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THE SPONTANEOUS BREAKDOWN OF CHIRAL SYMMETRY IN SEMI-CLASSICAL METHOD (I). BOUNDARY CONDITION ON THE PATH INTEGRAL

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ABSTRACT : It is suggested that the usual path integral representation of Euclidean vacuum amplitude in QCD must be supplemented by the explicit boundary condition corresponding to the spontaneous breaking of chiral SU(N) x SU(N). The analogy with quantum mechanical example naturally lead to the trial wave function of Nambu and Jona-Lasinio and this in trun gives the starting point for the self-consistent calculation.

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I. INTRODUCTION

One of the popular methods in the theoretical study of quantum chromodynamics (QCD) is (or used to be) the semi-classical method and its generalizations [1].

Such an approach gives very good qualitative description when applied to the quantum mechanics problems [2] , even if used in a rather crude form. But this technics seems to suffer various defects when applied to the problems of quantum field theory in general, and Q.C.D. in particular.

In latter case, while there have been the series of works on the possibility of the spontaneous breaking of chiral symmetry and the generation of quark masses [3], [4] many of which follow the classical observation of 't Hooft [5], one is also worried by the fact that, as soon as one tries to analyze the situation by semi-classical method even in its most general form [6], one gets the results completely contrary to the expectation [7].

It was Crewther who examined this and related problems ("U(1) problem") in the greatest detail [8] and his conclusion was that, even if one is to reject the most general assumption of semi-classical method such as the importance of classical solutions with finite Euclidean action, and thus the whole idea of integer topological numbers, one is still left with quite severe chiral selection rules which may minimize the significance of "gauge non invariance" of U(1) axial charge. Thus, in spite of observation by 't Hooft [5], one would be in difficulty so long as one does not admit the unwanted U(1) Goldstone boson [8], [9].

On the other hand, recently there appeared the series of works based on 1/N expansion [10] of QCD which have shown that the appearance of U(1) Goldstone boson, after all, may not be so disastrous and one can get on quite happily with normal current algebra type phenomenology as long as one does not really insist on the quantitative explanations of, for instance, η' mass or, indeed, pion decay constant [11].

At the same time, Witten has shown the possible unreliableness of semi-classical method in the problems of quantum field theory [12]. If one defines the semi-classical method as the Gaussian expansion around the arbitrary (well separated) real minima of the Euclidean action, Witten's idea was confirmed by the exact calculation by Lüscher and Berg on the special model [13], [14]. It is quite possible that one must interpret η' as U(1) Goldstone boson [11] [12] [15] and moreover that one cannot ask for the quantitative explanation beyond the consistency argument offered by 1/N approximation [11] [12].

However, even if the most familiar method of the dilute gas approximation is shown to be definitely misleading in some cases [13] [14], there seems to be still quite a few unsolved problems as well as the possibilities of computational improvement in the semi-classical technics in field theory [16].

In the following note, I would like to present the arguments to show that the conclusion of Crewther and others is not the most general one which one can expect within the framework of conventional QCD. Even the seemingly clear-cut conclusion from dilute gas approximation [7] of QCD may originate from the way in which basic "path integral" representation is written down without due regard for the boundary conditions.

The most "simple minded" semi-classical approximation consists in starting from the path integral representation of the Euclidean expectation value of operator (or the product of operators)

$\chi\,(\psi, \bar{\psi}, A_\mu)$ with respect of so-called θ-vacuum,

$$\langle \chi(\psi, \bar{\psi}, A_\mu) \rangle_\theta = \frac{1}{N} \int \mathcal{D}A_\mu\, e^{-S_\epsilon^{YM}} \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\ e^{-\frac{i\theta}{16\pi^2}\int \mathrm{Tr}\, {}^*FF}\ e^{-S_\epsilon^F} \cdot \chi\,(\psi, \bar{\psi}, A_\mu) \qquad (1)$$

(: Normalization factor)

where

$$S_\epsilon^{YM} = \text{pure Yang-Mills action} = \frac{1}{2g^2}\int d^4x\ \mathrm{Tr}\ \hat{F}_{\mu\nu}\hat{F}^{\mu\nu}$$

$$S_\epsilon^F = \int d^4x\ \bar{\psi}_{su}\delta_{st}(\delta_{uv}\partial_\mu + (\hat{A}_\mu)_{uv})\,\gamma^{\mu\prime}\psi_{tv} + \sum_{s \geq L+1}\int d^4x\ \bar{m}_s \bar{\psi}_{su}\psi_{su} \qquad (2)$$

with

$$\hat{A}_\mu = \frac{g}{i} A_\mu^a \cdot T^a \qquad (3)$$

$$[T^a, T^b] = i f^{abc}\, T^c \qquad (4)$$

$$\hat{F}^{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + [\hat{A}_\mu, \hat{A}_\nu]$$

and

$${}^*\hat{F}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\hat{F}_{\lambda\sigma}$$

The Euclidean γ matrices $\{\gamma_\mu'\}_{\mu=1}^{4}$ satisfy

$$\{\gamma_\mu', \gamma_\nu'\} = 2\delta_{\mu\nu}$$

The indices s,t... and u,v... refer to the flavours and colours of quarks and run for $1,\ldots,N_f$ and $1,\ldots,N$ respectively. The letter L in the second term of S_ϵ^F (3) represents the number of light quarks. Physically $L \sim 2$ [17].

Then assume the semi-classical boundary condition on the gauge fields integration

i.e. Assume that $\hat{A}_\mu(x)$ reduces to the pure gauge at [1] $|x| \to \infty$.

$$A_\mu(x) \to g^{-1}(x)\,\partial_\mu\, g(x) \quad \text{as } |x| \to \infty \tag{5}$$

This implies that the integration $\int \mathcal{D}A_\mu$ can be expressed as the sum of configurations with integer Pontryagin number, i.e.

$$\int \mathcal{D}A_\mu = \sum_{\nu=-\infty}^{\infty} \int [\mathcal{D}A_\mu]_{-\frac{1}{16\pi^2}\int \mathrm{Tr}\,{}^*\!\hat{F}\hat{F} = \nu} \tag{6}$$

Thus, the θ-vacuum expectation value (1) reduces to the Fourier series

$$\langle X \rangle_\theta = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} \langle X \rangle_\nu \tag{7}$$

where

$$\langle X \rangle_\nu = \frac{1}{N'} \int \mathcal{D}A_\mu \cdot \delta\left(-\frac{1}{16\pi^2}\int \mathrm{Tr}\,{}^*\!\hat{F}\hat{F}\, d^4x - \nu\right)$$
$$e^{-S_E^{YM}} \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; X \cdot e^{-S_E^F} \tag{8}$$
$$\nu = 0, \pm 1, \pm 2, \ldots$$

The trouble is that the calculation of (1) through (7) and (8) can cause the seeming desaster, as pointed out by several authors [1,7].

One can "diagonalize" the action S_E^F by introducing the gauge field dependent "quark variables" $(\psi_n, \bar{\psi}_n)$ corresponding to the Euclidean Dirac eigen value problems

$$(\partial\!\!\!/ + A\!\!\!/)\,\phi_n = \lambda_n \phi_n \tag{9}$$

and expansion of the fermionic co-ordinates [1]

$$\psi(x) = \sum_n \xi_n \phi_n(x)$$
$$\bar{\psi}(x) = \sum_n \phi_n^{*}(x) \eta_n \qquad (10)$$

One defines the integral over fermion fields $\int d\psi d\bar{\psi}$ as

$$\prod_x \int d\psi d\bar{\psi} = \prod_n \int d\eta_n d\xi_n \qquad (11)$$

The action S_E^F now takes the form

$$S_E^F = \sum_{\lambda_n \neq 0} \lambda_n \eta_n \xi_n \qquad (12)$$

Now the eigenvalue equation (9) in general has several solutions with $\lambda_0 = 0$ (zero modes). Number of zero modes is related to the Pontrjagin number ν of gauge fields, as

$$\mathcal{N} = \mathcal{N}_+ + \mathcal{N}_- \qquad (13)$$

with $\nu = \mathcal{N}_+ - \mathcal{N}_-$

where $\mathcal{N}_\pm$ represents the number of independent solutions of

$$(\partial + A)\phi = 0$$

with + ve or - ve chiriality.

$$\gamma_5 \phi_\pm = \pm \phi_\pm \qquad (14)$$

The last statement is the consequence of celebrated Atiyah-Singer's theorem.

Since the zero mode variables $(\xi_0^{\nu}, \eta_0^{\nu})_{\nu=1}^{\mathcal{N}}$ do not appear in the action S_e^F (of (12)), the integral

$$\int d\psi \, d\bar{\psi} \cdots\cdots$$

$$= \prod_{\nu=1}^{\mathcal{N}} \int d\eta_0^\nu d\xi_0^\nu \prod_{n\neq 0} \int d\eta_n d\xi_n \cdots\cdots$$

vanishes unless enough numbers of zero mode variables are supplied from the "integrand" $X(A_\mu, \psi, \bar{\psi})$.

Note that the number of "light" flavours L increases the number of zero modes of (4) from $\mathcal{N}$ to $L\mathcal{N}$

Now, for instance turn to the calculation of vacuum expectation value of

$$X^\pm = \bar{\psi}(1 \pm \gamma_5)\psi$$

which is supposed to be the measure of spontaneous beaking of chiral symmetry. In (7), the sector $\nu = 0$ does not contribute to

$$\langle X^\pm \rangle = \langle \bar{\psi}(1 \pm \gamma_5)\psi \rangle$$

because in this sector one has rigorous chiral selection rule

$$\chi(X) = 0 \qquad (15)$$

But for all other sectors, $|\nu| \geqslant 1$, the numbers of zero mode integration $\prod_{\nu=1}^{\mathcal{N}} \int d\eta_0^\nu d\xi_0^\nu$

$$2\mathcal{N} \geqslant 2L \qquad (16)$$

Thus, for the "physical" case of $L = 2$, the bilinear operators $\bar{\psi}(1 \pm \gamma_5)\psi$ can never absorbe all the zero mode integral and thus

$$< \bar{\psi}(1 \pm \gamma_5)\psi >_\nu = 0 \tag{17}$$

i.e. from (7)

$$< \bar{\psi}(1 \pm \gamma_5)\psi >_\theta = 0 \tag{18}$$

Of course, if one has taken enough numbers of these operators, e.g. $(\bar{\psi}(1 \pm \gamma_5)\psi)^L$, then one would not get the trivial zero for the integral. This means that chiral symmetry is still broken by the presence of non trivial topological sector of gauge field. Only, the chriality can be changed only by the large value

$$\Delta \chi = 2L \tag{19}$$

This kind of "paradoxe" is known for the long time and usually dismissed [1] as the faute of semi-classical nature of the calculation. On the other hand, since only explicit assumption here is the vacuum boundary condition on $\int \mathfrak{D} A_\mu$

$$\int \mathfrak{D} A_\mu = \sum_{\nu=-\infty}^{\infty} \int [\mathfrak{D} A_\mu]_\nu$$

it is not easy to set up alternative scheme which allows more meaningful results, in particular, to reach the spontaneous breaking of chiral symmetry $<\bar{\psi}\psi> \neq 0$.

There is also the difficulty that the resultant chiral selection rule (19) can be obtained under more general assumptions, even when the "topological" number ν is not restricted to the integer [8,9]. For instance, Crewther[8] has carried out very detailed analysis using only the current algebra with anomaly and it looks as if the naîve suggestion from the path integral method is closely followed.[19]

The simplest result of "naîve" analysis described above is when $\chi = 1$, i.e. one is dealing with simple Euclidean vacuum transition amplitude

$$\langle 1 \rangle_\theta \sim \lim_{t\to\infty} \langle e^{-Ht} \rangle_0$$

In this case, the counting zero mode immediately gives (in (7))

$$\langle 1 \rangle_\nu = 0 \qquad , (20)$$

for $\nu \neq 0$. This usually is interpreted as the absence of vacuum tunneling in the presence of massless fermion. But the more correct interpretation [41] would be that the vacuum of massless fermion is not stable in the presence of gauge fields with instantons (i.e. $\nu \neq 0$) Thus, after tunneling, it finds itself in the state with several pairs of real massless quark and anti-quarks. This means that the true vacuum in Q.C.D. cannot be expressed as the small perturbation of the Fock vacuum of massless quarks but rather the superposition of quark -antiquark states. Such situation is familiar in the many body theory and in fact the bases of B.C.S. theory of superconductivity [29].

In the following section, I shall analyze the origin of spontaneous symmetry breaking in a simple model from quantum mechanics (can be taken the most primitive kind of B.C.S. model) and establish the correct "path integral" representation for such model. The last section is devoted for the suggestion for Q.C.D. case.

II. THE PATH INTEGRAL IN THE QUANTUM MECHANICS

1/ A MODEL

To illustrate the possible modification to the path integral formalism of QCD, I would like to discuss a simpler model from quantum mechanics [38], [39] which shows the spontaneous symmetry breaking in the limit of ∞ degree of freedom.

Let us consider the system of $2N\Omega$ fermionic oscillators $\{a_s(k), b_s(k)\}$ with anti-commutation relations ($s = 1,\ldots,N$ and k can take Ω different values)

$$\{a_s(k), a^*_{s'}(k')\} = \delta_{ss'}\delta_{kk'}$$

$$\{b_s(k), b^*_{s'}(k')\} = \delta_{ss'}\delta_{kk'}$$

and

$$\{a_s(k), a_{s'}(k')\} = 0$$

$$\{b_s(k), b_{s'}(k')\} = 0$$

$$\{a_s(k), b^*_{s'}(k')\} = 0 \qquad \text{etc.} \tag{21}$$

which are coupled by the interaction hamiltonian

$$H = -\frac{g}{2\Omega}\sum_{\alpha=0}^{N^2-1}\left[J_\alpha^+ J_\alpha^- + J_\alpha^- J_\alpha^+\right] \tag{22}$$

where

$$J_\alpha^+ = \sum_k a_s^*(k)\,(\lambda_\alpha)_{ss'}\,b^*_{s'}(-k) = (J_\alpha^-)^* \tag{23}$$

The N^2 matrix λ_α satisfies

$$\lambda_0 = \sqrt{\frac{2}{N}}\;\mathbb{1}^{[N]}$$

$$\left.\begin{aligned} \mathrm{Tr}\,\lambda_i &= 0 \\ \mathrm{Tr}\,\lambda_i \lambda_j &= 2\delta_{ij} \end{aligned}\right\} \quad i, j = 1, 2 \ldots N^2-1 \tag{24}$$

These are the U(N) generalization of Gell'Mann's λ -matrix for SU(3)
They satisfy the completeness relation

$$\sum_{\alpha=0}^{N^2-1} (\lambda^\alpha)_{ss'} (\lambda^\alpha)_{tt'} = 2\,\delta_{st'}\,\delta_{s't} \tag{25}$$

The hamiltonian (22) is invariant under the transformations

A)

$$\begin{aligned} a_s(k) &\to (e^{-i\phi\cdot\lambda})_{ss'}\, a_{s'}(k) \\ b_s(k) &\to b_{s'}(k)\, (e^{i\phi\cdot\lambda})_{s's} \end{aligned}$$

("Diagonal subgroups") (26)

B)

$$\begin{aligned} a_s(k) &\to (e^{-i\phi'\cdot\lambda})_{ss'}\, a_{s'}(k) \\ b_s(k) &\to b_{s'}(k)\, (e^{-i\phi'\cdot\lambda})_{s's} \end{aligned}$$

("Chiral subgroups") (27)

A) and B) form a symmetry group of chiral U(N) x U(N).

One can show that, in the limit of $\Omega \to \infty$, the symmetry A) is preserved while chiral symmetry B) is "spontaneously broken". To prove this, one can, in principle, solve the model for finite Ω exactly exploiting the SU(2N) classification of the states. Then one can show that the energies of one part of "ground level" become degenerate for large Ω while the another part "run away" to ∞ .

However, to see the behaviour of the system in $\Omega \to \infty$ limit, it is much simpler to adopt the exact methoa proposed by Haag

[40] for the study of BCS theory. Here, of course, I leave all the mathematical detail for the relevant literature.

First of all, one notes that the operator [40]

$$Q_{\pm}^{\alpha} = \lim_{\Omega \to \infty} \frac{1}{\Omega} J_{\pm}^{\alpha} \tag{28}$$

commutes with arbitrary element of algebra

$$\sum f_{k,s} a_{k,s} + \sum g_{k,s} b_{k,s}$$

$$\sum f'_{k,s} a^{*}_{k,s} + \sum g'_{k,s} b^{*}_{k,s}$$

$$\sum |f_{k,s}|^2 < \infty \qquad \sum |g_{k,s}|^2 < \infty \tag{29}$$

This means that in the limit of $\Omega \to \infty$, $Q_{\pm}^{\alpha}$ are C-numbers in any given irreducible representation of the original algebra. (Schur's Lemma).

As $\Omega \to \infty$

$$Q_{\pm}^{\alpha} \to \Delta_{\pm}^{\alpha} \quad (\text{C-number}) \tag{30}$$

Then, taking the commutators between H and elements of algebra, one has

$$[H, a_s(k)] = g \sum_{\alpha=0}^{N^2-1} Q_{\alpha}^{-} (\lambda_{\alpha})_{ss'} b^{*}_{s'}(-k)$$

$$[H, a_s^{*}(k)] = -g \sum_{\alpha=0}^{N^2-1} Q_{\alpha}^{+} b_{s'}(-k) (\lambda_{\alpha})_{s's}$$

$$[H, b_s(-k)] = -g \sum_{\alpha=0}^{N^2-1} Q_{\alpha}^{-} a^{*}_{s'}(k) (\lambda_{\alpha})_{s's}$$

$$[H, b_s^{*}(-k)] = g \sum_{\alpha=0}^{N^2-1} Q_{\alpha}^{+} (\lambda_{\alpha})_{ss'} a_{s'}(k) \tag{31}$$

From these, one can see that the hamiltonians H can be written, in the limit of $\Omega \to \infty$ and within a given irreducible representation.

$$H \to H' = -g \left[\Delta_-^{\alpha} \sum_{k} a_s^*(k) \lambda_{ss'}^{\alpha} b_{s'}^*(-k) + \Delta_+^{\alpha} \sum_{k} b_{s'}(-k) \lambda_{s's}^{\alpha} a_s(k) \right] + \text{const}. \tag{32}$$

where $\Delta_{\pm}^{\alpha}$ are the constant depending on the particular irreducible representation. The Hamiltonian (32) is bilinear and can be diagonalized by canonical transformation of creation and annihilation operators.

First define the "chiral phase" by writing

$$a_s(k) = (e^{-i\beta\cdot\lambda})_{ss'} a'_{s'}(k)$$
$$b_s(k) = b'_{s'}(k) \, (e^{-i\beta\cdot\lambda})_{s's}$$

(Chiral transformation) (33)

with parameter β_α such that

$$e^{i\beta\cdot\lambda} (\Delta_-^{\alpha} \lambda^{\alpha}) e^{i\beta\cdot\lambda}$$
$$= e^{-i\beta\cdot\lambda} (\Delta_+^{\alpha} \lambda^{\alpha}) e^{-i\beta\cdot\lambda}$$

$$= \mathcal{A} \quad \text{(hermitian and +ve definite)} \tag{34}$$

One can always find such a unitary transformation $e^{i\beta\cdot\lambda}$.

Next, one diagonalizes $\mathcal{A}$ by

$$a'_s(k) = (e^{-i\alpha\cdot\lambda})_{ss'} a''_{s'}(k)$$

$$b_{s'}(k) = b^{*}_{s'}(k)\,(e^{i\alpha\cdot\lambda})_{s's}$$

(Diagonal subgroup) (35)

so that

$$e^{i\alpha\cdot\lambda}\mathcal{A}\,e^{-i\alpha\cdot\lambda} = \begin{bmatrix}\mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n\end{bmatrix} \qquad \mu_i > 0 \tag{36}$$

Lastly, the canonical transformation proper,

$$\begin{aligned} a''_s(k) &= \cos\gamma\,\tilde{a}_s(k) + \sin\gamma\,\tilde{b}^{*}_s(-k) \\ b''_s(-k) &= \cos\gamma\,\tilde{b}_s(-k) - \sin\gamma\,\tilde{a}^{*}_s(k) \end{aligned} \tag{37}$$

If one assumes $\Delta_\alpha^{*}\lambda_\alpha \neq 0$, then one can see that the choice

$$\begin{cases} \cos^2\gamma = \sin^2\gamma = \frac{1}{2} \\ \cos\gamma\sin\gamma = \frac{1}{2} \end{cases} \qquad (\text{ie } \gamma = \pi/4\ ) \tag{38}$$

reduces H' to the +ve diagonal form

$$H' = g\sum_s \mu_s\Big(\sum_k \hat{a}^{*}_s(k)\,\hat{a}_s(k) + \hat{b}^{*}_s(k)\,\hat{b}_s(k)\Big) + \text{const} . \tag{39}$$

Clearly, the ground state of hamiltonian H' is the Fock vacuum $|\psi_0\rangle$ of new operators $\hat{a}_s(k)$ and $\hat{b}_s(k)$

$$\begin{aligned} \hat{a}_s(k)\,|\psi_0\rangle &= 0 \\ \hat{b}_s(k)\,|\psi_0\rangle &= 0 \end{aligned} \tag{40}$$

Then $|\psi_0\rangle$ can be represented as the <u>formal</u> coherent state in terms of original Fock states,

$$
\begin{aligned}
|\Psi_0\rangle &= \prod_{k,s} [\cos\gamma + \sin\gamma\, a_s^{*}(k)\, b_{s'}^{*}(-k)]\,|0\rangle \\
&= (\cos\gamma)^{N\Omega} \prod_{k} (1+\tan\gamma\, a_s^{*}(k)(e^{-2i\beta\lambda})_{ss'}\, b_{s'}^{*}(-k))]\,|0\rangle \\
&= (\cos\gamma)^{N\Omega} \exp \sum_{k,s,s'} \{\tan\gamma\; a_s^{*}(k)(e^{-2i\beta\lambda})_{ss'}\, b_{s'}^{*}(-k)\}\,|0\rangle
\end{aligned}
\tag{41}
$$

where $|0\rangle$ is the Fock vacuum of original creation and annihilation operators

$$
\begin{aligned}
a_s(k)\,|0\rangle &= 0 \\
b_s(k)\,|0\rangle &= 0
\end{aligned}
\tag{42}
$$

The parameter β_α is the same as in (33).
In our case, $\gamma = \pi/4$ and

$$
|\Psi_0\rangle = \left(\frac{1}{\sqrt{2}}\right)^{N\Omega} \exp \sum_{k,s,s'} \{a_s^{*}(k)\,(e^{-i\beta\lambda})_{ss'}\, b_{s'}^{*}(-k)\}\;|0\rangle
$$

One solution (39) still contains unknown coefficients μ_i, i.e. $\Delta_\alpha^{\pm}$. This can be easily found out by calculating vecuum expectation value in the limit of $\Omega\to\infty$.

$$
\begin{aligned}
\Delta_\alpha^{\pm} &= Q_\alpha^{\pm} \quad (\sim \text{c-number}) \\
&= \langle\Psi_0|\, Q_\alpha^{\pm}\,|\Psi_0\rangle \\
&= \lim_{\Omega\to\infty} \frac{1}{\Omega}\,\frac{1}{2}\,\Omega\; \mathrm{Tr}\,[e^{\pm i\beta\lambda}\lambda_\alpha e^{\pm i\beta\lambda}] \\
&= \frac{1}{2}\,\mathrm{Tr}\left(e^{\pm i\beta\lambda}\lambda_\alpha e^{\pm i\beta\lambda}\right)
\end{aligned}
\tag{43}
$$

Thus $$\mathcal{M} = e^{-i\beta\cdot\lambda}\sum_{\alpha}\Delta_{\alpha}^{+}\lambda_{\alpha}e^{-i\beta\cdot\lambda} = \frac{1}{2}\sum_{\alpha}\mathrm{Tr}\left(e^{i\beta\cdot\lambda}\lambda_{\alpha}e^{i\beta\cdot\lambda}\right)$$
$$\times\left(e^{-i\beta\cdot\lambda}\lambda_{\alpha}e^{-i\beta\cdot\lambda}\right) = \mathbb{1} \tag{44}$$

where one has used the completeness relations of λ -matrices (25). Thus $\mu_s = 1$ for all S, and
$$\sum_{\alpha}\Delta_{\alpha}^{\pm}\cdot\lambda^{\alpha} = e^{\pm 2i\beta\cdot\lambda} \tag{45}$$

The undetermined parameters $\{\beta_{\alpha}\}_{\alpha=0}^{N^2-1}$ of U(N) group space represents the multiplicity of irreducible components in the limit of $\Omega\to\infty$.

In each irreducible sectors, the reduced hamiltonian has the same form
$$H' = g\sum_{s}\sum_{k}\left(\widehat{a}_s^{*}(k)\widehat{a}_s(k) + \widehat{b}_s^{*}(k)\widehat{b}_s(k)\right)$$

but they can be distinguished by the vacuum expectation values of appropriate operators, e.g.
$$\sum_{\alpha}{}_{\beta}\langle\Psi_0|\,Q_{\alpha}^{\pm}\,|\Psi_0\rangle_{\beta}\,\lambda_{\alpha} = e^{\pm 2i\beta\cdot\lambda}$$

Thus, one has the degeneracy of vacuum and the spontaneous breakdown of the chiral part of U(N) x U(N) symmetry. For given $\{\beta_{\alpha}\}$, the each irreducible subspace is still invariant under the U(N) subgroups given by the elements of transformation
$$a_s(k)\to\left(e^{-i\beta\cdot\lambda}e^{-i\alpha\cdot\lambda}e^{i\beta\cdot\lambda}\right)_{ss'}a_{s'}(k)$$
$$b_s(k)\to b_{s'}(k)\left(e^{i\beta\cdot\lambda}e^{i\alpha\cdot\lambda}e^{-i\beta\cdot\lambda}\right)_{s's}$$
$$(\beta_{\alpha}\ ;\ \text{given}) \tag{46}$$

The degeneracy can be removed if one adds the small perturbation

$$\Delta H = \varepsilon\mu \sum_{k} a_s^{*}(k)\left(e^{-2i\theta\cdot\lambda}\right)_{ss'} b_{s'}^{*}(-k) \tag{47}$$

Then the chiral transformation (33) to diagonalize the hamiltonian in the limit $\Omega \to \infty$ will be fixed as

$$e^{2i\beta\cdot\lambda} = e^{2i\theta\lambda}$$

and $Q_\alpha^{\pm}$ will take the well determined vacuum expectation value. The system is invariant under the $U(N)$ transformation (46). This is the equivalent of Dashen's theorem [42] in current algebra.

2/ THE PATH INTEGRAL FORMALISM [39]

As it is explained in the Appendix, it is easy to represent the Euclidean amplitudes with fermionic degree of freedom as the path integral (functional integral) on the appropriate Grassmann variables.

The problem is to define the path integral so that one can go smoothly to the symmetry breaking solutions in the limit $\Omega \to \infty$.

For instance, the naïve prescription suggests the representation [24] for the vacuum expectation value of the operators $X(a_s(k), b_s(k))$

$$\text{"}\langle X(t) \rangle\text{"} = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \langle e^{-H(t''-t)} X \, e^{-H(t-t')} \rangle$$

$$= \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \prod_{t=t'}^{t''} \prod_{\alpha, a, t} \prod_{k, s} \int d\eta_t^{\alpha}(k, s)\, d\xi_t^{\alpha}(k, s)$$

$$\exp - \int_{t'}^{t''} \Big[\sum \eta_t^{\alpha}(k, s) \frac{d}{dt} \xi_t^{\alpha}(k, s)$$

$$- \frac{g}{2\Omega} : J_+ J_- + J_- J_+ : (\eta_t, \xi_t) \Big] \times X(\eta_t, \xi_t)$$

$$\times 1/Z$$

(Z : normalization factor) (48)

Taking, for instance, $X = \frac{1}{\Omega} J_{\alpha}^{+}$, one can see easily that R.H.S. of (48) will not go to the symmetry breaking solutions in the limit $\Omega \to \infty$. In fact, for finite Ω,

$$"\langle X(t) \rangle" = 0 \qquad (49)$$

unless X(t) = invariant under the chiral transformation (27). Thus, in particular, $"\langle \frac{1}{\Omega} J_{\alpha}^{+} \rangle" = 0$ (for any finite Ω). Therefore, this quantity remains zero also in the limit of inifnite Ω. The addition of symmetry breaking term (48) does not help because then one can show that

$$\lim_{\delta\mu \to 0} \lim_{\Omega \to \infty} "\langle \frac{1}{\Omega} J_{\alpha}^{+} \rangle_{\text{in presence of } \Delta H}" = 0 \qquad (50)$$

The situation is a little different from the case of certain field theoretical model such as $\lambda(\phi^2 - c^2)^2$ potential in two dimensions, where one can demonstrate the existence of spontaneous symmetry breaking by adding small symmetry breaking term (or the "external magnetic field").

In the present case, the simple minded path integral (48) represents the expectation value with respect to the Focks vacuum of original creation and annihilation operators and not the mixture of the symmetry breaking states such as in case of scalor model in two dimen sion.

On the other hand, one can obtain the correct vacuum expectation value by applying the path integral representation for arbitrary matrix elements given in Appendix (A.22). As it has been shown in (41), the true vacuum $|\Psi_0\rangle$ can be formally expressed as the coherent state in terms of original Fock states. This expression contains the divergent coefficient as $\Omega \to \infty$. But this is not serious since such a factor can be cancelled in taking ratio with normalization factor, i.e. one can give the following path integral representation

$$
\begin{aligned}
&{}_\beta\langle \Psi_0 |\; X(t) \;| \Psi_0 \rangle_\beta \\
&= \frac{1}{Z} \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \prod_{t=t'}^{t''} \prod_{\alpha=\pm} \prod_{k,s} \int d\eta_t^{\alpha}(k,s)\, d\xi_t^{\alpha}(k,s) \\
&\qquad \exp \sum_k \xi_{t''}^{+}(-k,s)\,(e^{2i\beta\lambda})_{ss'}\,\xi_{t''}^{-}(k,s') \\
&\qquad \exp \sum_k \eta_{t'}^{-}(k,s)\,(e^{-2i\beta\lambda})_{ss'}\,\eta_{t'}^{+}(-k,s') \\
&\quad \exp\Big[-\int_{t'}^{t''} \Big[\sum_{\alpha,k,s} \eta_t^{\alpha}(k,s)\,\frac{d}{dt}\,\xi_t^{\alpha}(k,s) \\
&\qquad - \frac{g}{2\Omega} :(J_+ J_- + J_- J_+): (\eta_t, \xi_t) \Big] \\
&\quad \exp\Big[-\sum \eta_{t'}^{\alpha}(k,s)\,\xi_{t'}^{\alpha}(k,s) \Big] \\
&\quad X(\eta_t, \xi_t) \\
&\qquad\qquad \times\ 1/\big[\text{ same expression with } X \to 1 \big] \qquad (51)
\end{aligned}
$$

First two factors taken at the final and initial (Euclidean) time $t = t''$ and t' represent the wave function (and its conjugate) of the true vacuum $|\Psi_0\rangle_\beta$. (See (41)).

The last factor $\exp -\sum \eta_t^\alpha(k.s)\,\xi_t^\alpha(k.s)$ is of purely kinematical origin (See Appendix). Alternatively, one can write down the path integral directly in terms of new variables $\tilde{\xi}^\alpha$ and $\tilde{\eta}^\alpha$ corresponding to the canonically transformed operators $\tilde{a}$, $\tilde{b}$ and $\tilde{a}^\dagger$, $\tilde{b}^\dagger$ of (37). In this case, there is no question of vacuum wave function. However, now one can transform back the Grassmann variables $\tilde{\xi}^\alpha$ and $\tilde{\eta}^\alpha$ according to the canonical transformations (33), (35) and (37).

Then, one sees that the "kinematical part" of the Euclidean action will generate precisely the vacuum wave functions in (51), i.e

$$
\begin{aligned}
&\exp -\int_{t'}^{t''} \sum \tilde{\eta}_t^\alpha(k.s)\,\frac{d}{dt}\,\tilde{\xi}_t^\alpha(k.s) \\
&\times \exp -\sum_{k.s.\alpha} \tilde{\eta}_{t'}^\alpha(k.s)\,\tilde{\xi}_{t'}^\alpha(k.s) \\
&= \mathrm{const}\cdot \exp \int_{t'}^{t''} \sum_{k.s.\alpha} \eta_t^\alpha(k.s)\,\frac{d}{dt}\,\xi_t^\alpha(k.s) \\
&\quad \times \exp -\sum_{k.s.\alpha} \eta_{t'}^\alpha(k.s)\,\xi_{t'}^\alpha(k.s) \\
&\quad \times \Big\{ \exp \sum_k \xi_{t'}^a(-k.s)\left(e^{2i\beta\lambda}\right)_{ss'}\xi_{t'}^a(k.s') \\
&\qquad\quad \exp \sum_k \eta_{t'}^a(k.s)\left(e^{-2i\beta\lambda}\right)_{ss'}\eta_{t'}^c(-k.s') \Big\}
\end{aligned}
\tag{52}
$$

where the relations between $(\tilde{\eta}^\alpha, \tilde{\xi}^\alpha)_{\alpha=a,c}$ and $(\eta^\alpha, \xi^\alpha)_{\alpha=a,b}$ are defined through (33), (35) and (37).

With expression (51), one can study the spontaneous symmetry breaking of chiral U(N). For instance, changing the integration variables according to the local (time wise) version of chiral transformation (27), one obtains the generalized charge conservation

$$\langle \Psi_0 | \int_{-\infty}^{\infty} dt \frac{d}{dt} \Big\{ \sum a_s^*(k;t) (\lambda_\alpha)_{ss'} a_{s'}(k;t) - \sum b_s(k;t) (\lambda_\alpha)_{ss'} b_{s'}^*(k;t) \Big\}$$

$$, \quad X(t_0) | \Psi_0 \rangle_\theta$$

$$= i \Omega_{\alpha\beta}(\theta) \frac{\partial}{\partial \theta_\beta} {}_\theta\langle \Psi_0 | X(t_0) | \Psi_0 \rangle_\theta \tag{53}$$

where $\Omega_{\alpha\beta}$ is the $N^2 \times N^2$ matrices defined by

$$e^{-2i\theta'\lambda} = e^{i\lambda\delta\phi} e^{-2i\theta\lambda} e^{i\lambda\delta\phi}$$

$$\delta\theta_\alpha = (\theta' - \theta)_\alpha = \delta\phi_\beta \Omega_{\beta\alpha}(\theta) + O(\delta\phi^2) \tag{54}$$

$a_s(k,t)$, $b_s(k,t)$, etc., are the abuse of notation meaning

$$e^{Ht} a_s(k) e^{-Ht} , \qquad e^{Ht} b_s(k) e^{-Ht}$$

etc.

One may remark that the transformation such as (54) has appeared in the non-linear realization of chiral symmetry in the current algebra of Weinberg and Coleman, Weiss and Zumino [43].

I have discussed a way to promote the simple model of this section to relativistic field thaory in Appendix B.

III. THE PATH INTEGRAL IN QCD

The discussions at the end of last section suggest that the way to modify the path integral (1), so that the resultant chiral selection rules may be less severe, is to add the non trivial wave function which can contribute to the Euclidean path integral in the limit $t' \to +\infty$ and $t' \to -\infty$. Such a wave function must be able to induce the system to fall into one of degenerate vacua and thus must contain the germ of chiral SU(N) symmetry breaking in itself.

The discussion of Section I shows that the non trivial boundary condition on the gauge field integration $\mathcal{D}A_\mu$, although it breaks chiral U(1) symmetry [30], does not have enough symmetry breaking in it. Thus the simplest possibility would be to look for the wave functions which depend on the "fermionic" variables ψ and $\bar{\psi}$ at the boundary surface $t \to \mp\infty$. From the way in which our path integral is defined (i.e. as the generalization of (A.22) to infinitely many degrees of freedom), this wave function should express the relationship between the Fock vacuum of massless quarks and antiquarks and the true physical vacuum where the chiral symmetry is spontaneously broken and quarks are massive.

Now, just such a relationship has been considered in the classical paper by Nambu and Jona-Lasinio [31] introducing for the first time the "Goldstone pions" in the theory of strong interaction.

According to these authors, the chiral symmetry is spontaneously broken through the "super conducting" states where the massless quark and anti-quark pairs (nucleon-antinucleon of Ref. 31) of same helicity and opposite momenta form the "Cooper pairs" [35].

In analogy with the coherent trivial states of Refs.[26]

and [27], Nambu and Jona-Lasinio give the explicit expression in the simplest case of $L = 1$ [32]. (See also Appendix B.12).

$$|\Omega^m\rangle = \prod_{p,\lambda} \left\{ \sqrt{\tfrac{1}{2}(1+\beta_p)} + \sqrt{\tfrac{1}{2}(1-\beta_p)}\; a^*(p,\lambda)\, b^*(p,\lambda) \right\} |\Omega^0\rangle \tag{55}$$

p , quark momentum

λ , helicity

and $\beta_p = |p| / \sqrt{p^2 + m^2}$ (m : parameter),

$|\Omega^0\rangle \equiv |0\rangle$ is the Fock vacuum of the massless "nucleons" or quarks

$$a(p,\lambda)\,|\Omega^0\rangle = 0$$
$$b(p,\lambda)\,|\Omega^0\rangle = 0 \tag{56}$$

Writing
$$\sin\theta(p) = \sqrt{\tfrac{1}{2}(1-\beta_p)}$$
$$\cos\theta(p) = \sqrt{\tfrac{1}{2}(1+\beta_p)}$$

one sees that the formula (42) corresponds to the Bogoliubov transformation

$$\tilde{a}(p,\lambda) = \cos\theta(p)\, a(p,\lambda) + \sin\theta(p)\, b^*(-p,\lambda)$$
$$\tilde{b}(p,\lambda) = -\sin\theta(p)\, a^*(-p,\lambda) + \cos\theta(p)\, b(p,\lambda) \tag{57}$$

The new annihilation operators satisfy

$$\tilde{a}(p,\lambda)\,|\Omega^m\rangle = 0$$
$$\tilde{b}(p,\lambda)\,|\Omega^m\rangle = 0 \tag{58}$$

for all p and λ .

The parameter m , which is related to the Bogoliubov angle as

$$\tan\theta(p) = \left(\sqrt{p^2+m^2} - |p|\right)/m \tag{59}$$

corresponds to the spontaneously generated mass of quarks. This can be in principle calculated with self-consistent method [33]. (See Appendix B).

The chiral symmetry breaking trial state (55) of Nambu, Jona-Lasinio is of the form discussed in Section II. Moreover, if one calculates the overlap with Fock vacuum [31],

$$\langle 0|\Psi_0\rangle \equiv \langle\Omega^0|\Omega^m\rangle$$

$$= \exp 4\pi\int_0^\infty dp\; p^2 \ln\sqrt{1+\beta_p} = 0 \tag{60}$$

because the exponent is negative at large momentum and diverges linearly with the ultra-violet limit of integral.

Before writing down the modified path integral which should replace(1), I generalize the Nambu-Jona-Lasinio representation (55) to chiral $SU(N_f)$ with $N_f > 1$

$$|\Omega^{(m)}\rangle_{\Omega,\alpha} = \prod_{p,\lambda,\alpha}\Big\{\cos\theta(p)$$

$$+\sin\theta(p)\sum_{s,t} a^{\dagger}_{s}(p,\lambda)\left(e^{2i\lambda\cdot\Omega\cdot T}\right)_{st} e^{2i\lambda\alpha}\, b^{\dagger}_{t}(-p,\lambda)\Big\}|\Omega^0\rangle \tag{61}$$

where the angle $\theta(p)$ is chosen as before, and

$\{T_a\}_{a=1}^{N_f^2-1}$, the generators of $SU(N_f)$ in the quark representation

$\{\Omega_a\}_{a=1}^{N_f^2-1}$, parametrize vacuum degeneracy with respect to the chiral part of

α : parametrize vacuum degeneracy with respect to chiral U(1). Corresponding to the global chiral transformation of the field operators

$$e^{iQ_r\cdot\Omega'}\psi_{su}\,e^{-iQ_r\cdot\Omega'} = [\,e^{i\mathcal{I}\cdot\Omega'\gamma_r}]_{st}\psi_{tu} \tag{62}$$

One has

$$e^{iQ_r\cdot\Omega'}|\Omega^{(m)}\rangle_{\Omega,\alpha} = |\Omega^{(m)}\rangle_{\Omega'',\alpha}$$

where Ω'' is given by

$$e^{i\Omega'\mathcal{I}}\,e^{2i\Omega\mathcal{I}}\,e^{i\Omega'\mathcal{I}} = e^{2i\Omega''\cdot\mathcal{I}} \tag{63}$$

$|\Omega''\rangle$ also breaks chiral U(1) which amounts to the change of parameter

$$\alpha \to \alpha + \alpha' \tag{64}$$

Now I can put the wave function corresponding to the trial state (61) into (1) and obtain the following modified path integral representation of the vacuum expectation value of operator $X(\psi,\bar{\psi},A_\mu)$

$$\begin{aligned}
&\langle 0\,|\,X(\psi,\bar{\psi},A_\mu)\,|\,0\rangle_{\alpha,\Omega} \\
&= \prod_{x,\tau}\int \mathcal{D}A_\mu(x,\tau)\,e^{-S_E^{YM}}\int\mathcal{D}\psi(x,\tau)\,\mathcal{D}\bar{\psi}(x,\tau)\;e^{-\frac{i\theta}{8\pi^2}\int \mathrm{Tr}\,\tilde{F}F} \\
&\cdot \prod_{p,\lambda=\pm 1}\left\{\cos\theta(p)+\sin\theta(p)[\eta_t^{2}(-p,\lambda)]_{\tau=\infty}(e^{-2i\lambda\Omega\cdot\mathcal{I}})_{ts}[\xi_s^{1}(p,\lambda)]_{\tau=\infty}\right\} \\
&\cdot \prod_{p,\lambda=\pm 1}\left\{\cos\theta(p)+\sin\theta(p)[\eta_s^{1}(p,\lambda)]_{\tau=-\infty}(e^{2i\lambda\Omega\cdot\mathcal{I}})_{st}[\xi_t^{2}(-p,\lambda)]_{\tau=-\infty}\right\} \\
&\quad\cdot e^{-S_E^F}\cdot\; X(\psi,\bar{\psi},A_\mu)
\end{aligned}$$

$$\times \quad 1/(\text{normalization factor}) \tag{65}$$

The letters (θ, Ω) in LHS indicate the degeneracy of vacuum with respect to θ as well as the direction in the space of chiral transformation Ω, the Grassmann variables $[\xi^{1,2}_{su}(p,\lambda)]_\tau$ and $[\eta^{1,2}_{su}(p,\lambda)]_\tau$ correspond to the Fourier components of local 4-component Dirac variables $\psi(x,\tau)$ and $\psi^{*}(x,\tau)$ at the given Euclidean time τ. One can write

$$\psi_{su}(x,\tau) = \int d^3p\, e^{ipx}\Big\{[\xi^1_{su}(p,+1)]_\tau \begin{pmatrix} 0 \\ w^-(p)\end{pmatrix} + [\xi^1_{su}(p,-1)]_\tau \begin{pmatrix} w^+(p) \\ 0\end{pmatrix}$$

$$+ [\xi^2_{su}(-p,+1)]_\tau \begin{pmatrix} w^-(p) \\ 0\end{pmatrix} + [\xi^2_{su}(-p,-1)]_\tau \begin{pmatrix} 0 \\ w^+(p)\end{pmatrix}\Big\}$$

$$\psi^{*}_{su}(x,\tau) = \int d^3p\, e^{-ipx}\Big\{[\eta^1_{su}(p,+1)]_\tau (0, w^{-*}(p)) + [\eta^1_{su}(p,-1)]_\tau (w^{*}_{+}(p), 0)$$

$$+ [\eta^2_{su}(-p,+1)]_\tau (w^{*}_{-}(p), 0) + [\eta^2_{su}(-p,-1)]_\tau (0, w^{*}_{+}(p))\Big\}$$

$$(\psi^{*} = \bar{\psi}\gamma_4) \tag{66}$$

The vectors $\begin{pmatrix} 0 \\ w^{\pm}(p)\end{pmatrix}$, etc. are the massless spinors in the representation where γ_5 matrix is diagonal. One can choose, for instance,

$$w_+(p) = \sqrt{\frac{|p|+p_3}{2|p|}}\begin{pmatrix} 1 \\ \frac{p_1+ip_2}{|p|+p_3}\end{pmatrix} \qquad w_-(p) = \sqrt{\frac{|p|+p_3}{2|p|}}\begin{pmatrix} \frac{p_1-ip_2}{|p|+p_3} \\ 1\end{pmatrix}$$

and

$$w_+ \otimes w_+^{*} = \frac{1}{2}\left(1 + \frac{p\cdot\sigma}{|p|}\right)$$

$$w_- \otimes w_-^{*} = \frac{1}{2}\left(1 - \frac{p\cdot\sigma}{|p|}\right) \tag{69}$$

(See Appendix A)

Note that one can write the coherent state (61) as formal unitary transformation

$$|\Omega^m\rangle_{\underline{\Omega},\alpha} = \exp i\,G(a,b,a^*,b^*;\underline{\Omega}\ \alpha)\,|0\rangle$$

where

$$G(a,b,a^*,b^*;\underline{\Omega},\alpha) = i\int d^3p\,\theta(p)$$
$$\sum_{\lambda=\pm 1}\left\{ a^*_{s\alpha}(p,\lambda)\left(e^{2i\lambda\underline{\Omega}\cdot\underline{\tau}}\right)_{st} e^{2i\lambda\alpha} b^*_{t\alpha}(p,\lambda) - b_{t\alpha}(-p,\lambda)\left(e^{-2i\lambda\underline{\Omega}\cdot\underline{\tau}}\right)_{ts} e^{-2i\lambda\alpha} a_{s\alpha}(p,\lambda)\right\}$$

(68)

although it is not-so simple to introduce the object like (68), which is not normal ordered, into the path integral. (Normal ordered form of (68) is, of course, juste original (61)).

The "current algebra" vacuum of Nambu-Jona-Lasinio breaks chiral SU(N) x SU(N) according to (63). It also breaks the chiral U(1), i.e. under the global transformation

$$\psi(x) \to e^{i\alpha'\gamma_5}\psi(x)$$
$$\bar\psi(x) \to \bar\psi(x)\,e^{i\alpha'\gamma_5}. \qquad (69)$$

one has

$$|\Omega^m\rangle_{\underline{\Omega},\alpha} \Rightarrow |\Omega^m\rangle_{\underline{\Omega},\alpha+\alpha'} \qquad (70)$$

Just as in QCD Lagrangian of quarks and gluons, the invariance under the chiral SU(N) x SU(N) implies automatically the chiral U(1) invariance (unlike the Gell-Mann-Levy linear σ -model with σ , $\vec\pi$ and N), the spontaneous breaking of former (by coupling to the wave functions(48)) entails the breaking of the latter.

The factors $a^*(p,\lambda)\,b^*(-p,\lambda)$ (λ -helicity), in (61) have the chirality 2λ . So to study the chiral transformation property of (61), it is convenient to write this as

$$|\Omega^m\rangle = \prod_{p} \left[\cos\theta(p) + \sin\theta(p)\, \hat{U}^{+}(p) \right] \times \prod_{p} \left[\cos\theta(p) + \sin\theta(p)\, \hat{U}^{-}(p) \right] \tag{71}$$

where the operators $\hat{U}^{\pm}(p)$ transform as

$$U^{\pm}(p) \rightarrow e^{\pm 2i\alpha'} U^{\pm}(p)$$

under (69).

(71) means that the Nambu-Jona-Lasinio's trial states are the coherent superposition of chiral U(1) eigenstates

$$|\Omega^m\rangle = \sum_{\nu=-\infty}^{\infty} \hat{W}_{2\nu} |0\rangle \tag{72}$$

where

$$\hat{W}_{\chi} \rightarrow e^{i\chi_\alpha'} \hat{W}_{\chi}$$

under (69).

The vacuum expectation value according to the modified expression (65), $\langle\Omega^m| \cdots\cdots |\Omega^m\rangle$ can be written as

$$\sum_{\nu_f, \nu_i = -\infty}^{\infty} \langle 0| \hat{W}^{*}_{2\nu_f} \cdots\cdots \hat{W}_{2\nu_i} |0\rangle$$

where $|0\rangle$ is still "Fack vacuum" or the factor 1 in the path integral.

The assumption of spontaneous breaking of chiral $SU(N_F)\times SU(N_F)$ (or the current algebra) is that the chiral sectors $\langle 0|\hat{W}^{*}_{2\nu_f} \cdots \hat{W}_{2\nu_i}|0\rangle$ with more or less arbitrary values of ν_f and ν_i should be able to contribute to the vacuum amplitude, and not just the ones with $|\nu_f - \nu_i| = 2L\nu$.

More explicitely, with the path integral (65), one can, in principle, justify the normal self-consistency calculation for the spontaneous symmetry breaking. As it is discussed in more details in the Appendix B , the formal effect of the presence of wave function such as (61) is just to change the mass of fermions implied by the path integral from 0 to m. Thus the "free propagator" which is inherent in the path integral (65) is

$$\frac{1}{\not p - m}$$

and not $\frac{1}{\not p}$. Thus it is meaningful to transform the action

$$S_E^F = \bar{\psi}_0 (\not\partial + \not A) \psi_0$$

to

$$\bar{\psi}_m (\not\partial - m + \not A) \psi_m \quad + m \bar{\psi}_m \psi_m \qquad (73)$$

(0 and m refer to the inherent mass used to define the path integral, see Appendix A -(c)).

Thus the natural consequence of the decomposition such as (73) is the self consistency equation [3], [4], [33], [34]

Also, this gives the bases of mechanism for spontaneous symmetry breaking of chiral symmetry proposed by 't Hooft [5] Callan, Dashen and Gross [3] if this scheme can be freed from the defect of dilute gas approximation.

IV. SUMMARY AND CONCLUSION

In this note, I have attempted to trace the certain inconsistencies in the manipulation of the path integral formalism of QCD. Taking account of the boundary condition (expressed in terms of "inherent" mass) on theirfermionic degree of freedom, I have suggested that the semi-classical method, even with the assumption of integer topological number, could lead to the meaningful conclusion.

Owing to the still remaining ambiguity in the path integral with respect to the removal of ultra-violet cut off, it is of course not possible to set up the rigorous scheme in the manner of BCS theory. But one seems to be naturally lead to the self-consistency calculations of usual kind instead of the "disasters" like ref [7]. The latter is still plagned by the fact that, to get the definite conclusion, one must rely on the dilute gas approximation.

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- APPENDIX A -

-:-:-:-:-

PATH INTEGRAL REPRESENTATION

a)

The path integral representation of the Euclidean fermionic amplitude can be introduced in the completely algebraic way for the finite number of degree of freedom. I follow the method proposed by Fradkin et al. (25) for the study of Ising model. The result does not differ from the standard treatment such as Faddeev (25).

The method can be illustrated for the simplest example, i.e one component fermionic oscillator. For the more complicated case, a all one needs is careful combinatories and I will only give the result.

Let us consider the quantum system of fermionic oscillator with Hamiltonian

$$H = \omega a^{*} a \qquad \text{(A.1)}$$

where the creation and annihilation operators a^{*} and a satisfy anti-commutation relation

$$a^{*} a + a a^{*} = 1$$

$$a^{2} = a^{*2} = 0 \qquad \text{(A.2)}$$

The problem is trivial since the Hilbert space here is just two-dimensional complex vector space spanned by

$|0\rangle$ (defined by $a|0\rangle = 0$, Fock vacuum = real vacuum)

and

$|1\rangle = a^{*}|0\rangle$

The operator H is then equivalent to 2x2 diagonal matrix

$$H \sim \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}$$

But let us imagine that we do not use even the simplest matrix calculus and to calculate the amplitude

$$\langle 0 | \; e^{-H(t''-t')} \; | 0 \rangle \tag{A.3}$$

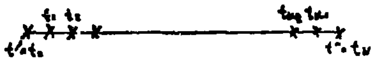
(Euclidean vacuum amplitude)

One has to apply the standard procedure for calculating quantum amplitude. First, one writes (3) as

$$\begin{aligned} &\langle 0 | \; e^{-H(t''-t')} | 0 \rangle \\ &= \sum_{n_N=0}^{1} \cdots \sum_{n_0=0}^{1} \langle 0 | n_N \rangle \langle n_N | \; e^{-H\Delta t} \; | n_{N-1} \rangle \\ &\times \langle n_{N-1} | e^{-H\Delta t} | n_{N-2} \rangle \cdots \langle n_{j+1} | e^{-H\Delta t} | n_j \rangle \langle n_j | e^{-H\Delta t} | n_{j-1} \rangle \\ &\times \cdots \times \; \langle n_1 | e^{-H\Delta t} | n_0 \rangle \langle n_0 | 0 \rangle \end{aligned}$$

$$\text{where} \quad \Delta t = (t''-t')/N \tag{A.5}$$

The interval $[t', t'']$ is segmented as



As usual, for large N one assumes the validity of

$$e^{-H\Delta t} \sim 1 - H\Delta t$$

Then, one can see easily that there is numerical equality

$$\langle n_i | 1-\Delta t H | n_j \rangle = (1-n_i)(1-n_j) + (1-\Delta t\omega) n_i n_j \tag{A.5}$$

(n_i and n_j both take the values 0 and 1).

One has then

$$\begin{aligned} &\langle 0 | e^{-H(t''-t')} | 0 \rangle \\ &= \sum_{n_0=0}^{1} \cdots \sum_{n_N=0}^{1} \Big\{ \langle 0 | n_N \rangle \times \prod_{j=1}^{N} \big[(1-n_j)(1-n_{j-1}) \\ &+ (1-\Delta t\omega) n_j n_{j-1} \big] \times \langle n_0 | 0 \rangle \Big\} + O(\Delta t) \end{aligned} \tag{A.6}$$

to evaluate RHS of (A.6), one must perform the sum $\sum_{n_0=0}^{1} \cdots \sum_{n_N=0}^{1}$ explicity. The difficulty for doing this is due to the fact that the same set of numbers $n_j = (0,1)$ appears in the two consecutive factors

$$\langle n_{j+1} | 1-\Delta t H | n_j \rangle$$

and

$$\langle n_j | 1-\Delta t H | n_{j-1} \rangle$$

This difficulty would be resolved if one could have written the expression

$$\langle n_i | 1-\Delta t H | n_j \rangle = (1-n_i)(1-n_j) + (1-\Delta t\omega) n_i n_j$$

as the product of two factors each depending only on n_i and n_j respectively

$$(1-n_i)(1-n_j) + (1-\Delta t\omega) n_i n_j \text{ "=" } F(n_i) F'(n_j) \tag{A.7}$$

In general, such a factorization is impossible. The observation of Fradkin et al. [14] quoted above is that one can do such a factorization if one introduces the Grassmann variables.

Indeed, taking the pair of generators of the Grassman algebras v_i and u_j with $v_i^2 = 0$, $u_i^2 = 0$, $\{v_i, u_j\} = 0$, One can easily verify

$$
\begin{aligned}
&(1-n_i)(1-n_j) + (1-\Delta t\,\omega)\, n_i n_j \\
&= -\int dv_i\, du_j \; \left(n_i + (1-n_i) v_i \right) \\
&\qquad\qquad \times \left(1 + (1-\Delta t\,\omega) v_i u_j \right) \\
&\qquad\qquad \times \left(n_j + (1-n_j) u_j \right) \\
&\equiv -\int dv_i\, du_j \; \left(n_i + (1-n_i) v_i \right) e^{(1-\Delta t\,\omega) v_i u_j} \\
&\qquad\qquad \times \left(n_j + (1-n_j) u_j \right)
\end{aligned}
\qquad (A.8)
$$

where the integral is constructed in the space of polynomials according to the familiar rule of Berezin [13].

$$
\int dv\, du \begin{pmatrix} 1 \\ v \\ u \\ uv \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (A.9)
$$

More precisely, one can define the integral with the help of anti-commuting differentiation on the Grassmann algebra [14]

$$
\begin{aligned}
&\int dv\, du \; e^{\lambda u v} (\cdots) \\
&\equiv \left(\lambda - \frac{\delta^2}{\delta u\, \delta v} \right) (\cdots) \Big|_{u=v=0}
\end{aligned}
\qquad (A.10)
$$

where u , v , $\frac{\delta}{\delta v}$, $\frac{\delta}{\delta u}$ must all anti-commute, with each other.

Also, as the special case of (8), one can express the "end" terms at $t''=t_N$ and $t'=t_0$ as

$$\langle 0 | n_N \rangle = \{ (1-n_{N+1})(1-n_N) + n_{N+1} n_N \}_{n_{N+1}=0}$$

$$= - \int d\bar{v}_{N+1} du_N \, \bar{v}_{N+1} \, e^{\bar{v}_{N+1} u_N} (n_N + (1-n_N) u_N) \quad .$$

(A.11a)

$$\langle n_0 | 0 \rangle = \{ (1-n_0)(1-n_{-1}) + n_0 n_{-1} \}_{n_{-1}=0}$$

$$= - \int d\bar{v}_0 \, du_{-1} (n_0 + (1-n_0)\bar{v}_0) \, e^{\bar{v}_0 u_{-1}} u_{-1} \qquad \text{(A.11b)}$$

Substituting (A.8) and (A.11) into (A.3), for the Euclidean vacuum amplitude and effecting the sum $\sum_{n_\nu} (\nu = N, N-1, \ldots 0)$ between consecutive factors, with

$$\sum_{n_j=0}^{1} (n_j + (1-n_j) u_j)(n_j + (1-n_j)\bar{v}_j)$$

$$= 1 + u_j \bar{v}_j = e^{-\bar{v}_j u_j} \qquad \text{(A.12)}$$

one ends up with the representation

$$\langle 0 | e^{-H(t''-t')} | 0 \rangle$$

$$= \prod_{j=0}^{N} \int d\bar{v}_j du_j \left[\int du_{-1} d\bar{v}_{N+1} \, e^{\bar{v}_{N+1} u_N} e^{\bar{v}_0 u_{-1}} u_{-1} \right]$$

$$\left[\prod_{j=0}^{N} e^{(1-\Delta t \omega)\bar{v}_j u_{j-1}} \right] \left[\prod_{j=0}^{N} e^{-\bar{v}_j u_j} \right] + O(\Delta t)$$

$$= \prod_{j=0}^{N} \int d\bar{v}_j du_j \, \exp\left[-\sum_{j=1}^{N} \{ \bar{v}_j (u_j - u_{j-1}) + \Delta t \omega \bar{v}_j u_{j-1} \} \right.$$

$$\left. - \bar{v}_0 u_0 \right] + O(\Delta t)$$

(A.13)

If the formal transition to continuum limit $u_j - u_{j-1} \sim \Delta t\, \dot{u}_{j-1}$ is in someway justified, (A.13) can go over to the familiar path integral formula.

$$\langle 0 | e^{-H(t''-t')} | 0 \rangle$$
$$= \prod_{t=t'}^{t''} \int d\sigma_t\, du_t \;\exp(-S_E - \sigma_{t''} u_{t'})$$

where "Euclidean action"

$$S_E = \int_{t'}^{t''} dt \left(\sigma \frac{du}{dt} + \omega \sigma u \right) \tag{A.14}$$

As a matter of fact, one cannot consider u_j and u_{j-1} are really close together for small Δt, owing to the form of fermionic propagator coming from (A.14). On the other hand, for the simple system like (A.1), one can write the path integral representation without the approximation $\Delta t \sim 0$, i.e. instead of approximate formula (A.5) (A.8), one can write directly

$$\langle n_i | e^{-H\Delta t} | n_j \rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\Delta t \omega} \end{bmatrix}$$
$$= (1-n_i)(1-n_j) + e^{-\Delta t \omega} n_i n_j$$
$$= -\int d\sigma_i\, du_j\, (n_i + (1-n_i)\sigma_i)\, e^{2K\sigma_i u_j} (n_j + (1-n_j) u_j)$$

with $\quad 2K = e^{-\Delta t \omega}$ (A.15)

In this way, one can write down the exact expression independently of smallness of Δt. (See K.G. Wilson Ref. 25).

$$\langle 0 | e^{-H(t''-t')} | 0 \rangle$$
$$= \prod_{j=0}^{N} \int d\sigma_j\, du_j \;\exp\left[-\sum_{j=0}^{N} \sigma_j u_j + 2K \sum_{j=1}^{N} \sigma_j u_{j-1} \right]$$
$$2K = e^{-\Delta t \omega}$$

(A.16)

Since the action of (A.16)

$$S_E = -\sum_{j=0}^{N} v_j u_j + 2K \sum_{j=1}^{N} v_j u_{j-1}$$

and the "limiting" action from (A.14)

$$S_E = \int_{t'}^{t''} dt \left[v \frac{d}{dt} u + \omega v u \right]$$

gives the same value for the correlation function

$$\langle a_i, a_j^* \rangle = \Pi \int dv\, du \quad u_j v_j \; \exp - S_E$$

one considers the (A.14) as being correct.

One can generalize the derivation of (A.13) or (A.14) to the matrix element between arbitrarily Fock state

$$|I\rangle = (\alpha_0 + \alpha_1 a^*) \; |0\rangle$$

$$|F\rangle = (\alpha_0' + \alpha_1' a^*) \; |0\rangle$$

Then

$$\langle F | e^{-H(t''-t')} | I \rangle$$

$$= \lim_{N\to\infty} \prod_{j=1}^{N} \int dv_j du_j \; [\alpha_0'^* - \alpha_1'^* u_N]$$

$$\exp - \sum_{j=1}^{N} \{ v_j (u_j - u_{j-1}) + \Delta t\, \omega v_j u_{j-1} \}$$

$$\exp - v_0 u_0 . \quad \times [\alpha_0 - \alpha_1 v_0]$$

$$\simeq \prod_{t=t'}^{t''} \int dv_t du_t \; \Psi_F(u_{t''}) \; \exp(-S_E - v_{t'} u_{t'})$$

$$\times \Psi_I(v_{t'})$$

(A.17)

where

$$\Psi_F(U_{t^*}) = \alpha_0'^* - \alpha_1'^* U_{t^*}$$

$$\Psi_2(V_{t'}) = \alpha_0 - \alpha_1 V_{t'}$$

b) General Case.

Now one can consider the set of fermionic creation and annihilation operators

$$\{ a_k^* \ , \ a_k \}_{k=1}^{\Omega}$$

with usual anti-commutation relations among them. For the finite Ω, the Fock space is complete and spanned by the set of base vectors

$$\begin{cases} |0\rangle \ ; \quad a_k |0\rangle = 0 \\ |\{n^\alpha\}_{\alpha=1}^{\Omega}\rangle \equiv |n^1 \cdots n^\Omega\rangle \\ \qquad \equiv (a_1^*)^{n_1} \cdots (a_\Omega^*)^{n_\Omega} |0\rangle \end{cases}$$

$$n^\alpha = 0 \text{ or } 1$$

$$\text{and } \sum_{\alpha=1}^{\Omega} n_\alpha \geqslant 1$$

(A.18)

Then, for the arbitrary normal ordered form

$$:H(a^*, a): = \sum_{\mu_i \nu_i = 0}^{1} H_{\mu_1 \cdots \mu_\Omega \nu_\Omega \cdots \nu_1} (a_1^*)^{\mu_1} \cdots (a_\Omega^*)^{\mu_\Omega} (a_\Omega)^{\nu_\Omega} \cdots (a_1)^{\nu_1}$$

(A.19)

one can write down the following Grassmann integral for the matrix element

$$
\begin{aligned}
&\langle \{n_i^\alpha\} \mid 1-\Delta t\, {:}H{:} \mid \{n_j^\alpha\} \rangle \\
&= \prod_{\alpha=1}^{\Omega} \int dv_i^\alpha\, du_j^\alpha\, e^{v_i^\alpha u_j^\alpha} \\
&\quad \{ n_i^\Omega + (1-n_i^\Omega) v_i^\Omega \} \cdots\cdots \{ n_i^1 + (1-n_i^1) v_i^1 \} \\
&\times \left[\, 1-\Delta t\, H_{\mu_1 \cdots \mu_\Omega \nu_\Omega \cdots \nu_1} \{ (v_i^1)^{\mu_1} \cdots\cdots ((-1)^{\Omega-1} v_i^\Omega)^{\mu_\Omega} \right. \\
&\quad \left. ((-1)^{\Omega-1} u_j^\Omega)^{\nu_\Omega} \cdots\cdots (u_j^1)^{\nu_1} \} \,\right] \\
&\times \{ n_j^1 + (1-n_j^1) u_j^1 \} \cdots\cdots \{ n_j^\Omega + (1-n_j^\Omega) u_j^\Omega \}
\end{aligned}
\tag{A.20}
$$

Using this formula and introducing new variables

$$
\eta_i^\alpha = (-1)^{\alpha-1} v_i^\alpha
$$

$$
\xi_i^\alpha = (-1)^{\alpha-1} u_i^\alpha \qquad \alpha = 1, 2, \ldots \Omega
$$

one obtains following path integral representation for the arbitrary amplitude, so long as the number of states Ω is even.

For

$$
|F\rangle = \sum_{n^\alpha=0}^{1} \alpha'_{n^1 \cdots n^\Omega} \mid \{n^\alpha\} \rangle
$$

$$
|I\rangle = \sum_{n^\alpha=0}^{1} \alpha_{n^1 \cdots n^\Omega} \mid \{n^\alpha\} \rangle
$$

and H given by (A.18)

$$
\langle F \mid e^{-H(t''-t')} \mid I \rangle
$$

$$= \prod_{j=0}^{N} \prod_{\alpha=1}^{\Omega} \int d\eta_j^{\alpha} d\xi_j^{\alpha}$$

$$\Psi_F \left(\{\xi_N^{\alpha}\}_{\alpha=1}^{\Omega} \right)$$

$$\exp - \Big[\sum_{j=1}^{N} \{ \eta_j^{\alpha} (\xi_j^{\alpha} - \xi_{j-1}^{\alpha})$$

$$+ \Delta t \sum_{\mu,\nu=0}^{1} H_{\mu_1 \cdots \mu_\Omega \nu_\Omega \cdots \nu_1} (\eta_j^1)^{\mu_1} \cdots (\eta_j^{\Omega})^{\mu_\Omega} (\xi_{j-1}^{\Omega})^{\nu_\Omega} \cdots (\xi_{j-1}^1)^{\nu_1} \} \Big]$$

$$\times \exp - \sum_{\alpha} \eta_0^{\alpha} \xi_0^{\alpha} \cdot$$

$$\times \Psi_I \left(\{\eta_0^{\alpha}\}_{\alpha=1}^{\Omega} \right) \qquad + O(\Delta t)$$

(A.21)

where
$$\Psi_F(\{\xi_N^{\alpha}\}_{\alpha=1}^{\Omega}) = \alpha'^{*}_{n_1 \cdots n_\Omega} (\xi_N^{\Omega})^{n_\Omega} \cdots (\xi_N^{1})^{n_1}$$

$$\Psi_I(\{\eta_0^{\alpha}\}_{\alpha=1}^{\Omega}) = \alpha_{n_1 \cdots n_\Omega} (\eta_0^{1})^{n_1} \cdots (\eta_0^{\Omega})^{n_\Omega}$$

This can be written as the continuum limit

$$\langle F | e^{-H(t''-t')} | I \rangle$$

$$= \prod_{t=t'}^{t''} \prod_{\alpha=1}^{\Omega} \int d\eta_t^{\alpha} d\xi_t^{\alpha}$$

$$\Psi_F(\{\xi_{t''}^{\alpha}\}_{\alpha=1}^{\Omega})$$

$$\exp - S_E \cdot \exp - \sum_{\alpha} \eta_{t'}^{\alpha} \xi_{t'}^{\alpha} \cdot$$

$$\times \Psi_I(\{\eta_{t'}^{\alpha}\}_{\alpha=1}^{\Omega})$$

(A.22)

where the Euclidean action S_E takes the expected form

$$S_E = \int_{t'}^{t''} dt \left[\sum_{\alpha=1}^{n} \eta_t^{\alpha} \frac{d\xi_t^{\alpha}}{dt} + H(\eta_\alpha(t), \xi_\alpha(t)) \right] \tag{A.23}$$

One must keep the kinematical factor $\exp - \sum_{\alpha=1}^{n} \eta_{t''}^{\alpha} \xi_{t''}^{\alpha}$
to make the correspondence with canonical formalism (see for instance Ch. II, b)

c) F..eld Theory.

It is not trivial to generalize the preceding discussion to the case of infinite degree of freedom. In fact, there are problems peculiar for the fermionic fields to define the Euclidean amplitude as the path integral. At the elementary level, this is connected to the fact that the free fermion Green's function is not well defined at equal time.

Thus, to make sense of the expression like (I), one must imagine, in absc.ce of exact theory, some emergency prescription. The lattice cut-off method with finite volume seems to be most hopeful way to eventually reach the more s.tisfactory definition and is tacitly assumed throughout the formal discussion .f CH. I and III. Here, of course, one optimistically assumed th... e consequence of most of formal manipulation survives independen... the detail of the lattice" QCD formalisM. It is also hoped that ... .r manipulation such as ζ function regularization method[20] to derive the current algebra selection rule eg (19) can be eventually justified.[19]

One can formally define the functional integral over the massless Dirac fields as the limit of fermionic oscillators discussed above.
One ..arts from the second quantized form at fixed time

$$\psi(\underline{x}) = \int d^3p \; e^{ip\cdot x} \Big\{ a(p, \lambda=1) \begin{pmatrix} 0 \\ w^-(p) \end{pmatrix}$$

$$+ \, a(p, \lambda=-1) \begin{pmatrix} w^+(p) \\ 0 \end{pmatrix}$$

$$+ \, b^*(-p, \lambda=1) \begin{pmatrix} w^-(p) \\ 0 \end{pmatrix} + b^*(-p, \lambda=-1) \begin{pmatrix} 0 \\ w^+(p) \end{pmatrix} \Big\}$$

(A.2 4)

and its conjugate $\psi^+(\underline{x})$

The vectors $\begin{pmatrix} w^\pm(p) \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ w^\pm(p) \end{pmatrix}$ are the massless spinors in the representation where γ_5 matrix is diagonal, e.g.

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \text{(A.25)}$$

The two-dimensional spinor $w^\pm(p)$ satisfies

$$\frac{p\cdot\sigma}{|p|} w_\pm(p) = \pm w_\pm(p)$$

$$w_\pm^*(p)\cdot w_\pm(p) = 1 \; , \quad w_\pm^*(p)\cdot w_\mp(p) = 0$$

$$w_\pm(p) \otimes w_\pm^*(p) = \frac{1}{2}\left(1 \pm \frac{p\cdot\sigma}{|p|}\right) \qquad \text{(A.26)}$$

For free hamiltonian

$$H = \int d^3p \sqrt{p^2} \left[a^*(p,\lambda) a(p,\lambda) + b^*(p,\lambda) b(p,\lambda) \right] \qquad \text{(A.27)}$$

one has the collections of oscillators such as been discussed before.

One introduces the Grassmann integration variables for given Euclidean time τ,

$$\xi_\tau^1(p,\lambda) \; , \; \xi_\tau^2(p,\lambda) \; , \; \eta_\tau^1(p,\lambda) \text{ and } \eta_\tau^2(p,\lambda)$$

corresponding to the Fock representation

$$a(p,\lambda),\ b^*(p,\lambda),\ a^*(p,\lambda) \text{ and } b(p,\lambda)$$

Then, write

$$\int d\psi\, d\bar{\psi} \underset{\text{def.}}{\sim} \prod_{\tau} \prod_{\alpha,p,\lambda} \int d\eta_\tau^{\alpha}(p,\lambda)\, d\xi_\tau^{\alpha}(p,\lambda) \qquad \text{(A.28)}$$

So long as one has some way of cutting-off the momenta p (ultra-violet as well infra-red) (A.28) is well defined. Certain details of such a cut-off method are explained in the Appendix B.

Definition such as (A.27) and (A.28) refers to the Fock space of massless fermions and not expected to be very convenient one. The improper "canonical" transformation discussed in Ch.II and III (Bogoliubov transformation applied by Nambu and Jona-Lasinio) has the effect of transforming them into the massive second quantization, i.e. the decomposition

$$\psi(x) = \int d^3p\ e^{ip\cdot x}\Big\{ \sum_\lambda a(p,\lambda) u(p,\lambda)$$

$$+ \sum_\lambda b^*(-p,\lambda) v(-p,\lambda) \Big\}$$

$$(\not{p} - m)\, u(p,\lambda) = 0$$

$$(\not{p} + m)\, v(p,\lambda) = 0$$

where m is the parameter in the canonical transformation (55)~(57).

On the other hand, precise meaning of the co-ordinate such as (10) is unclear.[37] Even the free field equivalent of (10)

$$(\not{\partial} - m)\, \phi_\lambda = \lambda \phi_\lambda$$

$$\psi = \sum \xi_\lambda \phi_\lambda$$

$$\bar{\psi} = \sum \eta_\lambda \phi_\lambda^*$$

and

$$\prod_{\tau, x} \int d\psi(x,\tau)\, d\bar{\psi}(x,\tau) = \prod_{\lambda} \int d\eta_\lambda\, d\xi_\lambda$$

seems to be of formal nature.

- APPENDIX B -

-:-:-:-:-

THE GENERALIZATION AND THE CONTNUM LIMIT OF THE MODEL OF CHAPTER II

The model discussed in the preceeding paragraphs has been taken from nuclear physics (non-abelian generalization of Racah model) and is seemingly irrelevent for the relativistic field theory.

But a slight generalization of the model brings it close to the theory of free relativistic particles obeying Fermi statistics.

First of all, one identifies the parameters k as the space components $\underline{p}$ of momentum of the particles placed on the finite 3-dimensional cubic lattice with lattice separation a and total volume $V = L^3$.

Taking usual periodic boundary condition, the allowed momenta are of the form

$$\underline{p} = \frac{2\pi}{L}(\underline{n}). \qquad \underline{n} \in \mathbb{Z}^3$$
$$|p_i|_{max} = \frac{2\pi}{a} \tag{B.1}$$

Thus,

$$\Omega = \left(\frac{L}{a}\right)^3 = \frac{V}{a^3} \quad = \text{number of lattice points}$$

Now, introduce the creation (and annihilation) of "massless fermions"

$$\{ a_s^{*}(\underline{p},\lambda),\ b_s^{*}(\underline{p},\lambda) \}$$

where $\lambda = \pm 1$ represent the "helicity" of the particles.

Define

$$A_i^s(p) = a_s(p,+1) W_i^-(p) + b_s^*(-p,-1) W_i^+(p)$$
$$B_i^s(-p) = a_s(p,-1) W_i^{+*}(p) + b_s(-p,+1) W_i^{-*}(p)$$
$$(i = 1, 2.) \qquad \text{B.2)}$$

and their conjugate $A_i^*(p)$ and $B_i^*(-p)$

Then the anti-commutation relations

$$\{a_s(p,\lambda), a_{s'}^*(p',\lambda')\} = \delta_{p,p'}\delta_{\lambda\lambda'}\delta_{ss'}$$
$$\{b_s(p,\lambda), b_{s'}^*(p',\lambda')\} = \delta_{p,p'}\delta_{\lambda\lambda'}\delta_{ss'} \quad \text{etc.} \qquad (B.3)$$

imply

$$\{A_i^s(p), A_{i'}^{s'*}(p')\} = \delta_{pp'}\delta_{ii'}\delta_{ss'}$$
$$\{B_i^s(p), B_{i'}^{s'*}(p')\} = \delta_{pp'}\delta_{ii'}\delta_{ss'} \quad \text{etc.} \qquad (B.4)$$

Defining

$$J_\alpha^+ = \sum_{i,p} A_i^{s*}(p) (\lambda^\alpha)_{ss'} B_i^{s'}(-p)$$
$$= (J_\alpha^-)^* \qquad (B.5)$$

one can introduce the analogue of the hamiltonian (x2) as

$$H_{int} = -\frac{g}{2\Omega}\left[J_\alpha^+ J_\alpha^- + J_\alpha^- J_\alpha^+ \right] \qquad (B.6)$$

At the same time, one generalizes the model by ad ding the bilinear term

$$H_0 = \sum_{p,s,i} \sqrt{p^2}\left[A_i^{s*}(p) A_i^s(p) + B_i^{s*}(p) B_i^s(p) \right]$$
$$= \sum_{p,\lambda,s} \sqrt{p^2}\left[a_s^*(p,\lambda) a_s(p,\lambda) + b_s^*(p,\lambda) b_s(p,\lambda) \right] + \text{const} .$$

(B.7)

and the total hamiltonian of the system is defined as

$$H = H_0 + H_{int}$$

Now, the discussion of §II.a) about the spontaneous symmetry breaking in the limit $\Omega \to \infty$ needs only little modification in the presence of additional bilinear term $(B.7)$. The quartic part $(B.6)$ is reduced to the bilinear form exactly as in (32) . Only change is the proper canonical transformation (Bogoliubov transformation) (39) and here one must introduce the momentum depending angle γ_p^s , which should diagonalyze the reduced hamiltonian

$$H'_{int} = \sum_{p,i,s} \sqrt{p^2}\left[A_i^{s''*}(p) A_i^{s''}(p) + B_i^{s''*}(p) B_i^s(p) \right]$$
$$- g \sum_s \mu_s \sum_{p,i} \left[A_i^{s'*}(p) B_i^{s''*}(-p) + B_i^{s'}(-p) A_i^{s'}(p) \right]$$

(B.8)

From this, one obtains the condition

$$\tan 2\gamma_p^s = g\mu_s / \sqrt{p^2}$$
$$\sin 2\gamma_p^s > 0$$

(B.9)

for the analogues of the transformations (3.2) for each p and s.

Then the calculation of the vacuum expectation value $\Delta^{\alpha}_{s} = \langle \Psi_0 | J_n^{s}/\Omega | \Psi_0 \rangle$ gives the consistency equations

$$\mu_s = 0$$

or

$$1 = \frac{g}{2\Omega} \sum_{p} \frac{1}{\sqrt{p^2 + g^2 \mu_s^2}} \tag{B.10}$$

Thus a symmetry breaking solution can be obtained by

$$\mu_s = \mu \qquad \text{(independent of } s \text{)}$$

and

$$1 = \frac{g}{2\Omega} \sum_{p} \frac{1}{\sqrt{p^2 + g^2 \mu^2}} \tag{B.11}$$

Note that the "Bogoliubov angle" γ_p can be written as

$$\sin 2\gamma_p = \sqrt{\tfrac{1}{2}(1-\beta_p)}$$

$$\cos 2\gamma_p = \sqrt{\tfrac{1}{2}(1+\beta_p)}$$

$$\beta_p = \frac{|p|}{\sqrt{p^2 + g^2 \mu^2}} \tag{B.12}$$

One can consider the limit when V is large and $a \to 0$. In this limit, one can replace the sum $\sum_p$ by the integral $L^3 \int d^3p$ and the Kronecker's delta $\delta_{p,p'}$ by $\frac{(2\pi)^3}{V} \delta^3(p-p')$. Lastly, if one renormalizes the creation and annihilation operator $a_s(p,\lambda)$, etc. by

$$\hat{a}_s(p,\lambda) = \frac{V}{(2\pi)^3} a_s(p,\lambda)$$

then

$$\{ \hat{a}_s(\underline{p},\lambda), \hat{a}_{s'}^{*}(\underline{p}',\lambda') \} \to \delta^3(\underline{p}-\underline{p}')\delta_{\lambda\lambda'}\delta_{ss'}$$

It is convenient, in the limit $V \to \infty$ and $a \to 0$, to introduce the local "Dirac" field

$$\Psi_s(\underline{x}) = \int d\underline{p}\, e^{i\underline{p}\underline{x}} \left[\begin{Bmatrix} 0 \\ \hat{a}_s(\underline{p},+1)w^-(\underline{p}) + \hat{b}_s^{*}(-\underline{p},-1)w^+(\underline{p}) \end{Bmatrix} + \begin{Bmatrix} \hat{a}_s(\underline{p},-1)w^+(\underline{p}) + \hat{b}_s^{*}(-\underline{p},+1)w^-(\underline{p}) \\ 0 \end{Bmatrix} \right] \tag{B.13}$$

and

$$\bar{\Psi}_s(\underline{x}) = \Psi_s^{*}(\underline{x})\gamma_0$$

Then, it is easy to convince oneself that in the limit of $\Omega \to \infty$

$$J_\alpha^{\pm} \sim \int d^3\underline{x}\; \bar{\Psi}(\underline{x})\lambda_\alpha \frac{1\pm\gamma_5}{2} \Psi(\underline{x}) \tag{B.14}$$

Thus

$$H_{int} \sim -\frac{g}{\Omega}\Big\{ \sum_\alpha \int d^3\underline{x}\; \bar{\Psi}(\underline{x})\lambda_\alpha \frac{1+\gamma_5}{2} \Psi(\underline{x}) \cdot \int d^3\underline{y}\; \bar{\Psi}(\underline{y})\lambda_\alpha \frac{1-\gamma_5}{2}\Psi(\underline{y}) + H.C. \Big\}$$

$$= -\frac{g}{\Omega}\Big\{ \sum_\alpha \int d^3\underline{x}\; \bar{\Psi}(\underline{x})\lambda_\alpha \Psi(\underline{x}) \int d^3\underline{y}\; \bar{\Psi}(\underline{y})\lambda_\alpha \Psi(\underline{y}) - \sum_\alpha \int d^3\underline{x}\; \bar{\Psi}(\underline{x})\gamma_5\lambda_\alpha \Psi(\underline{x}) \int d^3\underline{y}\; \bar{\Psi}(\underline{y})\gamma_5\lambda_\alpha\Psi(\underline{y}) \Big\} \tag{B.15}$$

Ω , number of lattice points.

After taking limit $\Omega \to \infty$, and reducing $H_{tot} = H_0 + H_{int}$ to the bilinear form, one obtains

$$H'_{tot} = \int d^3x \left\{ i\bar{\Psi}_s \Gamma \partial \Psi_s - g\mu\, \bar{\Psi}_s(x)\left[\cos 2\beta\cdot\lambda + i\gamma_5 \sin 2\beta\cdot\lambda\right]_{ss'} \Psi_{s'}(x) \right\} \tag{B.16}$$

$\{\beta_\alpha\}$; $U(N)$ parameter (arbitrary)

where μ is determined by the continuum version of the consistency equation (B·11)

$$\frac{g}{\lambda}\left(\frac{\pi^3}{\Lambda^2}\right)\int_{-\Lambda}^{\Lambda} \frac{d^3p}{\sqrt{p^2+g^2\mu^2}} = 1 \tag{B.17}$$

where $\Lambda = \pi/a$ corresponds to the usual momentum cut-off. RHS stays finite in the limit of $\Lambda \to \infty$, if g is renormalized on

$$\frac{g}{\Lambda} = \tilde{g} \quad (\text{finite}) \tag{B.18}$$

Since, according to (B·16) , $M^2 = g^2\mu^2$ represents the mass of resultant free fermions, the equation for the Bogoliubov transofmration (B·9, 12) is nothing but the transformation proposed by Nambu and Jona-Lasinio(55~57). In terms of new canonical variables $\tilde{a}_s(p,\lambda)$ and $\tilde{b}_s(p,\lambda)$, one can write

$$\Psi(x) = e^{2i\beta\cdot\lambda\,\gamma_5}\, e^{i\alpha\cdot\lambda} \cdot \int d^3p \left\{ e^{ip\cdot x}\sum_\lambda \tilde{a}(p,\lambda)\,u(p,\lambda) + e^{-ip\cdot x}\sum_\lambda \tilde{b}^\dagger(p,\lambda)\,v(p,\lambda) \right\} \tag{B.19}$$

with $(\not{p} - M)\, u(p,\lambda) = 0$

$(\not{p} + M)\, v(p,\lambda) = 0$

i.e. $\Psi(\underline{x})$ reduces to free Dirac field at given time.

On the other hand, eq. B.17 is consistent with $\lim_{\Lambda\to\infty} M$ = finite, only if $M = 0$ (when g_c) $g = \frac{\Lambda}{(2\pi)^4}$). g_c is the upper limit of values of g for which symmetry breaking solution (B.17) exists. The vacuum expectation value $<\bar{\psi}\psi>$ at this point is equal to zero.

Thus, one can go to smooth "relativistic" limit in this model only at the critical point. The resultant theory is free massless fermions. The analogous situation can be found in two dimensional Ising model [23,36].