L^{element}$,

and $L^{element}$ is the length of the reference line inside the deflector element. The behavior of the particles in $\tilde{y}$-direction is the same as for a drift-space element with the length $L^{element}$, as expected.

For the $\tilde{x}$-coordinate, the following equation is exact, if phase-difference effects in the arguments of the functions are neglected (Appendix).

$$\tilde{x}(t) = \frac{\sqrt{x_c'^2(t) + z_c'^2(t)}}{z_c'(t)}\left(x_p(t) - x_c(t)\right) \tag{13}$$

with $x_c' = \frac{dx_c}{dt}$ and $x_p(t)[x_c(t)]$ is the displacement of the particle (beam) from the z-axis at some time t.

For small values $x_c'(t)$, we get

$$\tilde{x}(t) = x_p(t) - x_c(t) \ . \tag{14}$$

Inserting this approximation into Eq. (11) and using Eq. (6), we obtain

$$\frac{d^2\tilde{x}}{ds^2} = \frac{qE_x}{mv_c^2}\left(\frac{1}{\gamma_p} - \frac{1}{\gamma_c}\right) \simeq \frac{-qE_x}{mv_c^2\gamma_c}\left(\frac{\Delta_\gamma}{\gamma_c}\right) = a_p \tag{15}$$

with $\gamma_p = \gamma_c + \Delta_\gamma$. The solutions of Eq. (15) can be written as

$$\tilde{x}(s) = \tilde{x}(0) + s\tilde{x}'(0) + \frac{a_p}{2} s^2 \tag{16}$$

and $0 \leq s \leq L^{element}$. The movement in $\tilde{x}$-direction is the same as in a drift-space element plus an additional velocity dependent term, which couples the transverse and longitudinal motion.

The functions $\tilde{x}(s)$ and $\tilde{y}(s)$ are both independent of any specific reference line $x_c(z)$.

For the phase difference ΔL, we obtain in first-order approximation (Appendix)

$$\Delta L^{end} = \Delta L^{begin} - \left(\frac{\Delta v}{v_c}\right) L^{element} + \tilde{x}'(0) I_1 + \left(\frac{\Delta\gamma}{\gamma_c}\right) I_2 \tag{17}$$

with $$I_{1,2} = \int_0^{L_{rf}} \frac{x_c'(L)}{\sqrt{1 + x_c'^2(L)}} \left(1, \frac{-qE_x}{mv_c^2\gamma_c} L\right) dL .$$

The first two terms of ΔL^{end} are the terms of drift-space element with length $L^{element}$; the next two terms are specific for the deflector element. Again we obtain a coupling between the transverse and longitudinal motion of the particles. The formula for ΔL^{end} is correct for any arbitrary function $x_c(L)$, but is not independent of the specific form of $x_c(L)$. For our funneling concept, we use $x_c(L)$, given in Eq. (10). The resulting formula of ΔL^{end} is given in Sec. IV, where we list all the definitions used.

III. THE rf-DEFLECTOR WITH A TIME-VARYING FIELD

Let us assume the existence of an electromagnetic field inside the rf-deflector of the following form:

$$\vec{E}(\vec{r},t) = (A \sin \omega t,\ 0,0)$$
$$\vec{B}(\vec{r},t) = 0 \quad . \qquad (18)$$

Equation (18) is correct, if edge effects and contributions of the magnetic field to the Lorentz-force are neglected.

The coordinates of the rf-deflector and the definition of the particle properties are the same as in Sec. II.

The equation of motion in x-direction for a particle inside the deflector

$$\frac{d^2x}{dt^2} = \frac{q}{m\gamma} A \sin \omega t \qquad (19)$$

has for $v_z = \frac{dz}{dt}$ = const solutions of the form

$$x(L) = x(0) + Lx'(0) + e(L)$$
$$x'(L) = \qquad x'(0) + f(L) \qquad (20)$$

with $L = z-z_0$ or $0 \leqslant L \leqslant L_{rf}$

$$e(L) = \frac{-q}{m\gamma} \frac{A}{\omega^2} \left[\sin\left(\frac{\omega L}{v_z} + \rho\right) - \frac{\omega L}{v_z} \cos \rho - \sin \rho \right]$$

$$f(L) = \frac{de(L)}{dL} = -\frac{q}{m\gamma} \frac{A}{\omega^2} \frac{\omega}{v_z} \left[\cos\left(\frac{\omega L}{v_z} + \rho\right) - \cos \rho \right] \quad .$$

The parameter $A \sin \rho$ is the electric-field strength of the rf-deflector when the particle enters.

For funneling two symmetric beams, we would like to have

$$\begin{pmatrix} x_c \\ x_c' \end{pmatrix}^{\text{beam 1}} = \begin{pmatrix} x_c \\ x_c' \end{pmatrix}^{\text{beam 2}} = 0 \qquad (21)$$

at the end of the deflector.

Using the restriction of Eq. (21) for one reference line, this beam trajectory $x_c(L)$ is given by (for $|v_z^c| \cong v_c$)

$$x_c(L) = \frac{-q}{m\gamma_c} \frac{A}{\omega^2} \left[\sin\left(\frac{\omega L}{v^c} + \rho\right) - \sin\left(\frac{\omega L_{rf}}{v^c} + \rho\right) + \frac{\omega}{v^c} (L_{rf} - L) \cos\left(\frac{\omega L_{rf}}{v^c} + \rho\right) \right]$$

$$x_c'(L) = \frac{-q}{m\gamma_c} \frac{A}{\omega^2} \frac{\omega}{v^c} \left[\cos\left(\frac{\omega L}{v^c} + \rho\right) - \cos\left(\frac{\omega L_{rf}}{v^c} + \rho\right) \right]. \tag{22}$$

If the frequency ω of the electric field is chosen as

$$\omega \Delta t = (2n + 1)\pi \qquad n = 0,1,2, \ldots \tag{23}$$

where Δt is the time difference between the two beam centers arriving at some point, then Eq. (21) is fulfilled for both beam centers. For the first beam, the value of the electric-field strength is $+ A \sin \rho$, whereas for the second beam, the field strength is given by $A \sin (\rho + \omega\Delta t) = - A \sin \rho$.

To get a smooth movement of the particles, we would like no beam-trajectory oscillations around the z-axis. Therefore $x_c'(L)$ should not be zero except at the points $L = 0$ (parallel injection into the deflector) and $L = L_{rf}$ (end of the deflector). These oscillations do not occur if ρ is limited in the interval $- \pi/2 \leqslant \rho < + \pi/2$ and we choose

$$\frac{\omega L_{rf}}{v_c} + \rho \leqslant \pi \qquad \text{for } \rho \geqslant 0$$

$$|\rho| < \frac{\omega L_{rf}}{v_c} + \rho \leqslant \pi \qquad \text{for } \rho \leqslant 0 . \tag{24}$$

In both cases, $x_c'(0) \neq 0$ and $x_c'(L_{rf}) = 0$. With the choice

$$\frac{\omega L_{rf}}{v_c} = 2|\rho| \text{ and } - \frac{\pi}{2} \leqslant \rho \leqslant 0 , \tag{25}$$

a parallel beam ($x_c'=0$) is brought to the z-axis without any oscillations between.

In every case, the field-amplitude A is positive for $x_c' \leq 0$, $x_c \geq 0$ and this sign combination is the result of using a bending magnet.

For a positive value of ρ, the absolute value of x_c' is maximal in the beginning, whereas for a negative value of ρ, the maximum of $|x_c'|$ is located inside the deflector. We expect a small emittance increase for $\rho > 0$, but more increase for $\rho < 0$, especially for parallel injection into the deflector element.

Equation (20) is only used for the description of the beam centers, because here $v_z(t)$ has to be constant. For the other particles in a bunch, we use a curved coordinate system $(\tilde{x},\tilde{y},s)$ and with the same approximations as in Sec. II, we obtain

$$\frac{d^2\tilde{y}}{ds^2} = 0$$
$$\frac{d^2\tilde{x}}{ds^2} = \frac{qA}{mv_c^2}\left[\frac{1}{\gamma_p}\sin\left(\frac{\omega}{v_c}s + \omega t_0 + \rho\right) - \frac{1}{\gamma_c}\sin\left(\frac{\omega}{v_c}s + \rho\right)\right] \tag{26}$$

where $A \sin \rho[A \sin(\rho + \omega t_0)]$ is the field strength when the beam center (particle) enters the rf-deflector. In the beginning, ωt_0 is proportional to a phase difference $\Delta\phi$.

The solutions of Eq. (26) can be written as (in first-order approximation in ωt_0 and $\Delta\gamma/\gamma_c$)

$$\tilde{y}(s) = \tilde{y}(0) + s\tilde{y}'(0)$$
$$\tilde{x}(s) = \tilde{x}(0) + s\tilde{x}'(0) + (\omega t_0)\,\alpha(s) + \left(\frac{\Delta\gamma}{\gamma_c}\right)\beta(s) \tag{27}$$
$$\tilde{x}'(s) = \tilde{x}'(0) + (\omega t_0)\,\lambda(s) + \left(\frac{\Delta\gamma}{\gamma_c}\right)\mu(s)$$

with

$$0 \leq s \leq L^{\text{element}}$$

$$\gamma_p = \gamma_c + \Delta\gamma$$

$$
\begin{pmatrix} \alpha(s) \\ \beta(s) \\ \lambda(s) \\ \mu(s) \end{pmatrix} = \frac{-qA}{m\omega^2 \gamma_c} \begin{pmatrix} 1, & 0, & -1, & s\frac{\omega}{v_c} \\ 0, & -1, & s\frac{\omega}{v_c}, & 1 \\ 0, & -\frac{\omega}{v_c}, & 0, & \frac{\omega}{v_c} \\ -\frac{\omega}{v_c}, & 0, & \frac{\omega}{v_c}, & 0 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\omega}{v_c} s + \rho\right) \\ \sin\left(\frac{\omega}{v_c} s + \rho\right) \\ \cos\rho \\ \sin\rho \end{pmatrix} .
$$

As in Sec. II, we obtain a coupling between the transverse and longitudinal motion and the results for $\tilde{x}$ and $\tilde{y}$ are independent of the reference line $x_c(L)$. For the length difference ΔL, proportional to the phase difference $\Delta\phi$, we get (Appendix)

$$
\Delta L^{end} = \Delta L^{begin} - \left(\frac{\Delta v}{v_c}\right) L^{element} + \tilde{x}'(0) I_3 + (\omega t_0) I_4 + \left(\frac{\Delta_\gamma}{\gamma_c}\right) I_5 \qquad (28)
$$

with:

$$
I_{3,4,5} = \int_0^{L_{rf}} \frac{(1, \lambda(L), \mu(L))}{\sqrt{1 + x_c'^2(L)}} x_c'(L)\, dL \quad .
$$

The first two terms in ΔL^{end} are the terms of a drift-space element; the last three terms are specific for the time-varying rf-deflector. The parameter ωt_0 is proportional to ΔL^{begin} and the functions $[\lambda(L), \mu(L)]$ are given in Eq. (27).

For our funneling arrangement, the reference line $x_c(L)$ is given by Eq. (22) and the resulting form of ΔL^{end} is listed in Sec. IV.

IV. CONSTRUCTION OF THE SIX-DIMENSIONAL TRANSFER MATRICES

In this section, we would like to list the complete (6 x 6) matrix for both deflector elements, and to repeat all the definitions used.

The arrangement of the deflector is the same as in Fig. 1; the plates are parallel to the z-axis, their length in z-direction is L_{rf}, and they are thought to be infinite in y-direction.

The quantity $v_p (v_c)$ is the total velocity of a particle (beam center = synchronous particle); m and q are its rest mass and charge.

The velocity difference Δv is related to the momentum, or total energy, difference by

$$\Delta p = p_p - p_c = m\, c\, \gamma\beta \frac{c^2}{c^2 - v_c^2} \frac{\Delta v}{v_c}$$

$$\Delta W = W_p - W_c = m\, c^2\, \gamma \frac{v_c^2}{c^2 - v_c^2} \frac{\Delta v}{v_c} \tag{29}$$

with $v_p = v_c + \Delta v$,

where $p_p (p_c)$ and $W_p (W_c)$ are the momentum and total energy of a particle (beam center) and (β, γ) are the usual relativistic factors of the synchronous particle; c is the velocity of light.

The phase difference $\Delta\phi$ is given in terms of the used length difference ΔL

$$\Delta\phi = (\phi_p - \phi_c)^{\text{end,beginning}} = \frac{2\pi}{\beta\lambda} \Delta L^{\text{end,beginning}} \quad , \tag{30}$$

where λ is the wavelength of the accelerating field.

The beam center is moving along the reference line $x_c(L)$ in the (x,z)-plane. The particle coordinates $\tilde{x}(s)$ and $\tilde{y}(s)$ are the displacements of the particles from the reference line with path-length s and $(\tilde{x}', \tilde{y}')$ are the derivatives with respect to s.

Using these definitions, the two transfer matrices can be calculated.

The transfer matrix T is defined by

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \\ \Delta L \\ \frac{\Delta v}{v_c} \end{pmatrix}_{end} = T \begin{pmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \\ \Delta L \\ \frac{\Delta v}{v_c} \end{pmatrix}_{beginning} \qquad (31)$$

a) the T-matrix for a constant electric field

$$T = \begin{pmatrix} 1, & \tilde{L}, & 0, & 0, & 0, & \frac{a_1}{2}\tilde{L}^2 \\ 0, & 1, & 0, & 0, & 0, & a_1\tilde{L} \\ 0, & 0, & 1, & \tilde{L}, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & I_1, & 0, & 0, & 1, & I_2 - \tilde{L} \\ 0, & 0, & 0, & 0, & 0, & 1 \end{pmatrix}$$

with: $\vec{E} = (E,0,0)$

$$a_1 = \frac{-qE}{m\gamma_c} \frac{1}{c^2 - v_c^2}$$

$$\tilde{L} = \int_0^{L_{rf}} \sqrt{1 + x_c'^2(L)} \; dL$$

$$I_{1,2} = \int_0^{L_{rf}} \frac{x_c'(L)}{\sqrt{1 + x_c'^2(L)}} \left(1, \; - \frac{qEL}{m\gamma_c} \frac{1}{c^2 - v_c^2}\right) dL$$

and $x_c(L)$ is an arbitrary reference line, given by Eq. (7).

For a funneling arrangement, both $x_c(L_{rf})$ and $x_c'(L_{rf})$ should be zero. In this case, for $|v_z^c| \cong v_c$, the reference line is given by Eq. (10)

$$x_c(L) = \frac{qE}{m v_c^2 \gamma_c} (L_{rf} - L)^2 \quad .$$

A parallel beam ($x_c' = 0$) cannot be handled by a constant electric-field strength E.

b) the T-matrix for a time-varying field:

$$T = \begin{pmatrix} 1, & \tilde{L}, & 0, & 0, & \frac{\omega}{v_c}\alpha(\tilde{L}), & \frac{v_c^2}{c^2 - v_c^2}\beta(\tilde{L}) \\ 0, & 1, & 0, & 0, & \frac{\omega}{v_c}\lambda(\tilde{L}), & \frac{v_c^2}{c^2 - v_c^2}\mu(\tilde{L}) \\ 0, & 0, & 1, & \tilde{L}, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & I_3, & 0, & 0, & 1 + \frac{\omega}{v_c} I_4, & \frac{v_c^2}{c^2 - v_c^2} I_5 - \tilde{L} \\ 0, & 0, & 0, & 0, & 0, & 0 \end{pmatrix}$$

with:

$$\begin{pmatrix} \alpha(s) \\ \beta(s) \\ \lambda(s) \\ \mu(s) \end{pmatrix} = \frac{-qA}{m\omega^2 \gamma_c} \begin{pmatrix} 1, & 0, & -1, & s\frac{\omega}{v_c} \\ 0, & -1, & s\frac{\omega}{v_c}, & 1 \\ 0, & -\frac{\omega}{v_c}, & 0, & \frac{\omega}{v_c} \\ -\frac{\omega}{v_c}, & 0, & \frac{\omega}{v_c}, & 0 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\omega}{v_c} s + \rho\right) \\ \sin\left(\frac{\omega}{v_c} s + \rho\right) \\ \cos \rho \\ \sin \rho \end{pmatrix}$$

$$\tilde{L} = \int_0^{L_{rf}} \sqrt{1 + x_c'^2(L)} \; dL$$

$$I_{3,4,5} = \int_0^{L_{rf}} \frac{x_c'(L)}{\sqrt{1 + x_c'^2(L)}} \; [1, \lambda(L), \mu(L)] dL \quad .$$

The electric field strength is given by

$$\vec{E} = (A \sin \omega t, 0, 0)$$

and A $\sin\rho$ [A $\sin(\rho + \omega t_0)$] is the field strength of the beam center (particle) in the beginning. The parameter ωt_0 is related to the length difference $\Delta L^{beginning}$ by $\omega t_0 = (\omega/v_c)\Delta L^{beginning}$; $x_c(L)$ is an arbitrary reference line, given by Eq. (20).

For a funneling arrangement, the frequency ω is given by

$$\omega \Delta t = (2n + 1) \pi \qquad n = 0,1,2, \ldots$$

where Δt is the time difference between the two beam centers. The reference line $x_c(L)$ is given by Eq. (22):

$$x_c(L) = \frac{-q}{m\gamma_c} \frac{A}{\omega^2} \left[\sin\left(\frac{\omega L}{v_c} + \rho\right) - \sin\left(\frac{\omega L_{rf}}{v_c} + \rho\right) + \frac{\omega}{v_c}\left(L_{rf} - L\right)\cos\left(\frac{\omega L_{rf}}{v_c} + \rho\right)\right] \quad .$$

To avoid oscillations ($x_c' = 0$) around the z-axis, we have to choose for $(-\pi/2) \leqslant \rho \leqslant (+\pi/2)$

$$\frac{\omega L_{rf}}{v_c} + \rho \leqslant \pi \qquad \text{if } \rho > 0$$

$$|\rho| < \frac{\omega L_{rf}}{v_c} + \rho \leqslant \pi \qquad \text{if } \rho < 0$$

In both cases, $x_c'(0) \neq 0$.

A parallel beam ($x_c' = 0$) can be handled by the parameter choice

$$\frac{\omega L_{rf}}{v_c} = 2|\varphi| \quad \text{for} \quad -\frac{\pi}{2} \leq \varphi \leq 0 \ .$$

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APPENDIX

CALCULATION OF THE PHASE DIFFERENCE $\Delta\phi$ FOR AN ARBITRARY MOVEMENT OF THE BEAM CENTER

If the beam center (the synchronous particle) is not moving along the z-axis, it is convenient to use a curved coordinate system $(\tilde{x},\tilde{y},s)$ instead of a normal cartesian coordinate system (x,y,z).[3] In this appendix, we consider only a movement of the beam center in the (x,z)-plane.

The coordinates $(\tilde{x},\tilde{y},s)$ are defined in this way: the beam center is moving along an arbitrary reference line $x_c(z)$ with constant velocity v_c in the (x,z)-plane. The coordinate s is the path length of the reference line with $s' = v_c$, starting at some point $s = 0$. The unit vectors $(\vec{e}_{\tilde{x}}, \vec{e}_{\tilde{y}})$ are perpendicular to the tangential vector $\vec{s}_0$ of the reference line at every point s, and the coordinates $(\tilde{x},\tilde{y})$ are the displacements of a particle from this reference line.

In Fig. A-1, the displacement $\tilde{x}$ is shown for an arbitrary reference line $x_c(L)$ and particle coordinates (x_p, L_p). The vector $\vec{e}_{\tilde{x}}$ is perpendicular to the tangential vector $\vec{s}_0$ of the reference line at the point L_0 and in general L_0 and L_p are not equal. The difference $L_p - L_0$ is proportional to the phase difference $\Delta\phi$ of the particle.

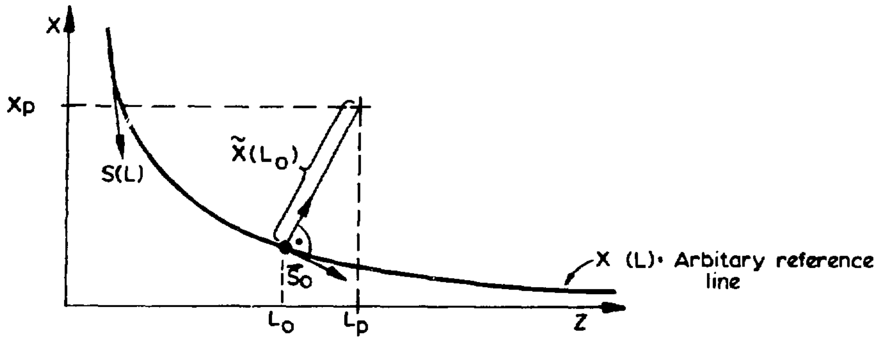

Fig. A-1. Curved Coordinate System

In some cases, the values $\tilde{x}(s) = \tilde{x}[s(L)]$, $\tilde{y}(s)$ and $x_c(L)$ are known, but unknown are the functions $x_p(L)$ and $y_p(L)$, the displacements of the particles from the z-axis. Using the definitions of Fig. A-1, we get for the parameter L_0 this complicated relation:

$$L_0 = L_p + \frac{1}{\sqrt{1 + x_c'^2(L_0)}} \quad \tilde{x}(L_0)x_c'(L_0) \tag{A-1}$$

with $\quad \tilde{x}(L_0) = \tilde{x}[s(L_0)] \quad .$

The displacement x_p and its derivative x_p' then are given as a function of the parameter L_0 by

$$x_p(L_p) = \frac{1}{\sqrt{1 + x_c'^2(L_0)}} \quad \tilde{x}(L_0) + x_c(L_0) \tag{A-2}$$

$$\frac{dx_p}{dL} = x_p' = \frac{(1 + x_c'^2)^{\frac{3}{2}} (\tilde{x}' + x_c') - \tilde{x}\, x_c'\, x_c''}{(1 + x_c'^2)^{\frac{3}{2}} (1 - \tilde{x}'\, x_c') - \tilde{x}\, x_c''} \tag{A-3}$$

with $x_p' = x_p'(L_p)$

$x_c' = x_c'(L_0) \quad , \quad x_c'' = x_c''(L_0)$

$\tilde{x} = \tilde{x}[s(L_0)]$

$\tilde{x}' = \frac{d\tilde{x}(s)}{ds} \Big|_{s(L_0)} = \frac{d\tilde{x}(L)}{dL} \Big|_{L_0}$.

For the value of the displacement y_p and its derivative y_p' we obtain

$$y_p(L_p) = \tilde{y}[s(L_0)]$$

$$\frac{dy_p}{dL} = y_p' = \tilde{y}' \cdot \frac{(1 + x_c'^2)^2}{(1 + x_c'^2)^{\frac{3}{2}} (1 - \tilde{x}' x_c') - \tilde{x}\, x_c''} \tag{A-4}$$

with $y_p' = y_p'(L_p)$ and all the $x_c(\tilde{x},\tilde{y}')$ - values are taken at the point $L_0[s(L_0)]$.

All of these kinetic formulas are exact for arbitrary functions $x_c(L)$, $\tilde{x}(s)$, and $\tilde{y}(s)$.

In a curved coordinate system, the phase difference $\Delta\phi$ is proportional to the time difference ΔT, at which the beam center and a particle are arriving at some point s_0. For an arbitrary element, located perpendicular to the z-axis in the interval $a \leqslant z \leqslant b$, we have the situation shown in Fig. A-2; the reference line is $x_c(z)$.

At the time t_0, a particle should arrive at the point s(a) and at the time $t_0 + \Delta T_p$, this particle should arrive at the point s(b).

The z-coordinates of beam are assumed to be $z = z_1$ at the time $t = t_0$ and $z = z_2$ at the time $t = t_0 + \Delta T_p$, as shown in Fig. A-2.

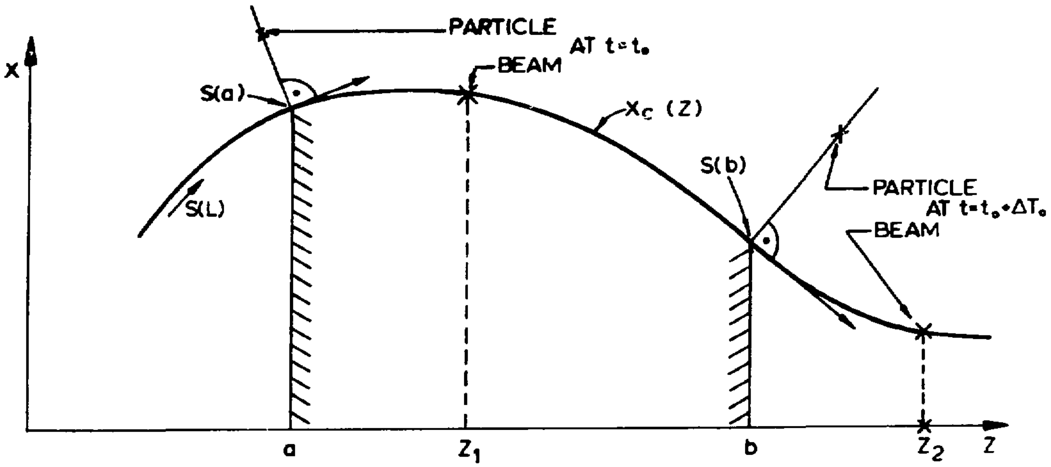

Fig. A-2. Coordinates for the Phase Difference Calculation

There exists a length difference ΔL^{end} between the beam and the particle at the end of the element:

$$\Delta L^{end} = \int_b^{z_2} \sqrt{1 + x_c'^2(z)}\; dz \quad . \tag{A-5}$$

ΔL^{end} is the length of the reference line between the points $z = b$ and $z = z_2$ and ΔL^{end} is proportional to the phase difference $\Delta\phi^{end}$.

Using the coordinate s with $s' = v_c$ = const, we obtain

$$\begin{aligned} \Delta L^{end} &= s(z_2) - s(b) \\ &= \left[s(z_2) - s(z_1)\right] + \left[s(z_1) - s(a)\right] + \left[s(a) - s(b)\right] \\ &= v_c\, \Delta T_p + \Delta L^{beginning} - \tilde{L} \quad , \end{aligned} \tag{A-6}$$

where $\tilde{L} = \int_a^b \sqrt{1 + x_c'^2(z)}\; dz$ is the effective total length of the element and

$$\Delta L^{end,\ beginning} = \left(s^{beam} - s^{particle}\right)^{end,\ beginning} \quad .$$

Equation (A-6) is exact for an arbitrary movement of the beam center in the (x,z)-plane and the only unknown parameter is the time ΔT_p, which is a quite complicated function of the parameters $\tilde{x}(s)$ and $x_c(z)$.

If the particle displacements $\tilde{x}(s)$ from the reference line are small at the beginning and at the end, then the parameter ΔT_p is approximately the flight time from the point z = a to the point z = b:

$$b - a = \int_{t_0}^{t_0+\Delta T_p} v_z^p(t)\, dt \qquad \text{(A-7)}$$

or:
$$\Delta T_p = \frac{1}{v_p} \int_0^{L_{rf}} \sqrt{1 + x_p'^2(L) + y_p'^2(L)}\; dL \qquad \text{(A-8)}$$

$$\text{with} \quad b = a + L_{rf} \text{ and } x_p' = \frac{dxp(L)}{dL}$$

where $x_p(L)$, $y_p(L)$ are the particle displacements from the z-axis.

The length difference ΔL can therefore be calculated, if the functions $x_p'(L)$ and $y_p'(L)$ are known.

Using the first-order approximation of Eqs. (A-3, A-4)

$$\begin{aligned} x_p'(L) &= \tilde{x}'(L) + x_c'(L) \\ y_p'(L) &= \tilde{y}'(L) \end{aligned} \qquad \text{(A-9)}$$

and the expansion $\frac{v_c}{v_p} = 1 - \frac{\Delta v}{v_c}$ with $v_p = v_c + \Delta v$,

we obtain the final result for ΔL^{end}:

$$\Delta L^{end} = \Delta L^{beginning} - \left(\frac{\Delta v}{v_c}\right)\tilde{L} + \int_0^{L_{rf}} \frac{\tilde{x}'(L) x_c'(L)}{\sqrt{1 + x_c'^2(L)}}\; dL \left(1 - \frac{\Delta v}{v_c}\right) \qquad \text{(A-10)}$$

where $\tilde{L} = \int_0^{L_{rf}} \sqrt{1 + x_c'^2(L)}\; dL$ is the effective length of the element.

Equation (A-10) is in first-order approximation correct for arbitrary reference line $x_c(L)$ and displacement $\tilde{x}(s)$ of the particle from this reference line. In general, Eq. (A-10) couples the longitudinal and transverse motion of the particles.

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