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PREQUANTISATION FROM PATH INTEGRAL VIEWPOINT *

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ABSTRACT

The quantum mechanically admissible definitions of the factor $\exp\left[i/\hbar \quad S(\gamma)\right]$ - needed in Feynman's integral - are put in bijection with the prequantisations of Kostant and Souriau. The different allowed expressions of this factor - the inequivalent prequantisations - are classified in terms of algebraic topology.

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1. INTRODUCTION

In [1] a first attempt was made to use the geometric techniques of Kostant and Souriau [2,3] ("K-S theory") in studying path integrals. The method was applied to Dirac's monopole and the Bohm - Aharonov experiment.

Here we intend to develop a more general theory. We show that a general symplectic system is quantum mechanically admissible (Q.M.A.S.) iff it is prequantisable with transition functions depending on space-time variables. If the configuration space is not simply connected, the different physical situations correspond to different prequantisations. A classification scheme [6] - implicitly recognised already by Kostant [2] and Dowker [5] - is presented.

The basic object of our considerations below is the factor

$$\exp\left[\frac{i}{\hbar} S(\gamma)\right] \tag{1}$$

where $S(\gamma)$ is the classical action along the path γ.

Our results contribute to the physical interpretation of prequantisation, and are hoped to provide physicists with a kind of introduction to this theory.

2. QUANTUM MECHANICALLY ADMISSIBLE SYSTEMS (Q.M.A.S.)

Let us restrict ourselves to classical systems (E, σ) with evolution space $E = T^*Q \times \mathbb{R}$ (Q is a configuration space) and presymplectic structure of the form

$$\sigma = d\Theta_0 + e\,\mathbb{F} \tag{2}$$

$d\Theta_0$ – where Θ_0 is the restriction to the energy surface $H = H_0(q,p,t)$ of the canonical 1-form of $T^*(Q \times \mathbb{R})$ – describes a free system ; $\mathbb{F}$ – a closed 2-form on space-time $X = Q \times \mathbb{R}$ – represents the external field coupled to our system by the constant e (cf. [3]).

If the system admits a Lagrangian function, then $\sigma = d\Theta$ and it is exactly this "Cartan 1-form" Θ which has to be integrated along paths in phase space (whose initial resp. final points project to the same $x = (q,t)$ resp. $x' = (q',t') \in X$) when computing a path integral in phase space :

$$S(\gamma) = \int_\gamma \Theta \tag{3}$$

Now, by Poincaré's lemma, for any point there exists a contractible neighbourhood U_j and a 1-form Θ_j defined here such that $\sigma|_{U_j} = d\Theta_j$. Thus we would be tempted to define $S_j(\gamma)$ by (3) even if no global Lagrangian – and consequently no global Θ – exists. It was pointed out in [1], that the different expressions $S_j(\gamma)$ and $S_k(\gamma)$ may be completely different. The following notion will be useful :

Definition (E, σ) is a quantum mechanically admissible system (Q.M.A.S.) iff there exists a collection $\{\mathcal{U}_j, \Theta_j\}$ of pairs of open contractible subsets $\mathcal{U}_j$ and 1-forms Θ_j defined there –

called local system in what follows - such that they are compatible, i.e. for any $\gamma \subset \mathcal{U}_j \cap \mathcal{U}_k$ we have

$$\exp\left[\frac{i}{\hbar}\int_\gamma \Theta_j\right] = c_{jk}(x,x')\cdot\exp\left[\frac{i}{\hbar}\int_\gamma \Theta_k\right] \qquad (4)$$

where the unitary factors c_{jk} depend only on the projections to space-time of the initial resp. end point of γ, but not on γ itself.

Clearly, in such situations the Feynman propagators corresponding to Θ_j resp. Θ_k will be related by unobservable phase factors.

In [1] we have shown that this happens iff

$$\frac{1}{2\pi\hbar}\int_S \sigma \in \mathbb{Z} \qquad (5)$$

for any 2-cycle S in space. Expressed in fiber bundle language we have (by Weil's lemma)

THEOREM [1]

A.) (E, σ) is Q.M.A. - iff prequantisable with transition functions depending on X. Then for any γ with end points in $\mathcal{U}_j$ we can define

$$\text{"}\exp\left[\frac{i}{\hbar} S(\gamma)\right]\text{"} \qquad (6a)$$

such that there exists phase factors c_{jk} with

$$\text{"}\exp\left[\frac{i}{\hbar} S_j(\gamma)\right]\text{"} = c_{jk}(x,x')\cdot\text{"}\exp\left[\frac{i}{\hbar} S_k(\gamma)\right]\text{"} \qquad (6b)$$

For $\gamma \subset \mathcal{U}_j \cap \mathcal{U}_k$, we have

$$\text{"}\exp\left[\frac{i}{\hbar} S_j(\gamma)\right]\text{"} = \exp\left[\frac{i}{\hbar} \int_\gamma \Theta_j\right] \tag{6c}$$

B.) Explicitely, we have the transition function $z_{jk} : \mathcal{U}_j \cap \mathcal{U}_k \to U(1)$ with

$$\Theta_j - \Theta_k = \frac{d z_{jk}}{i\, z_{jk}} \tag{6d}$$

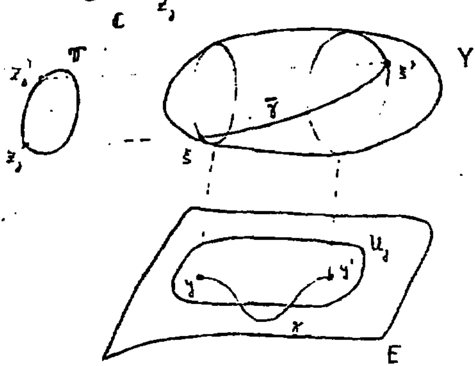
yielding

$$C_{jk}(x,x') = \frac{z_{jk}(x)}{z_{jk}(x')} \tag{6e}$$

Let γ be any path in E joining $y = (x,.)$ to $y' = (x',.)$. Denote (Y, ω, π) a prequantisation [3] of (E, σ). Lift γ to Y horizontally through a $\xi \in \pi^{-1}(y)$, denote ξ' the end point of this horizontal lift $\bar{\gamma}$. If $y, y' \in \mathcal{U}_j$, we can write locally $\xi = (y, z_j)$, $\xi' = (y', z_j')$

The expression (6a) is then

$$\text{"}\exp\left[\frac{i}{\hbar} S_j(\gamma)\right]\text{"} = \frac{z_j}{z_j'} \tag{7}$$



3. GEOMETRIC EXPRESSION FOR THE INTEGRAND

Now we can give a completely coordinate free form to the integrand in Feynman's expression. Following a suggestion of Friedmann and Sorkin [8] let us consider any path $\tilde{\gamma} \subset Y$ projecting to γ . Write $\tilde{\gamma}(0) = \xi \simeq (y, z_i)$ $\tilde{\gamma}(1) = \xi' \simeq (y', z_i')$

Lemma

$$"\exp\left[\frac{i}{\hbar} S_i(\gamma)\right]" \cdot \frac{z_i'}{z_i} = \exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}} \omega\right] \tag{8}$$

The product of two coordinate-dependent quantities is thus coordinate-independent !

Now all we need is to remember that the wave functions can be represented by complex functions on Y satisfying [3]

$$\psi\left(\underline{z}_Y(\xi)\right) = z \cdot \psi(\xi) \tag{9}$$

(where $\underline{z}_Y$ denotes the action of $U(1)$ on Y) rather than merely functions on Q ; the usual wave functions are the local representants of these objects obtained as

$$\psi(\xi) = z_j \cdot \psi_j \qquad \xi \in \pi^{-1}(U_j) \tag{10}$$

Thus, we get finally the geometric formula for the time evolution

$$\left(U_{t'-t}\psi\right)(\xi') = \int_Q dq \int_{P_{xx'}} \mathcal{D}\gamma \; \exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}} \omega\right]\psi(\xi) \tag{11}$$

where ,

Note that $\exp[\frac{i}{\hbar} \int_{\tilde{\gamma}} \omega] \cdot \psi(\xi)$ is, in fact, a function of γ, independently of the choice of $\tilde{\gamma}$ supposing $\tilde{\gamma}(1) = \xi'$ is held fixed.

Remarks.

1. We do not try to give a geometric definition for "$\mathcal{D}_\gamma$". An attempt in this direction was made by Simms [9] .
2. The introduction of the bundle (Y, ω, π) allows for devel-opping a generalized variational formalism [8] and makes it easy to study conserved quantities.

4. A CLASSIFICATION SCHEME [6]

If the underlying space is not simply connected, we may have more than one prequantisation and thus several inequivalent meanings of (1). (two local systems are said to be equivalent if their union is again an admissible local system).

The general construction for all the prequantisations are found in Souriau [3]. Denote $(\tilde{E}, \Pi_1, q)$ the universal covering of E ; define $\tilde{\sigma} = q^* \sigma$. Π_1, the first homotopy group of E, acts then on $\tilde{E}$ by symplectomorphisms.

Let us choose a "reference prequantisation" (Y_0, ω_0, π_0) of (E, σ). As $(\tilde{E}, \tilde{\sigma})$ is simply connected, it has a unique prequantisation $(\tilde{Y}, \tilde{\omega}, \tilde{\pi})$, which can be obtained from (Y_0, ω_0, π_0) as

$$(\tilde{Y}, \tilde{\omega}, \tilde{\pi}) = q^*(Y_0, \omega_0, \pi_0) \tag{12}$$

If $\chi : \Pi_1 \to U(1)$ is a character, then Π_1 admits an isomorphic lift to $(\tilde{Y}, \tilde{\omega}, \tilde{\pi})$ of the form

$$\hat{g}^{\chi}(\tilde{x}, \xi) = \left(g(\tilde{x}),\ \underset{Y_0}{\underline{\chi(g)}}\,(\xi)\right) \tag{13}$$

$g \in \Pi_1$,

Now, Souriau has shown that

$$(Y_\chi, \omega_\chi, \pi_\chi) := (\tilde{Y}, \tilde{\omega}, \tilde{\pi}) / \hat{\Pi}_1^{\chi} \tag{14}$$

is a prequantisation of (E, σ), and all prequantisations can be obtained in this way. The inequivalent prequantisations are thus in (1-1) correspondence with the characters of the homotopy group.

In [1] we rederived this theorem from our path-integral consideration noting that we are always allowed to add a closed but not exact 1-form α to Θ_0, which – due to non simply connectedness – may change the propagator in an inequivalent way. The corresponding character is then

$$\chi(q) = \exp\left[\frac{i}{\hbar}\oint_{[\gamma]=q}\alpha\right] \quad . \tag{15}$$

For instance, in the Bohm-Aharonov experiment [4] $\pi_1 = \mathbb{Z}$ and all the characters have this form.

This is, however, not the general situation. A physically interesting counter-example is that of identical particles [3], [10].

<u>Example</u>

Consider two identical particles moving in 3-space. The appropriate configuration space is then [11] $Q = \tilde{Q}/\mathbb{Z}_2$ where

$$\tilde{Q} := \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{q_1 = q_2\}$$

which has the homotopy group $\pi_1 = \mathbb{Z}_2$.

E is then $T^*Q \times \mathbb{R}$ with $\sigma = d\Theta_0^{(1)} + d\Theta_0^{(2)}$

$\pi_1 = \mathbb{Z}_2$ has two characters :

$$\chi_1(\tau) = 1 \qquad \text{and} \qquad \chi_2(\tau) = -1$$

where τ is the interchange of two configurations.

Thus we have two prequantum lifts of π_1 and two prequantisations, one of which is trivial, while the second is twisted. The first corresponds to bosons, the second to fermions. Now, it is easy to see that $\mathcal{X}_2$ is not of the form (15) :

PROPOSITION

If the homotopy group is finite, $|\pi_1| < \infty$, then $H^1(E, \mathbb{R}) = 0$, i.e. every closed 1-form is exact.

Proof. Let α be a closed 1-form on E , define $\tilde{\alpha} = q^*\alpha$, $\tilde{\alpha} = d\tilde{f}$ for $\tilde{E}$ is simply connected ; define

$$\hat{h} := \frac{1}{|\pi_1|} \cdot \sum_{g \in \pi_1} g^* \tilde{f}$$

$\tilde{h}$ is invariant under $g \in \pi_1$ and projects thus to a $h : E \to R$. On the other hand $\tilde{\alpha} = d\tilde{h} = dq^* h = q^* \alpha$, and thus $\alpha = dh$.

The general situation can be treated by algebraic topological means [11] . Consider the exact sequence of groups

$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{2\pi i} U(1) \to 0 \tag{16}$$

giving rise to the long exact sequence

$$\ldots \to \underset{}{H^1(E,\mathbb{Z})} \xrightarrow{1} \underset{\text{closed / exact}}{H^1(E,\mathbb{R})} \xrightarrow{2} \underset{\text{characters}}{H^1(E,U(1))} \xrightarrow{3} \underset{\text{Chern class}}{H^2(E,\mathbb{Z})} \xrightarrow{4} \underset{\text{curv. class}}{H^2(E,\mathbb{R})} \to \tag{17}$$

$$\left[\sigma/2\pi t\right]$$

We can make the following observations :

1) σ defines, by (5), an integer-valued element of $H^2_{dR}(E,\mathbb{R})$ which, by de Rham's theorem, is just $H^2(E, \mathbb{R})$.

2) The bundle is topologically completely characterized by its Chern class which sits in $H^2(E,\mathbb{Z})$. Thus we have as many distinct bundles as elements in the kernel of 4 .

3) As $U(1)$ is commutative, a character of π_1 depends only on $\pi_1/[\pi_1,\pi_1]$, which is known to be $H_1(E,\mathbb{Z})$.

On the other hand, the Theorem on Universal Coefficients [11] p. 76, yields that

$$\mathrm{Hom}(H_1(E,\mathbb{Z}), U(1)) \simeq H^1(E, U(1)) \tag{18}$$

Thus $H^1(E, U(1))$ is just the <u>set of all characters</u> classifying the different prequantisations.

4) Under quite general conditions, we have

$$\begin{aligned} H^i(E,\mathbb{Z}) &\cong \mathbb{Z}^{b_i} \oplus \mathrm{Tors}\, H^i \\ H_i(E,\mathbb{Z}) &\cong \mathbb{Z}^{b_i} \oplus \mathrm{Tors}\, H_i \end{aligned} \tag{19}$$

where $\mathrm{Tors}\, H^i$ and $\mathrm{Tors}\, H_i$ are groups whose elements are all of finite order;

5) The kernel of the map $H^i(E,\mathbb{Z}) \to H^i(E,\mathbb{R})$ is just $\mathrm{Tors}\, H^i$, the image of $\mathbb{Z}^{b_i}$ is a basis in $H^i(E,\mathbb{R})$;

6) Again, by the Theorem on Universal Coefficients,

$$\mathrm{Tors}\, H^2(E,\mathbb{Z}) \simeq \mathrm{Tors}\, H_1(E,\mathbb{Z}) \quad (\sim \mathrm{Tors}\, \Pi_1/[\Pi_1,\Pi_1]) \tag{20}$$

Thus 2), 5), 6) give us

<u>PROPOSITION</u>

The topologically distinct prequantum bundles are labelled by the elements of (20).

7) According to 5), the image of $H^1(E,\mathbb{Z})$ in $H^1(E,\mathbb{R})$ under 1) is made up of integer multiples of a basis. Thus $H^1(E,\mathbb{R})/\mathrm{im}\ H^1(E,\mathbb{Z}) \simeq (S^1)^{b_1}$ and we get the exact sequence

$$0 \to (S^1)^{b_1} \to H^1(E, U(1)) \to \mathrm{Tors}\, H_1(E,\mathbb{Z}) \to 0 \tag{21}$$

Now by de Rham's theorem, to any element of $H^1(E,\mathbb{R})$ we can associate a closed 1-form $\alpha/_{2\pi\hbar}$ such that its value on $g \in H_1(E,\mathbb{R})$ is

$$\frac{1}{2\pi\hbar} \oint_\gamma \alpha \tag{22}$$

where the homology class of γ is g.

Next, by (21), the image of $(S^1)^{b_1}$ in $H^1(U(1))$ is *composed of characters of the form*

$$\chi(g) = \exp\left[\frac{i}{\hbar} \oint_\gamma \alpha\right] \tag{23}$$

As $(S^1)^{b_1}$ is connected, and Tors $H^2(E,\mathbb{Z})$ is finite, we have

PROPOSITION

The characters of the form (23) make up the connected component containing $\chi \equiv 1$ of the group of characters;

8) Let us choose a basis $\alpha_1, \ldots \alpha_{b_1}$ in $H^1_{dR}(E,\mathbb{R})$ and pick up a $\chi_k \in H^1(E,U(1))$ corresponding to each element of Tors H^2 = Tors H_1.

PROPOSITION

Any character can be written as

$$\chi(g) = \exp\left[\frac{i}{\hbar} \sum_{j=1}^{b_1} a_j \oint_{[\gamma]=g} \alpha_j\right] \cdot \chi_k \tag{24}$$

where $a_j \in \mathbb{R}$ (mod 2π). The χ_k's can be chosen in such a way that they form a subgroup of the group of characters, however, there is no canonical choice for them.

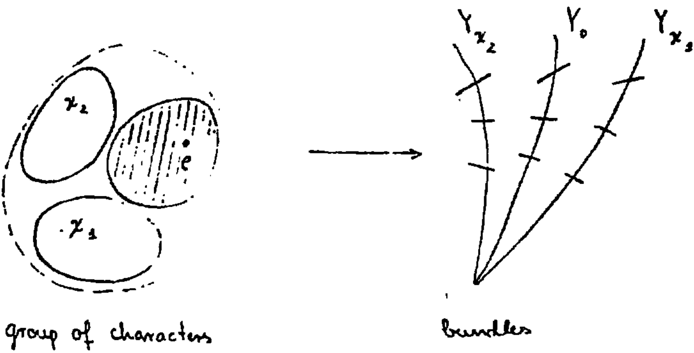
Finally, we get the following refinement of Souriau's construction (14)

PROPOSITION

Y_{χ_1} and Y_{χ_2} are topologically identical iff χ_1 and χ_2 belong to the same component of the group of characters.

The different connection forms on the same bundle are labelled by the elements of the connected component containing the identity character $\chi \equiv 1$.

Proof : If a character is of the form (23) then, by carrying smoothly the coefficients to 0, the bundle has to change also smoothly. On the other hand, the Chern class has to change discretely. Consequently, it remains constant.



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