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Improved Krook Model Collision Operator for Trapped-Particle Mode Calculations

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Abstract

An improved Krook model collision operator is presented for use in calculations of low-frequency microinstabilities in toroidal geometry. A comparison is made with results from calculations which use a Lorentz collision operator and with a previously employed Krook operator. The results with the new Krook operator agree with the results obtained using the Lorentz operator in the small collision frequency limit, whereas those with the old Krook operator do not.

I. Introduction

The study of microinstabilities so far can be divided into two categories: (i) simplified geometry or dynamics with a realistic collision operator involving velocity space derivatives, such as slab geometry using the Lorentz operator; and (ii) realistic geometry and dynamics with a simple collision operator that is algebraic in velocity space, such as toroidal geometry and a Krook model operator.[1,2] The advantage of using an algebraic collision operator for realistic geometries is that a closed-form solution of the gyrokinetic equation can be obtained, thereby reducing the dimensionality of the problem to be solved numerically by two, since the velocity variables only appear parametrically. We must incorporate into the algebraic operator as much of the essential physics, described by the differential collision operators, as possible. In this paper we present a Krook model collision operator which gives better agreement with the Lorentz operator results than previous Krook operators.

Previous toroidal calculations[1,2] have employed $\nu_\alpha \sim \nu_{\alpha i}/|(1-\varepsilon) - \Lambda|$, where ν_α is the effective collision frequency of species α, $\nu_{\alpha i}$ is the familiar Spitzer collision frequency, ε is the local inverse aspect ratio, and Λ is a dimensionless pitch angle variable. In this form the collision frequency diverges at the velocity-space boundary between trapped and circulating particles. It is the precise nature of the divergence which is important. This form automatically accounts for the enhancement in the effective collision frequency of trapped and barely circulating particles It is quite accurate in the large collisionality limit but becomes quite inadequate in the small collisionality limit.[1] An operator of the form $\nu_\alpha \sim \varepsilon \nu_{\alpha i}/|(1-\varepsilon) - \Lambda|^2$ has all of the desired properties of the old operator but is

much more accurate in the small collisionality limit. The accuracy of the operators is determined by comparison with an analytic solution of Rosenbluth et al.[3] for a simple form of the trapped-ion mode in which a Lorentz collision operator was employed.

The outline of this paper is as follows: In Section II we analyze the new collision operator by calculating analytically the eigenfrequencies of the trapped-ion mode in the radially local limit and then compare the results with the Rosenbluth results. One operator valid in both the high and low collisionality regimes is then constructed. In Section III we use the new collision operator to calculate the eigenfrequencies of the trapped-electron mode and compare the results with the results obtained using the old operator. A summary of results and some concluding remarks are given in Section IV. We note here that the new operator is not only applicable to the trapped-particle drift modes but can be applied to any of the tokamak low-frequency microinstability calculations.

II. Analysis of the New Collision Operator and Application to the Trapped-Ion Mode

Marchand et al.[1] derived a linear eigenmode equation which describes the trapped-ion instability in a large-aspect-ratio toroidal system. In the radially local limit, when the ion banana width is much smaller than the radial wavelength and the equilibrium scale length, the eigenmode equation is

$$\sum_m C_{pm}\, \phi_m(r) = 0, \tag{1}$$

where the $\phi_m(r)$ are the coefficients of the poloidal Fourier components of the perturbed electrostatic potential, and r is the radial distance from the magnetic axis. To obtain a form for the matrix elements C_{pm} of the matrix C which are analytically tractable, the following assumptions are made: (i) the plasma is in the banana regime,

$$\frac{\nu}{\epsilon}\Big|_{i,e} \ll \bar{\omega}_b\Big|_{i,e} , \tag{2}$$

where $\nu_{i,e}$ is the ion (electron) collision frequency and $\bar{\omega}_{bi,e}$ is the typical ion (electron) bounce frequency; (ii) the aspect ratio is large,

$$\epsilon \equiv \frac{r}{R_o} \ll 1 , \tag{3}$$

where R_o is the plasma major radius; (iii) the dissipative limit (within the banana regime) is appropriate

$$\frac{\nu_{ii}}{\epsilon} \ll |\omega| \ll \frac{\nu_{ei}}{\epsilon} , \tag{4}$$

where ω is the mode frequency; and (iv)

$$|\omega| \ll \bar{\omega}_{bi} . \tag{5}$$

In investigating the accuracy of our new model collision operator, to be discussed below, either species could be in either collisionality limit, $\nu_{\alpha i}/\epsilon|\omega| \ll 1$ or $\nu_{\alpha i}/\epsilon|\omega| \gg 1$. However, the trapped-ion mode encompasses both limits among the two species involved; this is convenient.

Further considering a weakly ballooning mode which is localized halfway between rational surfaces, the elements of the C matrix become

$$C_{pm} = \frac{\sqrt{\varepsilon}}{\pi^{5/2}} \int_{1-\varepsilon}^{1+\varepsilon} \frac{d\Lambda}{L_b} \int_0^\infty dY\, \sqrt{Y}\, \exp(-Y)\, \frac{\omega - \omega_{*Te}}{i\nu_e}\, J_{op} J_{om}$$

$$+ \frac{\tau\sqrt{\varepsilon}}{\pi^{5/2}} \int_{1-\varepsilon}^{1+\varepsilon} \frac{d\Lambda}{L_b} \int_0^\infty dY\, \sqrt{Y}\, \exp(-Y)$$

$$\times \left(\frac{\omega - \omega_{*Ti}}{\omega} - i\, \frac{\nu_i}{\omega}\, \frac{\omega - \omega_{*Ti}}{\omega + i\nu_i} \right) J_{op}\, J_{om}$$

$$+ \frac{2\tau\sqrt{\varepsilon}}{\pi^{5/2}} \sum_{n>0} \int_{1-\varepsilon}^{1+\varepsilon} \frac{d\Lambda}{L_b} \int_0^\infty dY\, \sqrt{Y}\, \exp(-Y)$$

$$\times \frac{\omega(\omega - \omega_{*Ti})}{\omega^2 - n^2\omega_{bi}^2}\, J_{np}\, J_{nm} + \frac{\tau\sqrt{\varepsilon}}{\pi^{5/2}} \int_0^{1-\varepsilon} \frac{d\Lambda}{L_t} \int_0^\infty dY\, \sqrt{Y}\, \exp(-Y)$$

$$\sum_n \frac{\omega(\omega - \omega_{*Ti})}{\omega^2 - [n + S^{(m)}\omega_{ti}]^2}\, K_{npm} K_{nmm} - (1 + \tau)\, \delta_{pm}\,, \tag{6}$$

from Ref. 1 where $Y = E/T$ is the dimensionless energy variable, ω_{ti} is the ion transit frequency, τ is the electron to ion temperature ratio,

$$\omega_{*T} = \omega_*[1 + \eta(Y - 3/2)], \tag{7.a}$$

$$\omega_* = -\frac{m}{r}\frac{cT}{eB_o}\frac{1}{n}\frac{dn}{dr}\,, \tag{7.b}$$

$$\eta = \frac{d(\ell n\ T)/dr}{d(\ell n\ n)/dr}\ , \tag{7.c}$$

m is the poloidal mode number, B_o the toroidal magnetic field on the magnetic axis, and n and T are the equilibrium particle density and temperature respectively. L_b and L_t are integrals over the poloidal angle ϑ,

$$L_b = \frac{\sqrt{\varepsilon}}{2\pi} \int_{-\vartheta_o}^{\vartheta_o} d\vartheta \left[1 - \frac{\Lambda}{h(\vartheta)}\right]^{-1/2} \simeq \frac{\sqrt{2}}{\pi} K(\sin\theta_o/2) \tag{8.a}$$

$$L_t = \frac{\sqrt{\varepsilon}}{2\pi} \int_{-\pi}^{\pi} d\vartheta \left[1 - \frac{\Lambda}{h(\vartheta)}\right]^{-1/2} \simeq \frac{2}{\pi} (1 - \cos\theta_o)^{-1/2} K_t'(2/\cos\theta_o)^{1/2} \tag{8.b}$$

where $h(\vartheta) = 1 + \varepsilon \cos\vartheta$, $\vartheta_o = \arccos[(\Lambda - 1)/\varepsilon]$ and $K(x)$ is the complete elliptic integral of the first kind. Also, $S^{(m)}(r) = \ell q(r) - m$, where ℓ is the toroidal mode number and $q(r)$ is the local safety factor. J_{nm} is a trapped-particle orbit integral and K_{nmp} is an untrapped-particle orbit integral both of which are described in Ref. 1 and will not be discussed here. For our purpose we only need the result

$$J_{om} \simeq \pi/\sqrt{2\varepsilon}\ . \tag{9}$$

The successive terms in Equation (6) are the trapped-electron, the collisionless and collisional time-averaged (n = 0) trapped-ion, the non-time-averaged (n≠0) trapped-ion, the circulating-ion, and the adiabatic contributions.

To obtain the eigenfrequency from Eqs. (1) and (6) we only need to specify the collision terms ν_e and ν_i. Following a suggestion of Marchand, we take ν_e and ν_i to be Krook model collision operators with the form

$$\nu_e(Y,\Lambda) = a_e \varepsilon \frac{\nu_{ei}}{Y^{3/2}} \frac{1 + H(\sqrt{Y})}{|(1 - \varepsilon) - \Lambda|^2} \tag{10.a}$$

and

$$\nu_i(Y,\Lambda) = a_i \varepsilon \frac{\nu_{ii}}{Y^{3/2}} \frac{H(\sqrt{Y})}{|(1 - \varepsilon) - \Lambda|^2} , \tag{10.b}$$

where $H(x) \equiv \exp(-x^2)/(\sqrt{\pi}x) + [1 - 1/(2x^2)]\ \mathrm{erf}(x)$ and $\mathrm{erf}(x)$ is the error function. Equations (10.a) and (10.b) are different from the Krook collision operators used in previous analyses of tokamak trapped-particle modes[1,2] where

$$\nu_e(Y,\Lambda) = \frac{\nu_{ei}}{Y^{3/2}} \frac{1 + H(\sqrt{Y})}{|(1 - \varepsilon) - \Lambda|} \tag{11.a}$$

and

$$\nu_i(Y,\Lambda) = \frac{\nu_{ii}}{Y^{3/2}} \frac{H(\sqrt{Y})}{|(1 - \varepsilon) - \Lambda|} \tag{11.b}$$

were employed. The new collision operators give new values for the time-averaged trapped-particle contributions to the growth rate. These contributions are defined in Ref. 1 as γ_e and γ_i. Proceeding as in Ref. 1, the new values for γ_e and γ_i are

$$\frac{\gamma_e}{\omega_o} = 2^{5/2} \frac{\sqrt{\pi}}{a_e} \frac{\lambda^{(2)}}{(1 + \frac{\alpha^{(1)}}{\tau})} \frac{\omega_{*e}}{\nu_{ei}} \varepsilon^{3/2} \left[1 + \eta_e\left(\frac{\alpha^{(2)}}{\alpha^{(1)}} - \frac{3}{2}\right)\right] , \tag{12.a}$$

$$\frac{\gamma_i}{\omega_o} = \frac{-2^{3/4}}{\pi} \frac{\sqrt{a_i}}{(\lambda^{(o)})^{3/2}} \left[\frac{\nu_{ii}}{\omega_{*e}\varepsilon^{3/2}} \frac{(1 + \tau)}{2}\right]^{1/2} f\left(\frac{\nu_{ii}}{\omega_{*e}\varepsilon^{3/2}} \frac{(1 + \tau)}{2}\right) \beta^{(o)}$$

$$\times \left[1 + \eta_i \left(\frac{\beta^{(1)}}{\beta^{(o)}} - \frac{3}{2}\right)\right] , \tag{12.b}$$

where the lowest order (real) mode frequency is

$$\omega_o = 0.577 \frac{2\sqrt{\epsilon}\omega_{*e}}{1+\tau} , \tag{13}$$

and

$$\lambda^{(n)} = \int_0^1 \frac{k^n dk}{K(\sqrt{1-k})} , \tag{14.a}$$

$$\chi^{(n)} = \int_0^\infty dY\, Y^{n+1} \frac{\exp(-Y)}{1+H(\sqrt{Y})} , \tag{14.b}$$

and

$$\beta^{(n)} = \int_0^\infty dY\, Y^{n-1/4} [H(\sqrt{Y})]^{1/2} \exp(-Y) . \tag{14.c}$$

f(x) was calculated numerically, and can be approximated to within 4% by the function g(x) in the range $10^{-12} < x < 10^{-3}$ where

$$f(x) = x^{-1/2} \int_0^1 \frac{dk}{K(\sqrt{1-k})} \cdot \frac{k^2}{k^2 + x^2} , \tag{15.a}$$

and

$$g(x) = 2.79[\ell n(17/\sqrt{x})]^{-1.055} . \tag{15.b}$$

A very similar dispersion relation was obtained by Rosenbluth *et al.*[3] Using a variational principle and a Lorentz collision operator, they found

$$\omega_o = 0.58 \frac{2\sqrt{\epsilon}}{1+\tau} \omega_{*e} \ , \tag{16.a}$$

$$\frac{\gamma_e}{\omega_o} = 0.84 \ (1 + 1.41\eta_e) \frac{2}{1+\tau} \frac{\epsilon^{3/2}\omega_{*e}}{\nu_{ei}} \ , \tag{16.b}$$

and

$$\frac{\gamma_i}{\omega_o} = - 0.31 \ (1 - 0.57\eta_i) \left(\frac{\nu_{ii}}{\epsilon^{3/2}\omega_{*e}} \frac{1+\tau}{2}\right)^{1/2}$$

$$\times \left\{\ln\left[17\left(\frac{\epsilon^{3/2}\omega_{*e}}{\nu_{ii}} \frac{2}{1+\tau}\right)^{1/2}\right]^{-3/2}\right\} \ . \tag{16.c}$$

We perform the integrals in Equations (14.a) - (14.c) numerically and then choose a_e and a_i so that the numerical coefficients leading Eqs. (12.a) - (12.b) are the same as those leading Eqs. (16.a) - (16.c). The results are

$$a_e = 1.31 \tag{17.a}$$

and

$$a_i = 9.42 \times 10^{-3} \ . \tag{17.b}$$

Inserting Eq. (17) into Eq. (12) we have

$$\frac{\gamma_e}{\omega_o} = 0.84 \ (1 + 1.41\eta_e) \frac{2}{1+\tau} \frac{\omega_{*e}\epsilon^{3/2}}{\nu_{ei}} \ , \tag{18.a}$$

and

$$\frac{\gamma_i}{\omega_o} = -0.31\,(1 - 0.57\eta_i)\left[\frac{\nu_{ii}}{\omega_{*e}\varepsilon^{3/2}}\,\frac{(1+\tau)}{2}\right]^{1/2}$$

$$\times \left\{\ell n\left[17/\left(\frac{\nu_{ii}}{\omega_{*e}\varepsilon^{3/2}}\,\frac{(1+\tau)}{2}\right)^{1/2}\right]\right\}^{-1.055} . \qquad (18.b)$$

Comparing Eqs. (18.a) - (18.b) to Eqs. (16.a) - (16.c) we see that the new operator gives a definite improvement over the old operator, i.e., the ion collisional growth rate now has the correct dominant scaling in γ_{ii} and ε; the old operator did not. The relative error between (16.c) and (18.b) is $\lesssim 25\%$ in the range indicated above of $x \equiv \nu_{ii}(1+\tau)/(2\omega_{*e}\varepsilon^{3/2})$.

Finally, from Eqs. (10.a) - (10.b) and Eqs. (17.a) - (17.b), we construct an operator which makes a smooth transition from $|\omega| \ll \nu_\alpha$ to $|\omega| \gg \nu_\alpha$. The result is

$$\nu_\alpha = \frac{\nu_{\alpha i}}{Y^{3/2}}\,\frac{\delta_{\alpha e} + H(\sqrt{Y})}{|(1-\varepsilon) - \Lambda|^2}\,\frac{0.111\,\frac{|\omega|}{\nu_{\alpha i}} + 1.31}{11.79\,\frac{|\omega|}{\nu_{\alpha i}} + 1} \qquad (19)$$

where α labels the species. ν_α as defined above gives the same contributions, in both the high and low collisionality limits, as would be obtained with a Lorentz collision operator.

III. Application to the Trapped-Electron Mode

To investigate the effect of the new collision operator we calculate the eigenfrequencies of the tokamak trapped-electron mode using the new operator

and compare the results with the results of using the old operator.[2] The code used is an improved version of that described in Ref. 4. The old electron collision operator is given by Eq. (11.a) while for the new operator we have

$$\nu_e(Y,\Lambda) = \frac{\nu_{ei}\varepsilon}{Y^{3/2}} \frac{1 + H(\sqrt{Y})}{|(1-\varepsilon) - \Lambda|^2} \frac{0.111 \frac{|\omega|}{\nu_{ei}} + 1.31}{11.79 \frac{|\omega|}{\nu_{ei}} + 1} , \tag{20}$$

with the corresponding operators being used for the ions. The real parts of the mode frequencies, ω_r/ω_{*e}, and the growth rates, γ/ω_{*e}, for each operator are shown as a function of $\nu_e^* \equiv \nu_{ei}/\varepsilon^{3/2}\omega_{te}$ in Fig. 1. The parameters used in the comparison are $\varepsilon = 0.1$, $T_e = 1$ keV, $T_i = 0.5$ keV, $q = 2.5$, $q'r/q = 1$, $n_i = n_e = 0$, $r_{ne}/r = 1$, $m_i/m_e = 3672$, $R_o = 132$ cm., and $k_\theta\rho_i = 0.63$, in standard notation. The density varies roughly proportionally to ν_e^*. On a semi-log scale the new collision operator has the effect of shifting the eigenfrequency curves to the right. This is because the dominant scaling of the collisional contribution to the low collisionality eigenfrequency scales as ν_{ei} for the old operator but as $\sqrt{\nu_{ei}}$ for the new operator.

IV. Conclusions

We have compared the dispersion relation of the trapped-ion mode obtained using a new Krook model collision operator with the results obtained using the more exact Lorentz operator. The new operator is seen to be accurate in both the high collision frequency limit ($|\omega| \ll \nu/\varepsilon$) and in the low collision frequency limit ($|\omega| \gg \nu/\varepsilon$). In particular, the dominant scaling of the growth rates in the parameter $\nu_{\alpha i}/\varepsilon|\omega|$ was correct in both limits, so that we

were able to construct one operator which is valid for any species and which makes a smooth transition from low to high collisionality. The Krook collision operator previously employed[1,2] did not have the correct dominant scaling of the growth rate in $\nu_{ai}/\epsilon|\omega|$ in the low collision frequency limit.

A comparison between the old and new collision operator was made by calculating the eigenfrequencies of the trapped-electron mode. When plotted on a semi-log scale the new operator has the effect of shifting the mode frequency curves to the right. The new Krook operator is a definite improvement over the old operator and can be used in any microinstability calculation in which a pitch angle dependent Krook operator is convenient.

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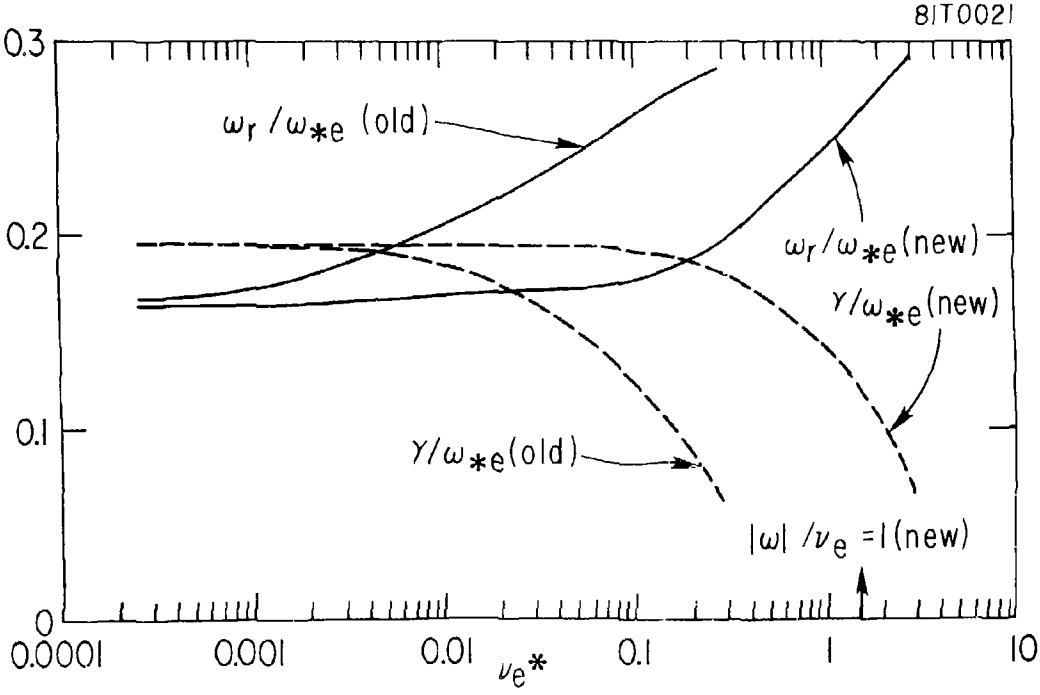

Fig. 1. Eigenfrequencies of the trapped-electron mode for the old and new Krook collision operators.