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FROM SHELL TO MEMBRANE THEORY

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Résumé :

Une nouvelle formulation de la théorie des membranes est donnée dans cet article. Les hypothèses qui permettent d'obtenir le modèle de Budiansky-Sanders ou celui des membranes, sont mises en évidence.

FROM SHELL TO MEMBRANE THEORY

Philippe Destuynder*

ABSTRACT.

A new formulation of the membrane theory is presented in this paper. The assumptions which allow the Budiansky - Sanders' model or the membrane theory to be deduced from the three-dimensional case are pointed out.

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1. - INTRODUCTION

A shell is a body which occupies in space a volume bounded by two curved surfaces and such that the distance between these two surfaces is small in comparison with its other dimensions [18]. This dimensional feature allows simplifications in the three-dimensional model for linear elasticity. It is shown in sections three and four of this paper how Kirchhoff-Love's kinematical assumptions lead to the well known Budiansky-Sanders's model [5]. In particular it is explained how the existence and uniqueness of a solution is a consequence of the three dimensional Korn's inequality. In section five the membrane model is deduced from the B.S. model by an asymptotic expansion. The small parameter being the thickness of the shell, (assumed to be uniform for the sake of simplicity). The mathematical study of this model is done in section six and seven. The existence and uniqueness of a solution is proved for a uniformly convex membrane shell.

The forthcoming section is devoted to the notations used in the sequel and is basic to the understanding of the other sections.

2. - NOTATIONS [21]

Let us consider a smooth surface imbedded in $\mathbb{R}^3$, say ω. We assume that there exists a map φ from an open set $\hat{\omega}$ of $\mathbb{R}^2$ onto ω which is at least C^3. The plane $\mathbb{R}^2$ containing $\hat{\omega}$ will be referred to a system of coordinates $(0; \xi^1, \xi^2)$, and $\mathbb{R}^3$ is referred to $(0; x_1, x_2, x_3)$.

Furthermore, the boundary $\hat{\gamma}$ of $\hat{\omega}$ corresponds by φ to the boundary γ of ω.

To the map φ corresponds a curvilinear system of coordinates on ω such that at any point $m = \varphi(\xi^1, \xi^2)$ of ω the vectors tangent to the coordinates lines are :

$$a_\alpha = \varphi_{,\alpha} \quad ; \quad \alpha = 1, 2, \text{ (where, } \alpha \text{ stands for } \frac{\partial}{\partial \xi^\alpha}.)$$

We assume that the vectors a_α are linearly independent and span the tangent plane $T_m(\omega)$ at m to ω. The unitary normal at m to ω is denoted by $N = N(m)$.

In the following, Greek indices will belong to the set {1,2}, and the summation convention will be assumed. Let $\{a^\alpha\}$ denote the dual basis of $\{a_\alpha\}$ defined by $a^\alpha a_\beta = \delta^\alpha_\beta$, (Kronecker's symbol). For any smooth function g defined on ω, we set :

$$\frac{\partial g}{\partial m} = \frac{\partial g}{\partial \xi^\alpha} a^\alpha = g_{,\alpha}\ a^\alpha \tag{2.1}$$

Let $\Pi = \Pi(m)$ be the projection from $\mathbb{R}^3$ on the tangent plane $T_m(\omega)$.

Clearly, for any point m of ω, the following identity holds :

$$Id = \Pi + \bar{N}\, N, \tag{2.2}$$

where Id is the identity in $\mathbb{R}^3$ and $\bar{N}$ is the transpose of N.

With use of (2.2), any vector field v of $\mathbb{R}^3$ defined on ω can be decomposed into a tangential component $v_t = \Pi v$ and a normal component $v_3 = \bar{N}\, v$.

The derivative of v_t with respect to the points of ω is defined by :

$$\frac{\partial v_t}{\partial m} = \frac{\partial v_t}{\partial \xi^\alpha} a^\alpha \tag{2.3}$$

If we set :

$$v_t = v^\alpha a_\alpha ,$$

then

$$\frac{\partial v_t}{\partial m} = v^\alpha{}_{,\beta}\ a_\alpha\ a^\beta + v^\alpha\ a_{\alpha,\beta}\ a^\beta . \tag{2.4}$$

The Christoffel symbols $\Gamma^\lambda_{\alpha\beta}$ are defined by :

$$\Pi\ a_{\alpha,\beta} = \Gamma^{\lambda}_{\alpha\beta}\ a_{\lambda},$$

or else :

$$\Gamma^{\lambda}_{\alpha\beta} = a^{\lambda}\ a_{\alpha,\beta}.$$

The covariant derivative of v_t is then, by definition :

$$\Pi\ \frac{\partial v_t}{\partial m} = (v^{\alpha},_{\beta} + \Gamma^{\alpha}_{\lambda\beta}\ v^{\lambda})\ a_{\alpha}\ a^{\beta} = v^{\alpha}\,|_{\beta}\ a_{\alpha}\ a^{\beta}. \tag{2.5}$$

The transpose of v_t, denoted $\bar{v}_t$ can also be represented in the basis a^{α}:

$$\bar{v}_t = v_{\alpha}\ a^{\alpha},$$

where

$$v_{\alpha} = g_{\alpha\beta}\ v^{\beta},$$

and

$$g_{\alpha\beta} = (a_{\alpha},\ a_{\beta}),\ \text{(scalar product in } \mathbb{R}^3).$$

A simple exercise [6] shows that :

$$v_{\alpha}\,|_{\beta} = v_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}\ v_{\lambda} = g_{\alpha\lambda} v^{\lambda}\,|_{\beta}, \tag{2.6}$$

where $v^{\lambda}\,|_{\beta}$ are the components of the matrix $\Pi \frac{\partial v_t}{\partial m}$ in the basis $\{a_{\alpha}\}$, defined by :

$$v^{\lambda}\,|_{\beta} = v^{\lambda},_{\beta} + \Gamma^{\lambda}_{\alpha\beta}\ v^{\alpha}\ .$$

We have for instance :

$$\bar{v}\ v = (v_{\alpha}\ a^{\alpha})\ .\ (v^{\beta}\ a_{\beta}\ ) = v_{\alpha}\ v^{\beta}\ a^{\alpha}\ a_{\beta} = v_{\alpha}\ v^{\alpha}.$$

The derivative $\frac{\partial N}{\partial m}$ of the unit normal N to the surface ω is called its curvature operator.

We can easily prove it is a symmetrical endomorphism of $T_m(\omega)$. We usually set :

$$\frac{\partial N}{\partial m} = -b_\beta^\alpha \, a_\alpha \, a^\beta. \tag{2.7}$$

Later on, we shall make use of the covariant divergence of a tangential vector field on ω or field of endomorphisms of $T_m(\omega)$.

For any vector field $v_t = v^\alpha a_\alpha$, let us define [6]:

$$\widehat{\operatorname{div}}\, v_t = \frac{1}{|g|^{1/2}} (v^\alpha |g|^{1/2}),_\alpha, \tag{2.8}$$

which is a scalar function on ω.

Similarly, for any field τ of endomorphisms of $T_m(\omega)$, $\tau = \tau_\beta^\alpha \, a_\alpha \, a^\beta$, let us define [6]:

$$\widehat{\operatorname{div}}\, \tau = \frac{1}{|g|^{1/2}} (- (|g|^{1/2} \, \tau_\beta^\alpha),\alpha) + \Gamma_{\lambda\beta}^\alpha \, \tau_\alpha^\lambda \tag{2.9}$$

which is a covector, ie. a linear form on the vectors.

In formulas (2.8) - (2.9), $|g|$ is the determinant of the tensor $g_{\alpha\beta}$.

In order to state our problem, we shall now define several spaces of functions defined on ω.

First, we set :

$$L^2(\omega) = \{f : \hat{f} = f \circ \varphi \in L^2(\tilde{\omega})\}, \tag{2.10}$$

with the norm :

$$\|f\|_{L^2(\omega)} = \{ \int_\omega |\hat{f}|^2 \, |g|^{1/2} \}^{1/2}.$$

Next we introduce :

$$\underset{\sim}{H}_t = \{v_t = v^\alpha \, a_\alpha : v^\alpha \in L^2(\omega)\}, \tag{2.11}$$

with the norm defined by :

$$\|v_t\|_{H_t} = \{\int_\omega \bar{v}_t \, v_t\}^{1/2} = \{\int_\omega v_\alpha \, v^\alpha\}^{1/2},$$

which is equivalent to :

$$\{\sum_{\alpha=1,2} \|v^\alpha\|^2_{L^2(\omega)}\}^{1/2}.$$

Using already introduced spaces, we define :

$$H^1(\omega) = \{f \in L^2(\omega) \; ; \; \overline{\frac{\partial f}{\partial m}} \in H_t\}, \tag{2.12}$$

which will be equipped with its natural norm :

$$\|f\|_{H^1(\omega)} = \{\|f\|^2_{L^2(\omega)} + \|\overline{\frac{\partial f}{\partial m}}\|^2_{H_t}\}^{1/2}$$

The tangential stress space is defined by :

$$\Sigma_t = \{\tau = \tau^\alpha_\beta \, a_\alpha \, a^\beta; \; \tau^\alpha{}_\beta \in L^2(\omega); \; \tau_{12} = \tau_{21}\}.$$

and provided with the norm :

$$\|\tau\|_{\Sigma_t} = \{\int_\omega \mathrm{Tr}\,(\bar{\tau}\,\tau)\}^{1/2} \tag{2.13}$$

where we have set : $\mathrm{Tr}(\sigma \cdot \tau) = \sigma^\alpha_\beta \, \tau^\beta_\alpha$ for arbitrary elements σ, τ of the space Σ_t.

The space of tangential displacements will be :

$$V_t = \{v_t = v^\alpha \, a_\alpha \; ; \; v^\alpha \in H^1_0(\omega)\}$$

provided with the norm :

$$\|v_t\|_{V_t} = \{\sum_{\alpha=1,2} \|v^\alpha\|^2_{H^1(\omega)}\}^{1/2}.$$

If we set :

$$\gamma(v) = \frac{1}{2}\left(\Pi \frac{\partial v_t}{\partial m} + \overline{\frac{\partial v_t}{\partial m}}\,\Pi\right),$$

it is proved in [7] that the mapping :

$$v_t \in \underset{\sim}{V}_t \to \left\{\int_\omega \mathrm{Tr}(\gamma(v)\ .\ \gamma(v))\right\}^{1/2}, \tag{2.15}$$

is a norm on $\underset{\sim}{V}_t$ equivalent to the one defined in (2.13) as soon as the Christoffel symbols $\Gamma^{\lambda}_{\alpha\beta}$ are small enough in $\mathcal{C}^0$ norm. The proof is based on a result due to P. Rougée [20] and on a comparison with an eigenvalue problem.

Finally we set :

$$H^2_0(\omega) = \{f : \hat{f} = f \circ \varphi_\epsilon \in H^2_0(\bar{\omega})\}, \tag{2.16}$$

with the norm :

$$\|f\|_{H^2(\omega)} = \|\hat{f}\|_{H^2(\bar{\omega})}$$

Referring again to [13], we note that this space can also be equipped with the equivalent norm :

$$f \to \left\|\Pi \frac{\partial}{\partial m}\left(\overline{\frac{\partial f}{\partial m}}\right)\right\|_{\underset{\sim}{\Sigma}_t}, \tag{2.17}$$

if the Christoffel symbols are small enough in $\mathcal{C}^0$ norm.

With the preceding notations, we have :

$$\Pi \frac{\partial}{\partial m}\left(\overline{\frac{\partial f}{\partial m}}\right) = (g^{\alpha\gamma} f_{,\alpha})_{|\beta}\ a_\alpha\ a^\beta,$$

where $g^{\alpha\gamma}$ is the inverse of the metric tensor $g_{\alpha\beta}$ (i.e. $g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$), and :

$$(g^{\alpha\gamma} f_{,\alpha})_{|\beta} = (g^{\alpha\gamma} f_{,\alpha})_{,\beta} + \Gamma^{\gamma}_{\alpha\beta}\, g^{\alpha\gamma}\ f_{,\alpha}.$$

We now introduce the closure $\bar{\Omega}^\varepsilon$ of the open set Ω^ε occupied by the shell :

(2.18) $\bar{\Omega}^{\epsilon} = \{M \in \mathbb{R}^3;\ OM = om + x_3 N \quad \text{where } -\epsilon \leq x_3 \leq \epsilon$ and m is a point of the surface $\bar{\omega}$; 0 being any origin in $\mathbb{R}^3\}$.

The real number ϵ is half the thickness of the shell. It is assumed to be constant and strictly positive. We suppose that ϵ is small enough such that to any point M of $\bar{\Omega}^{\epsilon}$ corresponds a unique couple (m, x_3) of the set $\bar{\omega} \times [-\epsilon, \epsilon]$, satisfying

$$OM = om + x_3 N.$$

At each point M of Ω^{ϵ} we have the identity :

(2.19) $$I_3 = \Pi + N \tilde{N},$$

where I_3 is the identity in $\mathbb{R}^3$ and $\tilde{N}$ the transposed of the vector N. It is to be mentioned that as Π is an orthogonal projection we have :

$$\Pi = \tilde{\Pi}.$$

Let v be a vector field defined on Ω^{ϵ}. At each point M of Ω^{ϵ}, v(M) is a vector of $\mathbb{R}^3$. Because of (2.19), we can split v into a tangential component :

(2.20) $$v_t = \Pi\, v,$$

and a normal component :

(2.21) $$v_3 = \tilde{N}\, v.$$

Thus we have :

(2.22) $$v = v_t + v_3 N.$$

In a similar way any symmetrical endomorphism field of $\mathbb{R}^3$, say τ, can be split into three components :

(i) $$\tau_t = \Pi\, \tau\, \Pi,$$

which is at each point (m, x_3) of $\bar{\Omega}^{\epsilon}$ a symmetrical endomorphism of $T_m(\omega)$;

(ii) $$\tau_s = \Pi\, \tau\, N,$$

(iii) $\tau_n = \bar{N}\,\tau\,N$

which a scalar.

Hence :

$$\tau = \tau_t + \tau_n\, N\bar{N} + N\bar{\tau}_s + \tau_s\,\bar{N}. \tag{2.23}$$

The curvature operator of the surface ω denoted by $\frac{\partial N}{\partial m}$ is a symmetrical endomorphism of $T_m(\omega)$. Hence it admits two real eigenvalues say $\frac{1}{R_1}$ and $\frac{1}{R_2}$ where R_1 and R_2 are the principal radius of curvature of the surface ω at the point m of $\bar{\omega}$.

The thin shell theory $\bar{\omega}$ assumes that :

$$\varepsilon \max\left(\frac{1}{|R_1|}, \frac{1}{|R_2|}\right) < 1. \tag{2.24}$$

If we define a symmetrical endomorphism of $T_m(\omega)$ by

$$\mu = I_2 + x_3 \frac{\partial N}{\partial m} \tag{2.25}$$

where I_2 is the identity in the tangent plane $T_m(\omega)$; then the thin shell assumption implies that :

$$\mu^{-1} = I_2 - x_3 \frac{\partial N}{\partial m} + x_3^2 \left(\frac{\partial N}{\partial m}\right)^2 + \ldots + (-x_3)^n \left(\frac{\partial N}{\partial m}\right)^n + \ldots \tag{2.26}$$

We shall also make use of the formula :

$$\det \mu = 1 + x_3\, \mathrm{Tr}\left(\frac{\partial N}{\partial m}\right) + x_3^2 \det \frac{\partial N}{\partial m}.$$

Let us now consider that g is a real function defined in Ω^ε. The derivative of g denoted $\frac{\partial g}{\partial M}$ is a linear form on $\mathbb{R}^3$. From the equality :

$$OM = om + x_3\, N,$$

we obtain by differentiating :

$$dM = dm + x_3 \frac{\partial N}{\partial m}\, dm + N\, dx_3,$$

where (dm, dx_3) denotes any element of the set $T_m(\omega) \times \mathbb{R}$. Which is the tangent space to the manifold $\bar{\omega} \times [-\varepsilon, \varepsilon]$ at (m, x_3), and dM the corresponding element of the tangent space $T_M(\Omega^\varepsilon)$ at the point M defined by :

$$OM = om + x_3 N$$

Hence on one hand :

$$\Pi \, d M = \mu dm, \tag{2.27}$$

or else under the thin shell assumption (2.24) :

$$dm = \mu^{-1} \Pi \, d M \tag{2.28}$$

and on the other hand :

$$dx_3 = \vec{N} \, d M. \tag{2.29}$$

Then setting by definition of $\frac{\partial g}{\partial m}$ and $\frac{\partial g}{\partial M}$

$$\frac{\partial g}{\partial M} dM = \frac{\partial g}{\partial m} dm + \frac{\partial g}{\partial x_3} dx_3,$$

we deduce

$$\frac{\partial g}{\partial M} = \frac{\partial g}{\partial m} \mu^{-1} \Pi + \frac{\partial g}{\partial x_3} \vec{N} \tag{2.30}$$

Let us now denote by v a vector field of $\mathbb{R}^3$ defined on Ω^ε. Then the derivative $\frac{\partial v}{\partial M}$ *is an endomorphism of* $\mathbb{R}^3$.

Following the steps (2.27) - (2.28) - (2.29) - (2.30), we have :

$$\frac{\partial v}{\partial M} dM = \frac{\partial v}{\partial m} dm + \frac{\partial v}{\partial x_3} dx_3,$$

then :

$$\frac{\partial v}{\partial M} = \frac{\partial v}{\partial m} \mu^{-1} \Pi + \frac{\partial v}{\partial x_3} \vec{N} \quad ,$$

and finally from (2.22) :

$$\frac{\partial v}{\partial M} = \frac{\partial v_t}{\partial m}\,\mu^{-1}\,\Pi + \frac{\partial v_t}{\partial x_3}\,\bar{N} + \frac{\partial}{\partial m}\,(v_3\,N)\,\mu^{-1}\,\Pi + \frac{\partial v_3}{\partial x_3}\,N\,\bar{N}.$$

If we notice that :

$$\frac{\partial}{\partial m}\,(v_3\,N) = v_3\,\frac{\partial N}{\partial m} + N\,\frac{\partial v_3}{\partial m},$$

we deduce that :

$$\frac{\partial v}{\partial M} = \frac{\partial v_t}{\partial m}\,\mu^{-1}\,\Pi + \frac{\partial v_t}{\partial x_3}\,\bar{N} + v_3\,\frac{\partial N}{\partial m}\,\mu^{-1}\,\Pi + N\,\frac{\partial v_3}{\partial m}\,\mu^{-1}\Pi + \frac{\partial v_3}{\partial x_3}\,N\,\bar{N}.$$

Making use of (2.19) we also have :

$$\frac{\partial v}{\partial M} = \Pi\,\frac{\partial v_t}{\partial m}\,\mu^{-1}\,\Pi + N\,\bar{N}\,\frac{\partial v_t}{\partial m}\,\mu^{-1}\,\Pi + \frac{\partial v_t}{\partial x_3}\,\bar{N} + v_3\,\frac{\partial N}{\partial m}\,\mu^{-1}\,\Pi + N\,\frac{\partial v_3}{\partial m}\,\mu^{-1}\,\Pi + \frac{\partial v_3}{\partial x_3}\,N\,\bar{N}.$$

But from :

$$\bar{N}\,v_t = 0,$$

we deduce by differentiating :

$$\bar{v}_t\,\frac{\partial N}{\partial m} = -\,\bar{N}\,\frac{\partial v_t}{\partial m},$$

and we obtain the final expression :

$$(2.31)\quad \frac{\partial v}{\partial M} = \Pi\,\frac{\partial v_t}{\partial m}\,\mu^{-1}\,\Pi + v_3\,\frac{\partial N}{\partial m}\,\mu^{-1}\,\Pi + \frac{\partial v_t}{\partial x_3}\,\bar{N} - N\,v_t\,\frac{\partial N}{\partial m}\,\mu^{-1}\,\Pi + \frac{\partial v_3}{\partial x_3}\,N\,\bar{N} + N\,\frac{\partial v_3}{\partial m}\,\mu^{-1}\,\Pi$$

In particular if σ denotes any symmetrical endomorphism of $\mathbb{R}^3$ we can write :

$$(2.32)\quad \mathrm{Tr}(\sigma\,\frac{\partial v}{\partial M}) = \mathrm{Tr}(\sigma_t\,\Pi\,\frac{\partial v_t}{\partial m}\,\mu^{-1}) + v_3\,\mathrm{Tr}(\sigma_t\,\frac{\partial N}{\partial m}\,\mu^{-1}) + \bar{\sigma}_s\,\gamma_s(v) + \sigma_n\,\gamma_n(v).$$

where we have set :

$$(2.33)\qquad \begin{cases} \gamma_s(v) = \dfrac{\partial v_t}{\partial x_3} + \mu^{-1}\,\overline{\dfrac{\partial v_3}{\partial m}} - \dfrac{\partial N}{\partial m}\,\mu^{-1}\,v_t, \\ \gamma_n(v) = \dfrac{\partial v_3}{\partial x_3}, \end{cases}$$

and σ_t, σ_s, σ_n are defined in (2.23).

In order to complete this set of notations let us remark that f is a "smooth function" defined on Ω^ε we have :

$$(2.34)\qquad \int_{\Omega^\varepsilon} f(M)\,d\Omega^\varepsilon = \int_\omega \int_{-\varepsilon}^{+\varepsilon} f(m, x_3)\,\det\mu\,d\omega\,dx_3$$
$$= \int_{\hat\omega} \int_{-\varepsilon}^{+\varepsilon} f(\xi^1, \xi^2, x_3)\,\det\mu\,|g|^{\frac{1}{2}} d\xi^1\,d\xi^2\,dx_3,$$

where $|g|$ is the determinant of the tensor $g_{\alpha\beta}$.

We are now able to formulate the three dimensional model in linear elasticity, for the shell occupying the set Ω^ε.

The displacements of the points of $\bar\Omega^\varepsilon$ are denoted by u.

The stress field denoted by σ is symmetrical; (i.e. $\sigma = \bar\sigma$). The applied forces are of two kinds :

(i) body forces, the density vector of which is f and surface forces acting on the upper and lower boundaries Γ_+^ε and Γ_-^ε . Their density are denoted by g^+ and g^-. The shell is supposed to be fixed on its lateral boundary $\Gamma_o^\varepsilon = \gamma \times]-\varepsilon, \varepsilon[$.

Hence

$$u = 0 \text{ on } \Gamma_o^\varepsilon.$$

The constitutive equation we are considering is a linear relationship between the stress σ and the linearized strain :

$$\gamma(u) = \frac{1}{2}\left(\frac{\partial u}{\partial M} + \overline{\frac{\partial u}{\partial M}}\right) .$$

It is known as Hooke's law and can be written :

$$\frac{1+\nu}{E}\,\sigma - \frac{\nu}{E}\,\mathrm{Tr}(\sigma)\,\mathrm{Id} = \gamma(u), \tag{2.35}$$

or else :

$$\sigma = \frac{E}{1+\nu}\left(\gamma(u) + \frac{\nu}{1-2\nu}\,\mathrm{Tr}(\gamma(u))\,\mathrm{Id}\right),$$

where E, (respectively ν), is the Young's modulus (respectively Poisson's coefficient). It corresponds to isotropic homogeneous material.

They satisfy for mechanical reasons [14]

$$E > 0, \quad 0 < \nu < \frac{1}{2}. \tag{2.36}$$

Let us define the set of admissible displacements by :

$$\underset{\sim}{V}^{\varepsilon} = \{v = (v^i),\ (\text{in the system of coordinates } (0; x^1, x^2, x^3));\ v^i \in \tilde{H}^1(\Omega^{\varepsilon});\ v^i = 0,\ \text{on } \Gamma_o^{\varepsilon}\},$$

and the stress fields set by :

$$\underset{\sim}{\Sigma}^{\varepsilon} = \{\tau = (\tau_{ij});\ (\text{in the system of coordinates } (0; x^1, x^2, x^3)),\ \text{such that } \tau_{ij} \in L^2(\Omega^{\varepsilon}) \text{ and } \tau_{ji} = \tau_{ij}\}.$$

Then we introduce a bilinear form $B(.,.)$ defined over $\underset{\sim}{\Sigma}^{\varepsilon} \times \underset{\sim}{V}^{\varepsilon}$ by :

$$B(\tau, v) = -\int_{\Omega^{\varepsilon}} \mathrm{Tr}\left(\tau \cdot \frac{\partial v}{\partial M}\right).$$

The Principal of Virtual Work [12] [2], can be written when the displacements are supposed to be small enough :

$$\forall\, v \in \underset{\sim}{V}^{\varepsilon}, \quad B(\sigma, v) = F(v) = -\int_{\Omega^{\varepsilon}} \bar{f}\, v - \int_{\Gamma_+^{\varepsilon} \cup \Gamma_-^{\varepsilon}} \bar{g}^{\pm}\, v. \tag{2.37}$$

where σ is the stress field on Ω^{ε}.

If we set for arbitrary elements τ, σ of the space $\underset{\sim}{\Sigma}^{\varepsilon}$

$$a(\sigma, \tau) = \int_{\Omega^\varepsilon} \frac{1+\nu}{E} \operatorname{Tr}(\sigma \cdot \tau) - \frac{\nu}{E} \operatorname{Tr}(\sigma) \cdot \operatorname{Tr}(\tau),$$

then both equations (2.35) and (2.37) can be resumed in Hellinger-Reissner's Principle which consists in finding an element (σ, u) in the space $\underset{\sim}{\Sigma}^\varepsilon \times \underset{\sim}{V}^\varepsilon$ such that :

$$(2.38) \qquad \begin{cases} 1 & \forall \tau \in \underset{\sim}{\Sigma}^\varepsilon, \quad a(\sigma, \tau) + B(\tau, u) = 0, \\ 2 & \forall v \in \underset{\sim}{V}^\varepsilon, \quad B(\sigma, v) = F(v). \end{cases}$$

The classical variationnal formulation of the equation of elasticity where only the displacement u appears is obtained by eliminating σ from the first equation (2.38).

The existence and uniqueness of a solution (σ, u) to (2.38) is very classical. With the Hellinger-Reissner's Principle it can be obtained from Brezzi's Theorem [4], that we recall hereafter.

THEOREM 2.1 (Brezzi). Let $\underset{\sim}{\Sigma}$ and $\underset{\sim}{V}$ be two Hilbert's spaces with norms $\| \cdot \|_{\underset{\sim}{\Sigma}}$ and $\| \cdot \|_{\underset{\sim}{V}}$.

Let $a : \underset{\sim}{\Sigma} \times \underset{\sim}{\Sigma} \to \mathbb{R}$; $B : \underset{\sim}{\Sigma} \times \underset{\sim}{V} \to \mathbb{R}$; $F : \underset{\sim}{V} \to \mathbb{R}$; $g : \underset{\sim}{\Sigma} \to \mathbb{R}$ be bilinear and linear continuous forms satisfying :

$$(2.39) \qquad \forall \tau \in \underset{\sim}{\Sigma} \; ; \; a(\tau, \tau) \geq C \, \| \tau \|^2_{\underset{\sim}{\Sigma}},$$

$$(2.40) \qquad \forall v \in \underset{\sim}{V}; \; \sup_{\tau \in \underset{\sim}{\Sigma}} \frac{B(\tau, v)}{\| \tau \|_{\underset{\sim}{\Sigma}}} \geq C \, \| v \|_{\underset{\sim}{V}},$$

(C denoting strictly positive constants independent on τ or v). Then there exists a unique element (σ, u) in the space $\underset{\sim}{\Sigma} \times \underset{\sim}{V}$ such that

$$\begin{cases} \forall \tau \in \underset{\sim}{\Sigma}, \; a(\sigma, \tau) + B(\tau, u) = g(\tau), \\ \forall v \in \underset{\sim}{V}, \; B(\sigma, v) = F(v). \end{cases}$$

∎

In the case of the elasticity we set $g = 0$, $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}^\varepsilon$, $\underset{\sim}{V} = \underset{\sim}{V}^\varepsilon$ and $a(.,.)$, $B(.,.)$, and $F(.)$ are defined above. Inequality (2.39) is a consequence of (2.36) and (2.40) is a consequence of Korn's inequality [1]

In the following sections we make use of the spaces :

$$\begin{cases} \underset{\sim}{V}_t^\varepsilon = \{v \in \underset{\sim}{V}^\varepsilon;\ \bar{N}\, v = 0\}\ , \\ V^\varepsilon = \{v \in \underset{\sim}{V}^\varepsilon;\ \Pi\, v = 0\}\ , \\ \underset{\sim}{T}_t^\varepsilon = \{\tau \in \underset{\sim}{T}^\varepsilon;\ \Pi \circ N = 0,\ \bar{N} \circ \Pi = 0,\ \bar{N} \circ N = 0\}, \end{cases}$$

with the norms induced by $\underset{\sim}{V}^\varepsilon$ or $\underset{\sim}{T}^\varepsilon$.

3. - THE KIRCHHOFF-LOVE'S ASSUMPTION

In order to simplify the model (2.38), Kirchhoff and Love [14] have introduced kinematical assumptions which should be satisfied by the solution u of (2.38) for a thin shell.

This assumptions is :

$$(3.5) \qquad \begin{cases} \gamma_s(u) = 0, \\ \gamma_n(u) = 0, \end{cases}$$

where (see section 2) :

$$\begin{cases} \gamma_s(u) = \dfrac{\partial u_t}{\partial x_3} + \mu^{-1}\, \overline{\dfrac{\partial u_3}{\partial m}} - \mu^{-1}\, \dfrac{\partial N}{\partial m}\, u_t, \\ \gamma_n(u) = \dfrac{\partial u_3}{\partial x_3}\ . \end{cases}$$

As we shall see this is the only mechanical assumptions needed for deriving the Budiansky-Sanders's model from the three dimensional one. First of all we have to characterize the Kirchhoff-Love displacement field.

THEOREM 3.1. Let us denote by $\underset{\sim}{V}^{\varepsilon}_{KL}$ the Kirchhoff-Love displacement field defined by :

$$\underset{\sim}{V}^{\varepsilon}_{KL} = \{v \in \underset{\sim}{V}^{\varepsilon}, \text{ such that: } \gamma_s(v) = 0, \gamma_n(v) = 0\}.$$

Then $\underset{\sim}{V}^{\varepsilon}_{KL}$ which can be equipped with the norm of $\underset{\sim}{V}^{\varepsilon}$ (because it is a closed subspace), is isomorphic to the space :

$$\underset{\sim}{W}^{\varepsilon} = \{v = (v_t, v_3); \text{ such that } v_3 \in H^2_o(\omega) \text{ and } v_t = \mu \underset{\sim}{v}_t - x_3 \frac{\partial \overline{v_3}}{\partial m} \text{ where } \underset{\sim}{v}_t \in \underset{\sim}{V}_t\},$$

which is equipped with its natural norm :

$$v \to \{ \|v_3\|^2_{H^2(\omega)} + \|v_t\|^2_{\underset{\sim}{V}_t} \}^{1/2}$$

Proof : There will be two steps.

STEP 1. Let us solve the equations (3.4) for an element v of $\underset{\sim}{V}^{\varepsilon}$. From the second equation we deduce that v_3 is independent of the coordinate x_3. Then the first equation leads to :

$$\frac{\partial v_t}{\partial x_3} - \mu^{-1} \frac{\partial N}{\partial m} v_t = - \mu^{-1} \overline{\frac{\partial v_3}{\partial m}},$$

or else :

$$\mu^{-1} \frac{\partial v_t}{\partial x_3} - \mu^{-2} \frac{\partial N}{\partial m} v_t = - \mu^{-2} \overline{\frac{\partial v_3}{\partial m}},$$

and then because $\frac{\partial}{\partial x_3} \mu^{-1} = - \frac{\partial N}{\partial m}$,

$$\frac{\partial}{\partial x_3} (\mu^{-1} v_t) = - \mu^{-2} \overline{\frac{\partial v_3}{\partial m}}.$$

This is an ordinary differential equation with respect to the coordinate x_3, where the unknown is $\mu^{-1} u_t$.

The general solution is the sum of a particular solution and an element $\underline{u}_t$ independent on x_3. One can check that $- x_3 \mu^{-1} \overline{\frac{\partial v_3}{\partial m}}$ is precisely a particular solution and therefore :

$$v_t = \mu \underline{v}_t - x_3 \overline{\frac{\partial v_3}{\partial m}} .$$

Then it is clear that the elements of $\underline{V}^\epsilon_{KL}$ are the same as those of $\underline{W}^\epsilon$. We have therefore algebraically :

$$\underline{V}^\epsilon_{KL} = \underline{W}^\epsilon .$$

STEP 2. Let us denote by j the natural embedding from $\underline{W}^\epsilon$ into $\underline{V}^\epsilon_{KL}$ both equipped with their respective norm. It is clear that j is linear and one to one. From step one it is also onto. Hence from Banach's Theorem [22], j is an isomorphism and this completes the proof of Theorem 3.1.

∎

We define now the space of stress fields which have no normal components by :

$$(3.6) \qquad \underline{\Sigma}^\epsilon_t = \{\tau \in \underline{\Sigma}^\epsilon \quad ; \ \tau_s = \Pi \sigma N = 0, \ \tau_n = \bar{N} \tau N = 0\},$$

and which is equipped with the norm of $\underline{\Sigma}^\epsilon$.

A key point in the sequel is the :

THEOREM 3.2. There exists a strictly positive constant C such that for any element v in the space $\underline{V}^\epsilon_{KL}$ we have :

$$(3.7) \qquad \sup_{\tau \in \underline{\Sigma}^\epsilon_t} \frac{B(\tau, v)}{\|\tau\|_{\underline{\Sigma}^\epsilon}} \geq C \ \|v\|_{\underline{V}^\epsilon}$$

∎

Proof : First we notice that for any (τ, v) in the space $\underline{\Sigma}^\epsilon \times \underline{V}^\epsilon$ we have

$$- B(\tau, v) = \int_{\Omega^\varepsilon} \{Tr\ (\tau_t \cdot \gamma_t(v)) + \bar{\sigma}_s \cdot \gamma_s(v) + \sigma_n \cdot \gamma_n(v)\}$$

where

$$\gamma_t(v) = \frac{1}{2} \left(\Pi \frac{\partial v_t}{\partial m} \mu^{-1} + \mu^{-1} \overline{\frac{\partial v_t}{\partial m}} \Pi\right) + v_3 \frac{\partial N}{\partial m} \mu^{-1}.$$

and $\gamma_s(v)$, $\gamma_n(v)$ are defined in (3.5).

Hence for any element v in the space $\underset{\sim}{V}^\varepsilon_{KL}$ we have :

$$\sup_{\tau \in \underset{\sim}{\Sigma}^\varepsilon_t} \frac{B(\tau, v)}{\|\tau\|_{\underset{\sim}{\Sigma}^\varepsilon}} = \sup_{\tau \in \underset{\sim}{\Sigma}^\varepsilon} \frac{B(\tau, v)}{\|\tau\|_{\underset{\sim}{\Sigma}^\varepsilon}}$$

and the Theorem 3.2 is a consequence of Korn's inequality, (section 2).

∎

REMARK 3.1. As a consequence of both Theorems 3.1 and 3.2 there exists a constant C strictly positive such that for any element $v = (v_t, v_3)$ of the space $\underset{\sim}{V}^\varepsilon_{KL}$ with $v_t = \mu \underset{\sim}{v}_t - x_3 \overline{\frac{\partial v_3}{\partial m}}$, we have :

$$(3.8) \quad \sup_{\tau \in \underset{\sim}{\Sigma}^\varepsilon_t} \frac{B(\tau, v)}{\|\tau\|_{\underset{\sim}{\Sigma}^\varepsilon}} \geq C \{\|\underset{\sim}{v}_t\|^2_{\underset{\sim}{V}_t} + \|v_3\|^2_{H^2(\omega)}\}^{1/2}.$$

4. - THE KIRCHHOFF-LOVE SHELL MODEL

If we introduce the Kirchhoff-Love assumptions; the approximated model is defined as follows.

The displacement u° is an element of the space $\underset{\sim}{V}^\varepsilon_{KL}$ and the couple (σ°, u°) of the space $\underset{\sim}{\Sigma}^\varepsilon \times \underset{\sim}{V}^\varepsilon_{KL}$ should satisfy the equations :

$$(4.1) \quad \left\{ \forall\ \tau_t \in \underset{\sim}{\Sigma}^\varepsilon_t \ ;\ a(\sigma^\circ_t, \tau_t) + B(\tau_t, u^\circ) = 0, \right.$$

$$(4.2) \quad \left. \forall\ v \in \underset{\sim}{V}^\varepsilon \ ;\ B(\sigma^\circ, v) = F(v), \right.$$

where :

$$\sigma_t^o = \Pi\, \sigma^o\, \Pi.$$

REMARK 4.1. We don't assume that the stresses are plane (ie. σ_s^o and σ_n^o are not null). But the constitutive equation (4.1) is a weaker form of $(2.38)_1$ because we consider only the projection of $(2.38)_1$ on $\underset{\sim}{\Sigma}_t^\varepsilon$.

The main result of this section is the :

THEOREM 4.1. Let us assume that the medium surface of the shell ω and the forces applied are "smooth enough". Then equations (4.1) - (4.2) admit a unique solution (σ^o, u^o) in the space $\underset{\sim}{\Sigma}^\varepsilon \times \underset{\sim}{V}_{KL}$.

Proof : There will be three steps for the sake of clarity. ∎

STEP 1. Let us restrict in the equation (4.2) the elements v to the space $\underset{\sim}{V}_{KL}^\varepsilon$. We have then :

$$\forall\ v \in \underset{\sim}{V}_{KL}^\varepsilon,\quad B(\sigma_t^o, v) = F(v).$$

We consider the following problem :

find (σ_t^o, u^o) element of the space $\underset{\sim}{\Sigma}_t^\varepsilon \times \underset{\sim}{V}_{KL}^\varepsilon$ such that :

$$\forall\ \tau_t \in \underset{\sim}{\Sigma}_t^\varepsilon,\quad a(\sigma_t^o, \tau_t) + B(\tau_t, u^o) = 0, \tag{4.3}$$

$$\forall\ v \in \underset{\sim}{V}_{KL}^\varepsilon,\quad B(\sigma_t^o, v) = F(v). \tag{4.4}$$

We shall deduce from Brezzi's Theorem [4], the existence and uniqueness of a solution to the equations (4.3) - (4.4).

It is clear that a(.,.) and B(.,.) are respectively two bilinear bicontinueous forms on $\underset{\sim}{\Sigma}_t^\varepsilon \times \underset{\sim}{\Sigma}_t^\varepsilon$ and $\underset{\sim}{\Sigma}_t^\varepsilon \times \underset{\sim}{V}_t^\varepsilon$. The coercivity condition for a(.,.) is clear because $\underset{\sim}{\Sigma}_t^\varepsilon$ is a subspace of $\underset{\sim}{\Sigma}^\varepsilon$, and for B(.,.) it is given by Theorem 3.2.

As F(.) is a linear continuous form on $\underset{\sim}{V}^{\varepsilon}$ it is obvious that F(.) is also linear and continuous on the subspace $\underset{\sim}{V}^{\varepsilon}_{KL}$ of $\underset{\sim}{V}^{\varepsilon}$.

STEP 2. We have determined at step 1 an element (σ_t^o, u^o) such that the equation (4.1) would be satisfied. We are now going to determine σ_s^o and σ_n^o such that the equation (4-2) would be satisfied for any v in the space $\underset{\sim}{V}^{\varepsilon}$ and not only in $\underset{\sim}{V}^{\varepsilon}_{KL}$.

Expliciting (4-2) we obtain first :

$$(4.5)\quad \forall\, v_t \in \underset{\sim}{V}^{\varepsilon}_t;\ \int_{\Omega^\varepsilon} \bar{\sigma}_s^o\, \gamma_s(v_t) = \int_{\Omega^\varepsilon} \bar{f}_t\, v_t + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} \bar{\bar{g}}_t^{\pm}\, v_t - \int_{\Omega^\varepsilon} \mathrm{Tr}\,(\sigma_t^o\, \Pi\, \frac{\partial v_t}{\partial m}\, \mu^{-1})$$

and because

$$\gamma_s(v_t) = \frac{\partial v_t}{\partial x_3} - \mu^{-1}\, \frac{\partial N}{\partial m}\, v_t = \mu\, .\, \frac{\partial}{\partial x_3}\,(\mu^{-1}\, v_t),$$

setting $q_t = \mu^{-1}\, v_t$, the equation (4.5) leads to :

$$(4.6)\quad \forall\, q_t \in \underset{\sim}{V}^{\varepsilon}_t \quad \int_{\Omega^\varepsilon} \bar{\sigma}_s^o\, \mu\, \frac{\partial}{\partial x_3}(q_t) = \int_{\Omega^\varepsilon} \bar{f}_t\, \mu\, q_t + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} \bar{\bar{g}}_t^{\pm}\, \mu\, q_t$$

$$- \int_{\Omega^\varepsilon} \mathrm{Tr}\,(\sigma_t^o\, \Pi\, \frac{\partial}{\partial m}(\mu\, q_t)\, \mu^{-1})$$

Then a simple computational procedure leads to the differential equation (see the introduction).

$$(4.7)\quad \begin{cases} -\dfrac{\partial}{\partial x_3}(\mu\, \sigma_s^o\, \det \mu) = \mu \left\{ f_t\, \det \mu + \overline{\widehat{\mathrm{div}}\,(\mu^{-1}\, \sigma_t^o\, \det \mu)} \right\}, \\ \sigma_s^o = -\, g_t^- \qquad \text{when } x_3 = -\varepsilon, \\ \sigma_s^o = g_t^+ \qquad \text{when } x_3 = +\varepsilon. \end{cases}$$

From the first two relationships we deduce that :

$$(4.8)\quad \sigma_s^o = -\bar{g}_t - \frac{\mu^{-1}}{\det \mu}\int_{-\epsilon}^{x_3} \mu\left(\overline{f_t + \widehat{\operatorname{div}}(\mu^{-1}\,\sigma_t^o \det \mu)}\right)$$

For $x_3 = +\epsilon$ we have :

$$\sigma_s^o(+\epsilon) = -\bar{g}_t^- - \frac{\mu^{-1}}{\det \mu}\int_{-\epsilon}^{+\epsilon} \mu\left(\overline{f_t + \widehat{\operatorname{div}}(\mu^{-1}\,\sigma_t^o \det \mu)}\right);$$

multiplying this relation by $\mu\,\underset{\sim}{v}_t$ for any $\underset{\sim}{v}_t$ in the space $\underset{\sim}{V}_t$ we obtain :

$$\int_{\Gamma_+^\epsilon} \bar{\sigma}_s^o(+\epsilon)\,\mu\,\underset{\sim}{v}_t = -\int_{\Gamma_-^\epsilon} \bar{g}_t\,\mu\,\underset{\sim}{v}_t - \int_{\Omega^\epsilon} \bar{f}_t\,\mu\,\underset{\sim}{v}_t + \int_{\Omega^\epsilon} \mathrm{Tr}(\sigma_t^o\,\pi\,\frac{\partial}{\partial m}(\mu\,\underset{\sim}{v}_t)\,\mu^{-1}).$$

which is equal to

$$\int_{\Gamma_+^\epsilon} \overline{g_t^+}\,\mu\,\underset{\sim}{v}_t$$

because of the equation (4.4) where we choose $v = (\mu\,\underset{\sim}{v}_t,\ 0)$.

Therefore :

$$\sigma_s^o(+\epsilon) = g_t^+,$$

and the equation (4.5) is always satisfied with σ_s^o defined in (4.8). Furthermore the smoothness of the data allows us to assume that σ_s^o has its components in the space $L^2(\Omega^\epsilon)$.

STEP 3. Expliciting (4.2) we have then :

$$\forall v_3 \in V^\epsilon,\ \int_{\Omega^\epsilon} \sigma_n^o \frac{\partial v_3}{\partial x_3} = \int_{\Omega^\epsilon} f_3\,v_3 + \int_{\Gamma_+^\epsilon \cup \Gamma_-^\epsilon} g_3^\pm\,v_3 - \int_{\Omega^\epsilon} \bar{\sigma}_s^o\,\mu^{-1}\,\overline{\frac{\partial v_3}{\partial m}}$$

which leads to :

$$(4.9)\quad \begin{cases} -\dfrac{\partial}{\partial x_3}(\det \mu\ \sigma_n^o) = f_3 + \widehat{\operatorname{div}}\,(\mu^{-1}\,\sigma_s^o \det \mu) \\ \sigma_n^o = -\bar{g}_3^- \quad \text{when } x_3 = -\epsilon, \\ \sigma_n^o = g_3^+ \quad \text{when } x_3 = \epsilon \end{cases}$$

The first two equations (4.9) give :

$$\sigma_n^o = - g_3^- + \frac{1}{\det \mu} \int_{-\varepsilon}^{x_3} \widehat{div}(\mu^{-1} \sigma_s^o \det \mu)$$

and therefore :

$$\sigma_n^o(+\varepsilon) = - g_3^- + \frac{1}{\det \mu} \int_{-\varepsilon}^{+\varepsilon} \widehat{div}(\mu^{-1} \sigma_s^o \det \mu) \tag{4.10}$$

Multiplying this relation by v_3 we obtain :

$$\int_{\Gamma_-^\varepsilon} \sigma_n^o(+\varepsilon)\, v_3 = - \int_{\Gamma_-^\varepsilon} g_3^- v_3 - \int_{\Omega^\varepsilon} \bar{\sigma}_s^o \mu^{-1} \overline{\frac{\partial v_3}{\partial m}},$$

which is equal to :

$$\int_{\Gamma_+^\varepsilon} g_3^+ v_3$$

because of the equation (3.4) where we choose $v = (- x_3 \overline{\frac{\partial v_3}{\partial m}}, v_3)$.

Finally

$$\sigma_n^o(+\varepsilon) = g_3^+$$

Here again the smoothness of the data allows us to assume that σ_n^o is an element of the space $L^2(\Omega^\varepsilon)$.

Finally the equation (4.4) is satisfied for any v in the space $\underset{\sim}{V}^\varepsilon$ and the proof of Theorem 4.1 is completed.

∎

5. - THE BUDIANSKY SANDERS' SHELL MODEL

The problem of which (σ_t^o, u^o) is the solution is in fact three dimensional because of the presence of μ^{-1} and det μ in the expression of $B(.,.)$.

As the thickness is small compared to the radius of curvature of ω, (it does not mean that ε is small with respect to be derivatives of $\frac{\partial N}{\partial m}$), we can write :

$$\mu = I + x_3 \frac{\partial N}{\partial m} \simeq Id,$$

and

$$\begin{cases} \mu^{-1} \simeq Id, \\ \det \mu \simeq 1. \end{cases}$$

This approximation leads to an approximation of the Kirchhoff-Love's model which is known as Budiansky-Sander's one and which is bidimensional in the sense that the unknowns can be expressed with respect to a finite number of functions defined on ω.

We set for arbitrary elements σ_t; τ_t; v

$$a^0(\sigma_t, \tau_t) = \int_\omega \int_{-\varepsilon}^{\varepsilon} \frac{1+\nu}{E} \mathrm{Tr}(\sigma_t \cdot \tau_t) - \frac{\nu}{E} \mathrm{Tr}(\sigma_t) \cdot \mathrm{Tr}(\tau_t),$$

$$B^0(\tau_t, v) = - \int_\omega \int_{-\varepsilon}^{\varepsilon} \mathrm{Tr}\left(\tau_t \Pi \frac{\partial v_t}{\partial m}\right) + \mathrm{Tr}\left(\tau_t \frac{\partial N}{\partial m}\right) v_3.$$

The Budiansky-Sanders's model consists in finding an element (σ_t^{00}, u^{00}) in the space $\underset{\sim}{\Sigma}_t^\varepsilon \times \underset{\sim}{V}_{KL}$ such that :

$$(5.1) \quad \begin{cases} \forall \tau_t \in \underset{\sim}{\Sigma}_t^\varepsilon, \quad a^0(\sigma_t^{00}, \tau_t) + B^0(\tau_t, u^{00}) = 0, \\ \\ (5.2) \quad \forall\, v \in \underset{\sim}{V}_{KL}^\varepsilon, \quad B^0(\sigma_t^{00}, v) = F(v). \end{cases}$$

The shear stress σ_s^{00} and the normal component σ_n^{00} being still given by the formulas (4.8) and (4.10).

THEOREM 5.1. Under the same assumptions as in Theorem 4.1, there exists a unique solution (σ_t^{00}, u^{00}) in the space $\underset{\sim}{\Sigma}_t^\varepsilon \times \underset{\sim}{V}_{KL}^\varepsilon$ to the equations (5.1) - (5.2).

∎

Proof : Once more it is a consequence of the Brezzi's Theorem [4] for which we have to check the coercivity of the bilinear forms $a^0(.,.)$ and $B^0(.,.)$, (section 2).

First we have for any element τ_t in $\underset{\sim}{\Sigma}_t^\varepsilon$

$$a^0(\tau_t,\tau_t) = \int_\omega \int_{-\varepsilon}^{\varepsilon} \frac{1+\nu}{E} \mathrm{Tr}\ (\tau_t\ \tau_t) - \frac{\nu}{E} \mathrm{Tr}\ (\tau_t)^2$$

$$= \int_{\Omega^\varepsilon} \frac{1}{\det \mu} \left(\frac{(1+\nu)}{E} \mathrm{Tr}(\tau_t \cdot \tau_t) - \frac{\nu}{E} \mathrm{Tr}(\tau_t)^2\right) \geq C \|\tau_t\|^2_{\underset{\sim}{\Sigma}_t^\varepsilon}$$

Next for any element v in the space $\underset{\sim}{V}_{KL}^\varepsilon$:

$$\sup_{\tau_t \in \underset{\sim}{\Sigma}_t^\varepsilon} \frac{B^0(\tau_t,v)}{\|\tau_t\|_{\underset{\sim}{\Sigma}_t^\varepsilon}} = \sup_{\tau_t \in \underset{\sim}{\Sigma}_t^\varepsilon} \frac{\int_\omega \int_{-\varepsilon}^{+\varepsilon} \mathrm{Tr}\ (\tau_t \amalg \frac{\partial v_t}{\partial m}) + \mathrm{Tr}\ (\frac{\partial N}{\partial m} \tau_t)\ v_3}{\|\tau_t\|_{\underset{\sim}{\Sigma}_t^\varepsilon}}$$

$$\geq (1 - \varepsilon \left|\left|\left|\frac{\partial N}{\partial m}\right|\right|\right|)\ C \sup_{\tau_t \in \underset{\sim}{\Sigma}_t^\varepsilon} \frac{\int_{\Omega^\varepsilon} \mathrm{Tr}\ (\tau_t \amalg \frac{\partial v_t}{\partial m} \mu^{-1}) + \int_{\Omega^\varepsilon} \mathrm{Tr}\ (\tau_t \frac{\partial N}{\partial m} \mu^{-1})\ v_3}{\|\tau_t\|_{\underset{\sim}{\Sigma}_t^\varepsilon}}$$

$$= (1 - \varepsilon \left|\left|\left|\frac{\partial N}{\partial m}\right|\right|\right|)\ C \sup_{\tau_t \in \underset{\sim}{\Sigma}_t^\varepsilon} \frac{B(\tau_t,\ v)}{\|\tau_t\|_{\underset{\sim}{\Sigma}_t^\varepsilon}}$$

and from Theorem 3.2 this last quantity is lower bounded by $C\|v\|$ The proof of Theorem 5.1 is then obtained from Brezzi's Theorem[4] $\underset{\sim}{V}^\varepsilon$.

■

A direct proof (which does not use the three dimensional Korn's inequality) has been given in [3] for a slightly different model known as Koiter's model.

---

* We have set :

$$\left|\left|\left|\frac{\partial N}{\partial m}\right|\right|\right| = \sup_{m \in \omega} \{\mathrm{Tr}\ (\frac{\partial N}{\partial m} \cdot \frac{\partial N}{\partial m})\}^{1/2} \quad \text{and therefore :}$$

$$\left|\left|\left| I - \mu \right|\right|\right| \leq \varepsilon \left|\left|\left|\frac{\partial N}{\partial m}\right|\right|\right| .$$

We give now an explicit form of the Budiansky-Sanders shell model as defined in (5.1) - (5.2) using a curvilinear system of coordinates (see section 2).

The tangential stress field σ_t^{oo} will be written :

$$\sigma_t^{oo} = \sigma_\beta^\alpha \, a_\alpha \, a^\beta, \tag{5.3}$$

the tangential displacement

$$u_t^{oo} = u^\alpha \, a_\alpha \tag{5.4}$$

which is such that

$$u_t^{oo} = \mu \, \underset{\sim}{u}_t^{oo} - x_3 \, \overline{\frac{\partial u_3^{oo}}{\partial m}} = \{(\delta_\beta^\alpha - x_3 \, b_\beta^\alpha) \, \underset{\sim}{u}^\beta - x_3 \, g^{\alpha\beta} \, u_{3,\beta}\} \, a_\alpha \tag{5.5}$$

therefore

$$u^\alpha = (\delta_\beta^\alpha - x_3 \, b_\beta^\alpha) \, \underset{\sim}{u}^\beta - x_3 \, g^{\alpha\beta} \, u_{3,\beta},$$

(the upper script index ^{oo}, is omitted on the components for sake of clarity in the notations).

We set

$$\gamma(u^{oo}) = \frac{1}{2} \left(\Pi \, \frac{\partial \underset{\sim}{u}_t^{oo}}{\partial m} + \overline{\frac{\partial \underset{\sim}{u}_t^{oo}}{\partial m}} \, \Pi \right) = \gamma_\beta^\alpha \, a_\alpha \, a^\beta, \tag{5.6}$$

with $\gamma_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha})$,

and

$$\chi(u^{oo}) = \gamma(\underset{\sim}{u}^{oo}) + \frac{\partial N}{\partial m} \, u_3^{oo} = \chi_\beta^\alpha \, a_\alpha \, a^\beta, \tag{5.7}$$

with $\chi_{\alpha\beta} = \gamma_{\alpha\beta} - b_{\alpha\beta} \, u_3$.

Introducing the tangential rotation θ^{oo} by

$$\theta^{oo} = \frac{\partial N}{\partial m} \underset{\sim}{u}^{oo}_t - \frac{\overline{\partial u_3^{oo}}}{\partial m} = \theta^\alpha a_\alpha, \tag{5.8}$$

we have :

$$\theta^\alpha = - \{ b^\alpha_\beta \underset{\sim}{u}^\beta + g^{\alpha\beta} u_{3,\beta} \}$$

and the change of curvature operator is defined by :

$$\rho(u^{oo}) = \gamma(\vartheta^{oo}) = \rho^\alpha_\beta a_\alpha a^\beta, \tag{5.9}$$

with

$$\rho_{\alpha\beta} = - \frac{1}{2} \{ b_{\alpha\gamma} \underset{\sim}{u}^\gamma|_\beta + b_{\alpha\gamma}|_\beta \underset{\sim}{u}^\gamma + b_{\beta\gamma} \underset{\sim}{u}^\gamma|_\alpha + b_{\beta\gamma}|_\beta \underset{\sim}{u}^\gamma \} - u_3|_{\alpha\beta}.$$

Then equation (5.1) which can be also written :

$$\frac{1+\nu}{E} \sigma^{oo}_t - \frac{\nu}{E} Tr(\sigma^{oo}_t) I_d = \underset{\sim}{\gamma}(u^{oo}) + x_3 \rho(u^{oo}) \tag{5.10}$$

leads to :

$$\sigma^\alpha_\beta = \frac{E}{1-\nu^2} \{ (1-\nu) \underset{\sim}{\gamma}^\alpha_\beta + \nu \underset{\sim}{\gamma}^\mu_\mu \delta^\alpha_\beta \} + \frac{E x_3}{1-\nu^2} \{ (1-\nu) \rho^\alpha_\beta + \nu \rho^\mu_\mu \delta^\alpha_\beta \}$$

We usually set :

$$\begin{cases} n^\alpha_\beta = \dfrac{1}{2\varepsilon} \displaystyle\int_{-\varepsilon}^{\varepsilon} \sigma^\alpha_\beta \, dx_3 \\ m^\alpha_\beta = \dfrac{3}{2\varepsilon^3} \displaystyle\int_{-\varepsilon}^{+\varepsilon} x_3 \, \sigma^\alpha_\beta \, dx_3 \end{cases} \tag{5.11}$$

which are called the resultant stress and bending moment.

Then equation (5.2) can be written :

(5.12) $\forall v = (\underset{\sim}{v}^{\alpha}, v_3) \in (H_o^1(\omega))^2 \times H_o^2(\omega)$,

$$\int_{\omega} n_{\beta}^{\alpha}\ \underset{\sim}{\gamma}_{\alpha}^{\beta}(v) + \frac{\varepsilon^2}{3}\int_{\omega} m_{\beta}^{\alpha}\ \rho_{\alpha}^{\beta}(v) = \int_{\omega} F^{i}\, v_i - \int_{\omega} M^{\alpha}\, v_{3,\alpha}$$

where $\gamma_{\beta}^{\alpha}(v)$ and $\rho_{\alpha}^{\beta}(v)$ are the components of $\gamma(v)$ and $\rho(v)$ and where we have set :

$$\begin{aligned}
F^{\alpha} &= \frac{1}{2\varepsilon}\Big\{\int_{-\varepsilon}^{+\varepsilon} f^{\beta}\,(\delta_{\beta}^{\alpha} - x_3\, b_{\beta}^{\alpha})\ \big|\ \delta_{\mu}^{\lambda} - x_3\, b_{\mu}^{\lambda}\ \big|\Big\} \\
&+ \frac{(g^{+})^{\beta}}{2\varepsilon}(\delta_{\beta}^{\alpha} - \varepsilon\, b_{\beta}^{\alpha})\ \big|\ \delta_{\mu}^{\lambda} - \varepsilon\, b_{\mu}^{\lambda}\ \big| \\
&+ \frac{(g^{-})^{\beta}}{2\varepsilon}\,(\delta_{\beta}^{\alpha} + \varepsilon\, b_{\beta}^{\alpha})\ \big|\ \delta_{\mu}^{\lambda} + \varepsilon\, b_{\mu}^{\lambda}\ \big|, \\
F^{3} &= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} f_3 \big|\, \delta_{\mu}^{\lambda} - x_3\, b_{\mu}^{\lambda}\ \big| + \frac{g_3^{+} + g_3^{-}}{2\,\varepsilon}\,, \\
M^{\alpha} &= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} x_3\, f^{\alpha}\ \big|\delta_{\mu}^{\lambda} - x_3\, b_{\mu}^{\lambda}\ \big| + \frac{(g^{+})^{\alpha} - (g^{-})^{\alpha}}{2}
\end{aligned} \tag{5.13}$$

(the notation $|d_{\beta}^{\alpha}|$, denotes the determinant of a tensor d_{β}^{α}).

The Budiansky-Sanders's model consists in finding an element $u = (\underset{\sim}{u}^{\alpha}, u_3)$ in the space $(H_o^1(\omega))^2 \times H_o^2(\omega)$ satisfying (5.11) and (5.12).

For a homogeneous material which follows the Hooke's law, equations (4.11) can be explicited as follows :

$$\begin{aligned}
n_{\beta}^{\alpha} &= \frac{E}{1-\nu^2}\{(1-\nu)\ \underset{\sim}{\gamma} + \nu\ \underset{\sim}{\gamma}_{\mu}^{\mu}\ \delta_{\beta}^{\alpha}\} \\
m_{\beta}^{\alpha} &= \frac{E}{(1-\nu^2)}\ \{(1-\nu)\rho_{\beta}^{\alpha} + \nu\ \rho_{\mu}^{\mu}\ \delta_{\beta}^{\alpha}\}
\end{aligned} \tag{5.14}$$

where E and ν denote the Young's modulus and Poisson's coefficient.

REMARK 5.1. Error estimates between the three dimensional solution (σ, u) and (σ^{oo}, u^{oo}) can be obtained via the asymptotic methods [7], the discussion rests on the boundary conditions on Γ_o^ε on the regularity of the medium surface ω and on the forces applied as well.

It would be too long to detail these results here. We mention a basic discussion concerning the validity of shell models by M. Dikmen [10].

6. - THE MEMBRANE THEORY FOR A THIN ELLIPTIC SHELL

We analyze in this section the asymptotic behaviour of the Budiansky-Sanders' shell model when the thickness is very small when compared to the other dimensions of the shell (radius of curvature, maximum length ...).

For sake of simplicity we consider as before that the membrane shell is made in a material which obeys Hooke's law. But the results could be extended to various materials.

The equations (4.14) and (4.12) can then be written in a variational form as follows. We define for arbitrary elements (n_t, p_t) and (m_t, q_t) of the space $\Sigma_t \times \Sigma_t$, the following bilinear forms :

$$(6.1)\qquad \begin{cases} a_o(n_t, p_t) = \displaystyle\int_\omega \frac{1+\nu}{E} \mathrm{Tr}\,(n_t\, p_t) - \frac{\nu}{E} \mathrm{Tr}\,(n_t)\,\mathrm{Tr}\,(p_t), \\ a_2(m_t, q_t) = \displaystyle\int_\omega \frac{1+\nu}{E} \mathrm{Tr}\,(m_t\, q_t) - \frac{\nu}{E} \mathrm{Tr}\,(m_t)\,\mathrm{Tr}\,(q_t), \end{cases}$$

and for arbitrary elements (n_t, m_t) of the space $\Sigma_t \times \Sigma_t$ and $v = (\underline{v}_t, v_3)$ of the space $\underline{V}_t \times H_o^2(\omega)$:

$$(6.2)\qquad \begin{cases} b_o(n_t, v) = -\displaystyle\int_\omega \mathrm{Tr}_2\Big(n_t\, \Pi\, \frac{\partial \underline{v}_t}{\partial m}\Big) - \int_\omega \mathrm{Tr}_2\Big(n_t\, \frac{\partial N}{\partial m}\Big)\, v_3 \\ b_2(m_t, v) = -\frac{1}{3} \displaystyle\int_\omega \mathrm{Tr}\,\Big(m_t\, \Pi\, \frac{\partial}{\partial m}\Big(\frac{\partial N}{\partial m}\, \underline{v}_t\Big)\Big) \\ \qquad\qquad + \frac{1}{3} \displaystyle\int_\omega \mathrm{Tr}\,\Big(m_t\, \Pi\, \frac{\partial}{\partial m}\Big(\overline{\frac{\partial v_3}{\partial m}}\Big)\Big) \end{cases}$$

With these new notations the problem (4.1) (4.2) can also be written as follows : (the old upperscript must be forgotten from now on),

find (n_t, m_t), $(\underset{\sim}{u}_t, u_3) \in \underset{\sim}{\Sigma}_t \times \underset{\sim}{\Sigma}_t \times \underset{\sim}{V}_t \times H_o^2(\omega)$,

such that :

$$(6.3)\qquad \begin{cases} \forall\ p_t \in \underset{\sim}{\Sigma}_t \quad , \ a_o(n_t, p_t) + b_o(p_t, u) = 0, \\ \forall\ q_t \in \underset{\sim}{\Sigma}_t \quad , \ a_2(m_t, q_t) + b_2(q_t, u) = 0, \\ \forall\ v = (\underset{\sim}{v}_t, v_3) \in \underset{\sim}{V}_t \times H_o^2(\omega), \ \ b_o(n_t, v) + \varepsilon^2\, b_2(m_t, v) = F(v) \end{cases}$$

The advantage of this notation is that the small parameter ε appears explicitely.

We set a priori :

$$(6.4)\qquad (n_t, m_t, \underset{\sim}{u}_t, u_3) = (n_t^o, m_t^o, \underset{\sim}{u}_t^o, u_3^o) + \varepsilon^2 (n_t^2, m_t^2, \underset{\sim}{u}_t^2, u_3^2) + \ldots$$

Introducing the expression in (6.3) and by equating the terms of same power in ε in the resulting expression, for arbitrary elements (p_t, q_t, v) we obtain :

$$(6.5)\qquad \begin{cases} a_o\,(n_t^o, p_t) + b_o(p_t, u^o) = 0, \\ a_2\,(m_t^o, q_t) + b_2(q_t, u^o) = 0, \\ b_o\,(n_t^o, v) = F(v). \end{cases}$$

The first question concerns the existence and uniqueness of a solution to the equations (6.5).

This is a non trivial problem and it is very connected to the nature of the medium surface ω, (parabolic, hyperbolic or elliptic). Whatever the case the second equation (6.5) leads to the relation :

$$(6.6)\qquad m_t^o = \frac{E}{(1-\nu^2)} \{(1-\nu)\ \rho(u^o) + \nu\ \mathrm{Tr}(\rho(u^o))\ \mathrm{Id}\},$$

and (n_t^o, u^o) are to be determined from the first and the third equation (6.5).

From the third equation (6.5) we deduce in particular that :

$$(6.7) \qquad \forall v_3 \in H_o^2(\omega), \quad \int_\omega \mathrm{Tr}\ (n_t^o \frac{\partial N}{\partial m})\ v_3 = \int_\omega F^3\ v_3 - \int_\omega M^\alpha\ v_{3,\alpha},$$

where F^3 and M^α have already been defined in (5.13).

This leads to :

$$(6.8) \qquad n_t^o = \frac{F^3 + \frac{1}{|g|^{1/2}} (M^\alpha |g|^{1/2})_{,\alpha}}{\mathrm{Tr}\ (\frac{\partial N}{\partial m} \cdot \frac{\partial N}{\partial m})} \frac{\partial N}{\partial m} + \underset{\sim}{n}_t^o = n_t^p + \underset{\sim}{n}_t^o$$

where

$$(6.9) \qquad \mathrm{Tr}\ (\underset{\sim}{n}_t^o \frac{\partial N}{\partial m}) = 0,$$

we introduce the space of tangential stress field satisfying (6.9) by :

$$(6.10) \qquad \underset{\sim}{y}_t = \{\tau_t \in \underset{\sim}{L}_t, \quad \mathrm{Tr}\ (\tau_t \frac{\partial N}{\partial m}) = 0\}.$$

Then the first equation (6.5) can be split into the following two relationships.

(i) $\qquad \forall\ \tau_t \in \underset{\sim}{y}_t \quad a_o(n_t^o, \tau_t) + b_o(\tau_t, u^o) = 0,$

and

(ii) $\qquad a_o(n_t^o, \frac{\partial N}{\partial m}) + b_o(\frac{\partial N}{\partial m}, u^o) = 0.$

The last one enables u_3^o to be determined by :

$$(6.11) \qquad u_3^o = (\frac{1+\nu}{E} \mathrm{Tr}\ (n_t^o \frac{\partial N}{\partial m}) - \frac{\nu}{E} \mathrm{Tr}\ (n_t^o)\ \mathrm{Tr}\ (\frac{\partial N}{\partial m}) - \mathrm{Tr}\ (\frac{\partial N}{\partial m}\ \Pi \frac{\partial \underset{\sim}{u}_t^o}{\partial m})) / \mathrm{Tr}\ (\frac{\partial N}{\partial m} \cdot \frac{\partial N}{\partial m}).$$

Then we have to solve the following system:find $(\underset{\sim}{n}_t^o, \underset{\sim}{u}_t^o) \in \underset{\sim}{y}_t^o \times \underset{\sim}{V}_t^o$ such that :

$$(6.12)\qquad \begin{cases} \forall\, \tau_t \in \mathcal{Y}_t^o\ ,\ \ a_o(\underset{\sim}{n}_t^o,\ \tau_t) + b_o(\tau_t,\ \underset{\sim}{u}_t^o) = -\, a_o(n_t^p,\ \tau_t), \\ \forall\, \underset{\sim}{v}_t \in \underset{\sim}{\mathcal{V}}_t^o\ ,\ \ b_o(\underset{\sim}{n}_t^o,\ \underset{\sim}{v}_t) = -\, b_o(n_t^p,\ \underset{\sim}{v}_t) - \int_\omega \bar{F}_t \cdot \underset{\sim}{v}_t, \end{cases}$$

where $F_t = F^\alpha\, a_\alpha$, F^α being defined in (5.13).

One can check straight forwardly that if there exists a solution then the component n_t^o is unique.

But that is not so clear as far as the component $\underset{\sim}{u}_t^o$ is concerned. This leads to the concept of inextensional movements [14].

We say that $\underset{\sim}{v}_t$ is an inextensional displacement field if :

$$(6.13)\qquad \forall\, \tau_t \in \mathcal{Y}_t^o \quad b_o(\tau_t,\ \underset{\sim}{v}_t) = 0,$$

another way to write (6.13) is :

$$(6.14)\qquad \Pi\, \frac{\partial \underset{\sim}{v}_t}{\partial m} + \overline{\frac{\partial \underset{\sim}{v}_t}{\partial m}}\, \Pi - 2\, \frac{\partial N}{\partial m}\, \mathrm{Tr}\ \left(\frac{\partial N}{\partial m}\, \Pi\, \frac{\partial \underset{\sim}{v}_t}{\partial m}\right) / \mathrm{Tr}\ \left(\frac{\partial N}{\partial m}\, \frac{\partial N}{\partial m}\right) = 0.$$

Using a system of curvilinear coordinates (6.14) can also be written :

$$(6.15)\qquad v_{\alpha|\beta} + v_{\beta|\alpha} - 2(b_{\alpha\beta}\, b^\mu_\delta\ v^\delta_{|\mu}) \Big/ (b^\mu_\delta\, b^\delta_\mu) = 0$$

where we set :

$$\underset{\sim}{v}_t = v^\alpha\, a_\alpha.$$

If we consider the system of coordinates generated by the lines of principal curvature [13], (6.15) leads to :

$$(6.16)\qquad \begin{cases} v_{1|1} - \dfrac{R_1\, R_2^2}{R_1^2 + R_2^2}\, \left(\dfrac{v_{1|1}}{R_1} + \dfrac{v_{2|2}}{R_2}\right) = 0, \\ v_{1|2} + v_{2|1} = 0 \\ v_{2|2} - \dfrac{R_1^2\, R_2}{R_1^2 + R_2^2}\, \left(\dfrac{v_{1|1}}{R_1} + \dfrac{v_{2|2}}{R_2}\right) = 0, \end{cases}$$

where $\frac{1}{R_1}$ and $\frac{1}{R_2}$ are the two eigenvalues of the curvature operator. It is a basic point of the membrane's theory that for a given shell the boundary conditions should be such that no solution of (6.14) would be admissible. Otherwise the membrane's theory should be rejected. Non linear models should then be used.

As a matter of fact non linear terms would become predominant and the hypotheses of small displacement is certainly not valid.

In order to prove an existence and uniqueness Theorem for the equation (6.12) we make use of Brezzi's Theorem [4]. Because the bilinear form $a_o(.,.)$ is obviously coercive on the space $\underset{\sim}{\Sigma}_t$ we have only to prove the so-called Brezzi's condition i.e. for any $\underset{\sim}{y}_t$ in the space $\underset{\sim}{V}_t$:

$$\sup_{\tau_t \in \underset{\sim}{y}_t} \frac{b_o(\tau_t, v_t)}{\|\tau_t\|_{\underset{\sim}{\Sigma}_t}} \geq C \|\underset{\sim}{y}_t\|_{\underset{\sim}{V}_t} , \tag{6.17}$$

where $C > 0$.

This inequality is not always true. It will be proved for an uniformly elliptic medium surface in Theorem 6.2. Inequality (6.17) will ensure when satisfied that the shell does not admit inextensional movements and we have as a consequence :

THEOREM 6.1. If the medium surface ω of the shell is such that (6.17) is true, then the membrane equations (6.12) have a unique solution $(\underset{\sim}{n}_t^o, \underset{\sim}{u}_t^o)$ in the space $\underset{\sim}{y}_t \times \underset{\sim}{V}_t$.

■

In order to satisfy (6.17) we assume first that the medium surface of the shell is uniformly elliptic. If we denote by $\frac{1}{R_1}$ and $\frac{1}{R_2}$ the two eigenvalues of the curvature operator $\frac{\partial N}{\partial m}$, it means that $\frac{1}{R_1}$ and $\frac{1}{R_2}$ have the same sign and that everywhere on ω we have :

$$\min(|\frac{1}{R_1}|; |\frac{1}{R_2}|) \geq C_o > 0. \tag{6.18}$$

where C_o is a strictly positive constant.

The surface curves which are tangent at each point of ω to the eigenvectors of $\frac{\partial N}{\partial m}$ are called the lines of curvature.

In the theory of surfaces it is proved that every surface can be referred to lines of curvatures, i.e., the lines of curvature of any given surface can be used as curvilinear coordinates of that surface. The coordinates curves are thus, in general, uniquely determined. The exceptional case arises when the surface has regions of constant curvature. Such regions represent parts of a sphere and on a sphere every curve can be considered as a line of curvature.

It is now proved that the inequality (6.17) is satisfied for a surface ω satisfying (6.18). For this purpose the following functional space is introduced :

$$(6.19)\quad \underset{\sim}{\mathcal{W}}_t = \{v_t \in \underset{\sim}{H}_t ;\ \Pi \frac{\partial v_t}{\partial m} + \overline{\frac{\partial v_t}{\partial m}} \Pi - 2 \frac{\partial N}{\partial m} \frac{\mathrm{Tr}(\frac{\partial N}{\partial m} \Pi \frac{\partial v_t}{\partial m})}{\mathrm{Tr}(\frac{\partial N}{\partial m} \cdot \frac{\partial N}{\partial m})} \in \underset{\sim}{L}_t\},$$

It is equipped with its canonical norm :

$$(6.20)\quad \|v_t\|_{\underset{\sim}{\mathcal{W}}_t} = \{\|v_t\|^2_{\underset{\sim}{H}_t} + \|\Pi \frac{\partial v_t}{\partial m} + \overline{\frac{\partial v_t}{\partial m}} \Pi - 2 \frac{\partial N}{\partial m} \cdot \frac{\mathrm{Tr}(\frac{\partial N}{\partial m} \cdot \Pi \frac{\partial v_t}{\partial m})}{\mathrm{Tr}(\frac{\partial N}{\partial m} \cdot \frac{\partial N}{\partial m})} \|^2_{\underset{\sim}{L}_t}\}$$

The first result proved hereafter concerns the definition of a trace on the boundary γ of ω of an element v_t in the space $\underset{\sim}{\mathcal{W}}_t$.

THEOREM 6.2. Let $v_t = v^\alpha a_\alpha$ be an element of the space $\underset{\sim}{\mathcal{W}}_t$ defined in (6.19). Then the restriction of v^α (or v_α), to the boundary γ of ω is continuously defined as an element of $H^{-1/2}(\gamma)$.

∎

Proof : Let $v_t = v^\alpha a_\alpha$ be an element of the space $\underset{\sim}{\mathcal{W}}_t$. The components v^α, or v_α, are elements of the space $L^2(\omega)$. But expressed with respect to the coordinates (ξ^1, ξ^2), v_α are also elements of the space $L^2(\bar{\omega})$.

From :

$$\Pi \frac{\partial v_t}{\partial m} + \overline{\frac{\partial v_t}{\partial m}} \Pi - 2 \frac{\partial N}{\partial m} \frac{\mathrm{Tr}(\frac{\partial N}{\partial m} \Pi \frac{\partial v_t}{\partial m})}{\mathrm{Tr}(\frac{\partial N}{\partial m} \cdot \frac{\partial N}{\partial m})} \in \underset{\sim}{L}_t,$$

using a system of curvilinear coordinates we obtain :

$$(6.21) \qquad \gamma_{\alpha\beta}(v_\tau) - b_{\alpha\beta} \cdot \frac{b^{\mu\nu}\,\gamma_{\mu\nu}(v_\tau)}{b^{\mu\nu}\,b_{\mu\nu}} \in L^2(\omega).$$

and for the system of coordinates generated by the lines of curvature the curvature tensor is diagonal and the equations (6.21) can be explicited as follows :

$$(6.22) \qquad \begin{cases} v_{1|1} - \dfrac{R_1 R_2^2}{R_1^2 + R_2^2}\left(\dfrac{v_{1|1}}{R_1} + \dfrac{v_{2|2}}{R_2}\right) \in L^2(\omega), \\ v_{1|2} + v_{2|1} \in L^2(\omega) \\ v_{2|1} - \dfrac{R_1^2 R_2}{R_1^2 + R_2^2}\left(\dfrac{v_{1|1}}{R_1} + \dfrac{v_{2|2}}{R_2}\right) \in L^2(\omega) \end{cases}$$

One can observe that the third equation (6.22) is equivalent to the first one. Because $v_\lambda \in L^2(\omega)$ and :

$$v_{\alpha|\beta} = v_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta} v_\lambda,$$

then from (6.22) we obtain :

$$\begin{cases} R_1\, v_{1,1} - R_2\, v_{2,2} \in L^2(\omega), \\ v_{1,2} + v_{2,1} \in L^2(\omega). \end{cases}$$

Taking the derivative of these equations with respect to ξ^1 and ξ^2 we deduce on the one hand :

$$(6.23)_1 \qquad \begin{cases} \left(\dfrac{R_1}{R_2} v_{1,1}\right),\ 1 + v_{1,22} \in H^{-1}(\hat{\omega}); \\ v_1 \in L^2(\hat{\omega}), \end{cases}$$

and on the other

$$(6.23)_2 \qquad \begin{cases} \left(\dfrac{R_2}{R_1} v_{2,2}\right),\ 2 + v_{2,11} \in H^{-1}(\hat{\omega}), \\ v_2 \in L^2(\hat{\omega}). \end{cases}$$

As R_1 and R_2 have the same sign, and satisfies (6.18), v_α are solution of an elliptic problem (set on $\hat{\omega}$).

For any function φ element of the space $H^2(\hat{\omega}) \cap H_o^1(\hat{\omega})$, we define the continuous linear form.

$$L_1(\varphi) = -\langle(\frac{R_1}{R_2} v_{1,1}),1 + v_{1,22}, \varphi\rangle + \langle v_1, (\frac{R_1}{R_2}\varphi_{,1}),1 + \varphi_{,22}\rangle$$

where $\langle , \rangle$ denotes the duality between $\mathcal{D}(\hat{\omega})$ and $\mathcal{D}'(\hat{\omega})$. For any function φ element of the space $H_o^2(\hat{\omega})$ we have $L_1(\varphi) = 0$. Hence $L_1(\varphi)$ depends only on $\frac{\partial\varphi}{\partial n}$, (normal derivative of φ on the boundary $\hat{\gamma}$ of $\hat{\omega}$).

Let g be an element of the space $H^{1/2}(\hat{\gamma})$. There exists [17] a continuous mapping from $H^{1/2}(\hat{\gamma})$ into $H_o^1(\hat{\omega}) \cap H^2(\hat{\omega})$, say $\mathcal{G}_1$, such that :

$$\frac{R_1}{R_2} b_1 \frac{\partial}{\partial\xi_1} \mathcal{G}_1(g) + b_2 \frac{\partial}{\partial\xi_2} \mathcal{G}_1(g) = g, \tag{6.24}$$

$b = (b_\alpha)$ denoting the unit outward normal to the boundary $\hat{\gamma}$).

We define a linear form on $H^{1/2}(\hat{\gamma})$ by :

$$\mathcal{L}_1(g) = L_1(\mathcal{G}_1(g))$$

which is clearly continuous, (because of the continuity of $\mathcal{G}_1$). Therefore $\mathcal{L}_1$ can be identified with an element, say, ℓ_1 of the space $H^{-1/2}(\hat{\gamma})$. We say that the trace of v_1 on the boundary $\hat{\gamma}$ of $\hat{\omega}$ is ℓ_1.

Because of (6.24) we check straight-forwardly that this definition is identical to the classical one when v_1 is smooth.

It is clear that the trace of v_2 can also be defined in a similar way.

From :

$$\langle v_1, g\rangle_{H^{-1/2}(\hat{\gamma})\times H^{1/2}(\hat{\gamma})} = \mathcal{L}_1(g) = \langle v_1, (\frac{R_1}{R_2}\mathcal{G}_1(g)_{,1})_{,1} + \mathcal{G}_1(g)_{,22}\rangle - \langle(\frac{R_1}{R_2} v_{1,1})_{,1} + v_{1,22}, \mathcal{G}_1(g)\rangle \tag{6.25}$$

we deduce

$$(6.26)_1 \quad \|v_1\|_{H^{-1/2}(\hat{\gamma})} = \sup_{g \in H^{1/2}(\hat{\gamma})} \frac{\langle v_1, g \rangle_{H^{-1/2}(\hat{\gamma}) \times H^{1/2}(\hat{\gamma})}}{\|g\|_{H^{1/2}(\hat{\gamma})}} \leq C\Big(\|v_1\|_{L^2(\hat{\omega})} + \|(\frac{R_1}{R_2} v_{1,1})_{,1} + v_{1,22}\|_{H^{-1}(\hat{\omega})} \Big),$$

and in a similar way :

$$(6.26)_2 \quad \|v_2\|_{H^{-1/2}(\hat{\gamma})} \leq C\Big(\|v_2\|_{L^2(\hat{\omega})} + \|(\frac{R_2}{R_1} v_{2,2})_{,2} + v_{2,11}\|_{H^{-1}(\hat{\omega})} \Big).$$

Combining $(6.26)_1$ - $(6.26)_2$ and using the fact that the derivation is continuous from $L^2(\hat{\omega})$ into $H^{-1}(\hat{\omega})$; we deduce the continuity of trace of v_t element of $\underset{\sim}{\mathcal{W}}_t$ into $(H^{-1/2}(\hat{\gamma}))^2$. This completes the proof of theorem 6.2.

∎

We define now the space :

$$\underset{\sim}{W}_t = \{ v_t \in \underset{\sim}{\mathcal{W}}_t ; \ v_t = 0 \text{ on } \gamma \}, \tag{6.27}$$

and $\underset{\sim}{W}_t$ is closed subspace of $\underset{\sim}{\mathcal{W}}_t$ from theorem 6.2. The next result is very similar to Korn's inequality [11].

THEOREM 6.3. Under assumption (6.18) the spaces $\underset{\sim}{V}_t$ and $\underset{\sim}{W}_t$ are isomorphic.

∎

Proof. Let us denote by j the natural embedding from $\underset{\sim}{V}_t$ into $\underset{\sim}{W}_t$, (it is clear that $\underset{\sim}{V}_t \subset \underset{\sim}{W}_t$). As j is linear, one to one and continuous we shall prove that j is an isomorphism iff we prove that j is onto, (Banach's Theorem [22]). Let $v_t = v^\alpha a_\alpha$ be an element of $\underset{\sim}{W}_t$.

On the one hand v_α is an element of the space $L^2(\hat{\omega})$ and $v_\alpha = 0$ on $\hat{\gamma}$. On the other from (6.22):

$$\begin{cases} (\frac{R_1}{R_2} v_{1,1})_{,1} + v_{1,22} \in H^{-1}(\hat{\omega}), \\ v_1 \in L^2(\hat{\omega}), \end{cases}$$

$$\begin{cases} (\frac{R_2}{R_1} v_{2,2})_{,2} + v_{2,11} \in H^{-1}(\hat{\omega}), \\ v_2 \in L^2(\hat{\omega}). \end{cases}$$

Let us introduce the element $\underline{v}_1$ of the space $H_o^1(\hat{\omega})$ such that :

$$(\frac{R_1}{R_2} \underline{v}_{1,1})_{,1} + \underline{v}_{1,22} = - (\frac{R_1}{R_2} v_{1,1})_{,1} - v_{1,22}.$$

Because the operator $A = - \frac{\partial}{\partial \xi^1} (\frac{R_1}{R_2} \frac{\partial}{\partial \xi^1}(.)) - \frac{\partial^2(.)}{(\partial \xi)^2}$, is elliptic, then $\underline{v}_1$ is uniquely defined. We set

$$w_1 = v_1 - \underline{v}_1 \tag{6.28}$$

Thus :

$$\begin{cases} w_1 \in L^2(\hat{\omega}), \\ A\, w_1 = 0, \\ w_1 = 0 \text{ on } \hat{\gamma}. \end{cases}$$

We assume now that the operator A is an isomorphism between $L^2(\hat{\omega})$ and $H_o^1(\hat{\omega}) \cap H^2(\hat{\omega})$. This is in fact a regularity assumption. It is satisfied as soon as R_α are elements of the space $\mathcal{C}^1(\hat{\omega})$, (this is satisfied as soon as the map φ is an element of the space $\mathcal{C}^3(\hat{\omega})$). The regularity of the boundary $\hat{\gamma}$ is also needed ($\mathcal{C}^1$), unless $\hat{\omega}$ is a convex set [17].

Let f be any element of the space $L^2(\hat{\omega})$ and φ the element of $H_o^1(\hat{\omega}) \cap H^2$ such that $A\varphi = f$. Then from (6.25) :

$$\langle w_1, f \rangle = \langle w_1, A\varphi \rangle = - \langle A\, w_1, \varphi \rangle = 0$$

(where $\langle , \rangle$ denote the duality between $\mathcal{D}(\hat{\omega})$ and $\mathcal{D}'(\hat{\omega})$).

As a consequence we deduce that $w_1 = 0$. Hence from (6.28) v_1 belongs to the space $H_o^1(\tilde{\omega})$. A similar proof ensures that v_2 is also an element of the space $H_o^1(\tilde{\omega})$ and therefore $v_t = v^\alpha a_\alpha$ is an element of $\underset{\sim}{V}_t$. The embedding j is thus onto and because of Banach's Theorem [22], is an isomorphism between $\underset{\sim}{V}_t$ and $\underset{\sim}{W}_t$.

∎

REMARK 6.1. The proof of Theorem 6.2 rests upon the fact that $v_\alpha = 0$ on the boundary of $\tilde{\omega}$. Otherwise the trace of v_α is only an element of $H^{-1/2}(\tilde{\gamma})$, and v_α would not be in general an element of the space $H^1(\tilde{\omega})$.

In order to prove the inequality (6.17) for a uniformly elliptic surface ω, we prove now :

THEOREM 6.4. Under the assumption (6.18) and if the Christoffel' symbols $\Gamma_{\alpha\beta}^{\lambda}$ are small enough in $\mathcal{C}^0$ norm, then the mapping :

$$(6.29)\quad v_t \in \underset{\sim}{V}_t \to \left\| \Pi \frac{\partial v_t}{\partial m} + \overline{\frac{\partial v_t}{\partial m}}\, \Pi - 2 \frac{\partial N}{\partial m} \frac{\mathrm{Tr}\left(\frac{\partial N}{\partial m} \Pi \frac{\partial v_t}{\partial m}\right)}{\mathrm{Tr}\left(\frac{\partial N}{\partial m} \cdot \frac{\partial N}{\partial m}\right)} \right\|_{\underset{\sim}{\Sigma}_t},$$

is a norm on the space $\underset{\sim}{V}_t$ which is equivalent to the canonical one defined in (6.20), (because of Theorem 6.3).

∎

Proof. The way the proof is given is very classical, (G. Duvaut, J.L. Lions [11]). We first prove that (6.29) defines a norm and by a compactness result we establish the equivalence between the norm (6.29) and (6.20).

STEP 1. Let us consider the equation

$$\Pi \frac{\partial v_t}{\partial m} + \overline{\frac{\partial v_t}{\partial m}}\, \Pi - 2 \frac{\partial N}{\partial m} \frac{\mathrm{Tr}\left(\frac{\partial N}{\partial m} \Pi \frac{\partial v_t}{\partial m}\right)}{\mathrm{Tr}\left(\frac{\partial N}{\partial m} \cdot \frac{\partial N}{\partial m}\right)} = 0.$$

Writing this relation in a system of coordinates generated by the lines of curvatures we obtain (see 6.22) :

$$\begin{cases} v_{1|1} - \dfrac{R_1 R_2^2}{R_1^2 + R_2^2}\left(\dfrac{v_{1|1}}{R_1} + \dfrac{v_{2|2}}{R_2}\right) = 0, \\ v_{1|2} + v_{2|1} = 0. \end{cases}$$

This is equivalent to :

$$\begin{cases} R_1\, v_{1|1} - R_2\, v_{2|2} = 0, \\ v_{1|2} + v_{2|1} = 0. \end{cases}$$

From the formula :

$$v_{\alpha|\beta} = v_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}\, v_{\lambda} ,$$

we deduce

$$\begin{cases} \dfrac{R_1}{R_2} v_{1,1} - v_{2,2} = \dfrac{R_1}{R_2}\, \Gamma^{\lambda}_{11}\, v_{\lambda} - \Gamma^{\lambda}_{22}\, v_{\lambda}; \\ v_{1,2} + v_{2,1} = 2\, \Gamma^{\lambda}_{12}\, v_{\lambda} . \end{cases}$$

And finally, we obtain :

$$(6.30) \quad -\left(\left(\frac{R_1}{R_2} v_{1,1}\right)_{,1} + v_{1,22}\right) = \left(\Gamma^{\lambda}_{22}\, v_{\lambda} - \frac{R_1}{R_2}\, \Gamma^{\lambda}_{11}\, v_{\lambda}\right)_{,1} - 2\left(\Gamma^{\lambda}_{12}\, v_{\lambda}\right)_{,2} ,$$

$$(6.31) \quad -\left(\left(\frac{R_2}{R_1} v_{2,2}\right)_{,2} + v_{2,11}\right) = \left(\Gamma^{\lambda}_{11}\, v_{\lambda} - \frac{R_2}{R_1}\, \Gamma^{\lambda}_{22}\, v_{\lambda}\right)_{,2} - 2\left(\Gamma^{\lambda}_{12}\, v_{\lambda}\right)_{,1} .$$

Let us now introduce two bilinear forms defined for arbitrary elements u_t, v_t of the space $\underset{\sim}{V}_t$, by :

$$(6.31)_1 \quad a(u_t, v_t) = \int_{\omega} \frac{R_1}{R_2}\, u_{1,1}\, v_{1,1} + u_{1,2}\, v_{1,2} + \int_{\omega} \frac{R_2}{R_1}\, u_{2,2}\, v_{2,2} + u_{2,1}\, v_{2,1},$$

and

$$(6.31)_2 \quad b(u_t, v_t) = \int_{\hat{\omega}} (R_1 \Gamma^\lambda_{11} - R_2 \Gamma^\lambda_{22})\, u_\lambda \left(\frac{v_{2,2}}{R_1} - \frac{v_{1;1}}{R_2}\right) - \int_{\hat{\omega}} 2\, \Gamma^\lambda_{12}\, u_\lambda (v_{2,1} + v_{1,2}),$$

where

$$u_t = u^\alpha a_\alpha \quad , \quad v_t = v^\alpha a_\alpha .$$

Equations (6.30) - (6.31) are equivalent to find an element u_t in the space $\underset{\sim}{V}_t$, such that :

$$(6.32) \qquad \forall\, v_t \in \underset{\sim}{V}_t, \quad a(u_t, v_t) = b(u_t, v_t).$$

Because the radius of curvature R_1 and R_2 have the same sign and satisfy the inequality :

$$\min (|R_1|, |R_2|) \geq C_o > 0$$

on the whole domain ω, the bilinear form $a(.,.)$ defined in $(6.31)_1$ is $\underset{\sim}{V}_t$ elliptic, i.e. there exists a strictly positive constant λ_o such that :

$$(6.33) \qquad \forall\, u_t \in \underset{\sim}{V}_t, \quad a(u_t, u_t) \geq \lambda_o \, \|u_t\|^2_{\underset{\sim}{V}_t}.$$

But if we set :

$$\Xi = \max_\lambda (\max (|R_1 \Gamma^\lambda_{11} - R_2 \Gamma^\lambda_{22}|_{L^\infty(\omega)}, \; 2\, |\Gamma^\lambda_{12}|_{L^\infty(\omega)}, \; 2\, |\Gamma^\lambda_{12}|_{L^\infty(\omega)})),$$

then there exists a strictly positive constant λ_1 such that :

$$(6.34) \qquad b(u_t, u_t) \leq \Xi \lambda_1 \, \|u_t\|_{\underset{\sim}{V}_t} \, \|u_t\|_{\underset{\sim}{H}_t},$$

Comparing (6.32) - (6.33) and ((6.34) we deduce :

$$\|u_t\|_{\underset{\sim}{V}} \leq \Xi \frac{\lambda_1}{\lambda_o} \|u_t\|_{\underset{\sim}{H}_t}$$

and if $\Xi \dfrac{\lambda_1}{\lambda_o}$ is smaller than 1. We must have $u_t = 0$ which proves that (6.29) defines a norm on the space $\underset{\sim}{V}_t$. It is worth noticing that the condition :

$$\Xi \frac{\lambda_1}{\lambda_o} < 1 ,$$

has a sense. It means that Ξ and $\frac{\lambda_1}{\lambda_o}$ does not vanish at the same time.

This is a consequence of the following two properties :

(i) the functions Ξ and $\frac{\lambda_1}{\lambda_o}$ are both continuous with respect to the map φ,

(ii) when ω is planar (i.e. φ is linear), then $\Xi = 0$ and $\frac{\lambda_1}{\lambda_o} > 0$.

STEP 2. We know that $\underset{\sim}{V}_t$ can be equipped with the norm (6.20). Hence the equivalence between (6.20) and (6.29) rests on the inequality :

$$(6.35)\quad \forall\ v_t \in \underset{\sim}{V}_t,\quad \|v_t\|_{\underset{\sim}{H}_t} \leq C \left\| \Pi \frac{\partial v_t}{\partial m} + \overline{\frac{\partial v_t}{\partial m}} \Pi - 2 \frac{\partial N}{\partial m} \frac{\mathrm{Tr}(\frac{\partial N}{\partial m} \cdot \Pi \frac{\partial v_t}{\partial m})}{\mathrm{Tr}(\frac{\partial N}{\partial m} \cdot \overline{\frac{\partial N}{\partial m}})} \right\|_{\underset{\sim}{L}_t} ,$$

where C denotes a strictly positive constant which of course should be independent of v_t.

Let us assume that (6.25) is not true. Then of any integer n, there exists an element v_t^n of the space $\underset{\sim}{V}_t$ such that :

$$\text{(i)}\quad \|v_t^n\|_{\underset{\sim}{V}_t} = 1,$$

$$\text{(ii)}\quad \left\| \Pi \frac{\partial v_t^n}{\partial m} + \overline{\frac{\partial v_t^n}{\partial m}} \Pi - 2 \frac{\partial N}{\partial m} \frac{\mathrm{Tr}(\frac{\partial N}{\partial m} \cdot \Pi \frac{\partial v_t^n}{\partial m})}{\mathrm{Tr}(\frac{\partial N}{\partial m} \cdot \overline{\frac{\partial N}{\partial m}})} \right\|_{\underset{\sim}{L}_t} \leq \frac{1}{n} .$$

From (i) and the weak compactness of the unit ball in an Hilbert' space [11] and from the compact embedding of $L^2(\omega)$ into $H^1(\omega)$, we deduce that we can extract a subsequence still denoted by v_t^n such that :

a) $v_t^n \to v_t^*$ in $\underset{\sim}{H}_t$ strongly,

b) $v_t^n \to v_t^*$ in $\underset{\sim}{V}_t$ weakly,

and

$$\text{c)}\quad \left\| \Pi \frac{\partial v_t^n}{\partial m} + \overline{\frac{\partial v_t^n}{\partial m}}\,\Pi - 2\,\frac{\partial N}{\partial m}\,\frac{\mathrm{Tr}\left(\frac{\partial N}{\partial m}\,\Pi\,\frac{\partial v_t^n}{\partial m}\right)}{\mathrm{Tr}\left(\frac{\partial N}{\partial m}\cdot\frac{\partial N}{\partial m}\right)} \right\|_{\Sigma_t} \leq \frac{1}{n}\,.$$

As a norm is a convex function, it is also semi-continuous for the weak topology [22].

Hence :

$$\left\| \Pi \frac{\partial v_t^*}{\partial m} + \overline{\frac{\partial v_t^*}{\partial m}}\,\Pi - 2\,\frac{\partial N}{\partial m}\,\frac{\mathrm{Tr}\left(\frac{\partial N}{\partial m}\,\Pi\,\frac{\partial v_t^*}{\partial m}\right)}{\mathrm{Tr}\left(\frac{\partial N}{\partial m}\,\frac{\partial N}{\partial m}\right)} \right\|_{\Sigma_t}$$

$$\leq \liminf \left\| \Pi \frac{\partial v_t^n}{\partial m} + \frac{\partial v_t^n}{\partial m}\,\Pi - 2\,\frac{\partial N}{\partial m}\,\frac{\mathrm{Tr}\left(\frac{\partial N}{\partial m}\,\Pi\,\frac{\partial v_t^n}{\partial m}\right)}{\mathrm{Tr}\left(\frac{\partial N}{\partial m}\cdot\frac{\partial N}{\partial m}\right)} \right\|_{\Sigma_t} = 0$$

From step 1 we deduce that $v_t^* = 0$ and the contradiction appears between a), and (i). This completes the proof of Theorem 6.4.

∎

As a consequence of Theorems (6.2) - (6.3) - (6.4) we formulate :

THEOREM 6.5. Under the assumptions (6.18) and if the Christoffel's symbols are small enough in $\mathcal{C}^0$ norm, then inequality (6.17) holds.

∎

7. - A COMPARISON BETWEEN THE BUDIANSKY-SANDERS' SHELL MODEL AND THE MEMBRANE MODEL

The equations (6.5) constitutes the membrane model for a thin shell. As a practical rule the solution of this model is done as follows.

The tangential components of the displacement, $\underset{\sim}{u}_t^0$ and the resultant tangential stress n_t^0 are solution of the equations (6.12).

The normal displacement is then given by (6.11) and the bending moments are given by (6.6). In this section we aim at giving a convergence result between the solution of the Budiansky-Sanders's model and the one of the membrane model. This will give a sense to the assumed asymptotic expansion (6.4).

Error estimates would be much more complicated because of a boundary layer effect wich appears or the displacement u_3 [7], [16].

THEOREM 7.1. Under the assumption of Theorem 6.1 the following convergences occur when ε tends to zero :

$$(7.1)\qquad \begin{cases} n_t \to n_t^0 & \text{in } \Sigma_t \text{ strongly,} \\ \underset{\sim}{u}_t \to \underset{\sim}{u}_t^0 & \text{in } \underset{\sim}{V}_t \text{ strongly,} \\ u_3 \to u_3^0 & \text{in } L^2(\omega) \text{ strongly.} \end{cases}$$

where $(n_t, \underset{\sim}{u}_t, u_3)$ is the solution of the B-S' model defined in (5.11) - (5.12) and $(n_t^0, \underset{\sim}{u}_t^0, u_3^0)$ is the solution of the membrane model defined in (6.5).

∎

REMARK 7.1. The convergence of the bending moments is a consequence of (7.1) and of the following equality valid for any ε:

$$m_t = \frac{E}{(1-\nu^2)} \{(1-\nu)\ \rho(u) + \nu\ \mathrm{Tr}(\rho(u))\ \mathrm{Id}\}.$$

But this convergence is very weak. The expression of $\rho(u)$ involves second order derivatives of u_3. From (7.1) we can only deduce the convergence of the components of m_t to the ones of m_t^0 in the space $H^{-2}(\tilde{\omega})$. As a consequence we observe that a membrane admits concentrated bending moments (i.e. Dirac's distribution). It is also the case for instance when $\frac{\partial N}{\partial m}$ is discontinuous, (see 5.9).

Proof of Theorem 7.1. There are three steps.

STEP 1. "A priori Estimate". Let us consider the equations (6.3) of which $(n_t, m_t, \underset{\sim}{u}_t, u_3)$ is the unique solution in the space $\Sigma_t \times \Sigma_t \times \underset{\sim}{V}_t \times H_0^2(\omega)$.

$$(7.2)\qquad \begin{cases} \forall p_t \in \Xi_t \ , \ a_o(n_t, p_t) + b_o(p_t, u) = 0, \\ \forall q_t \in \Xi_t \ , \ a_2(m_t, q_t) + b_2(q_t, u) = 0, \\ \forall v = (\underset{\sim}{v}_t, v_3) \in \underset{\sim}{Y}_t \times H_o^2(\omega), \ b_o(n_t, v) + \varepsilon^2 \, b_2(m_t, v) = F(v) \end{cases}$$

Choosing $p_t = n_t$, $q_t = m_t$ and $v = u$ we obtain by combining the equations :

$$a_o(n_t, n_t) + \varepsilon^2 \, a_2(m_t, m_t) = F(u)$$

or else because of the assumptions on the applied loads :

$$(7.3)\qquad \|n_t\|^2_{\Xi_t} + \varepsilon^2 \|m_t\|^2_{\Xi_t} \leq C(\|\underset{\sim}{u}_t\|_{\underset{\sim}{Y}_t} + \|u_3\|_{L^2(\omega)}).$$

Let us come back now to the first equation (7.2). If we choose for p_t any element of the space $\underset{\sim}{y}_t$ defined in (6.10), we obtain :

$$\forall p_t \in \underset{\sim}{Y}_t \quad b_o(p_t, u) \leq C \, \|n_t\|_{\Xi_t} \cdot \|p_t\|_{\Xi_t}$$

or else

$$\sup_{p_t \in \underset{\sim}{Y}_t} \frac{b_o(p_t, u)}{\|p_t\|_{\Xi_t}} \leq C \ \|n_t\|_{\Xi_t} ,$$

and from assumption (6.17) with $u = (\underset{\sim}{u}_t, u_3)$:

$$(7.4)\qquad \|\underset{\sim}{u}_t\|_{\underset{\sim}{Y}_t} \leq C \, \|n_t\|_{\Xi_t} .$$

We now choose $p_t = u_3 \frac{\partial N}{\partial m}$ and we obtain from the first equation (7.2) :

$$\int_\omega u_3^2 \, \mathrm{Tr}(\frac{\partial N}{\partial m} \cdot \frac{\partial N}{\partial m}) = a_o(n_t, u_3 \frac{\partial N}{\partial m}) - \int_\omega \mathrm{Tr}(\frac{\partial N}{\partial m} \, \Pi \, \frac{\partial \underset{\sim}{u}_t}{\partial m}) \, u_3,$$

which leads to the following relations because of the assumption (6.18) :

$$(7.5) \qquad \|u_3\|_{L^2(\omega)} \leq C\left(\|n_t\|_{\Sigma_t} + \|\underline{u}_t\|_{\underline{V}_t}\right).$$

Finally from (7.3) - (7.4) and (7.5), we deduce there exits a constant C independent of ε such that :

$$(7.6) \qquad \|n_t\|_{\Sigma_t} + \varepsilon\|m_t\|_{\Sigma_t} + \|\underline{u}_t\|_{\underline{V}_t} + \|u_3\|_{L^2(\omega)} \leq C.$$

STEP 2. "Weak convergence". The unit ball of an Hilbert-Space is weakly compact [22]. Hence from (7.6) we deduce there exists subsequence still denoted $(n_t, \underline{u}_t, u_3)$ such that when ε goes to zero :

$$\begin{cases} n_t \to n_t^* & \text{in } \Sigma_t \text{ weakly,} \\ \underline{u}_t \to \underline{u}_t^* & \text{in } \underline{V}_t \text{ weakly,} \\ u_3 \to u_3^* & \text{in } L^2(\omega) \text{ weakly.} \end{cases}$$

By taking the limit in the first equation (7.2) :

$$(7.7) \qquad \forall p_t \in \Sigma_t,\ a_o(n_t^*, p_t) + b_o(p_t, u^*) = 0,$$

(remembering that in the expression of $b_o(.,.)$ given in (6.2) there is no derivative of u_3).

The third equation (7.2) leads to :

$$(7.8) \qquad \forall v = (\underline{v}_t, v_3) \in \underline{V}_t \times H_o^2(\omega),\ b_o(n_t^*, v) = F(v),$$

(because from (7.6), $\varepsilon^2 m_t$ tends to zero).

The relationships (7.7) - (7.8) are nothing else but the membrane equations. Hence $(n_t^*, \underline{u}_t^*, u_3^*) = (n_t^o, \underline{u}_t^o, u_3^o)$ and from the uniqueness of the solution we conclude by a standard justification [11], that all the sequel $(n_t, \underline{u}_t, u_3)$ tends weakly to $(n_t^o, \underline{u}_t^o, u_3^o)$.

STEP 3. "Strong. Convergence". From (7.2) - (7.7) and (7.8) we obtain :

$$(7.9)\qquad C\,\|n_t - n_t^o\|^2_{\Sigma_t} \le a_o(n_t - n_t^o,\ n_t - n_t^o)$$

$$= - b_o(n_t - n_t^o,\ u) + b_o(n_t - n_t^o,\ u^o)$$

$$= \varepsilon^2\, b_2(m_t,\ u) + b_o(n_t - n_t^o,\ u^o).$$

As a consequence of step 2 and (7.6) this last expression tends to zero with ε.

Hence

$$(7.10)\qquad \lim_{\varepsilon\to 0} \|n_t - n_t^o\|_{\Sigma_t} = 0.$$

Then from :

$$(7.11)\qquad \forall \tau_t \in \Sigma_t\ ,\ b_o(\tau_t,\ u - u^o) = - a(n_t - n_t^o,\ \tau_t).$$

First of all we deduce with (6.17) :

$$C\,\|u_t - u_t^o\|_{V_t} \le \sup_{\tau_t \in \Sigma_t} \frac{b_o(\tau_t,\ u - u^o)}{\|\tau_t\|_{\Sigma_t}} \le C\,\|n_t - n_t^o\|_{\Sigma_t}$$

and therefore :

$$(7.12)\qquad \lim_{\varepsilon\to 0} \|u_t - u_t^o\|_{V_t} = 0.$$

If we set $\tau_t = (u_3 - u_3^o)\,\frac{\partial N}{\partial m}$ in (7.11) we obtain :

$$(7.13)\qquad \int_\omega (u_3 - u_3^o)^2\, \mathrm{Tr}\left(\frac{\partial N}{\partial m}\,\frac{\partial N}{\partial m}\right) \le$$

$$\le C\,\left(\|n_t - n_t^o\|_{\Sigma_t} + \|u_t - u_t^o\|_{V_t}\right)\,\|u_3 - u_3^o\|_{L^2(\omega)}$$

Finally from (6.18) - (7.10) - (7.12) and (7.13) we conclude :

$$\lim_{\varepsilon\to 0} \|u_3 - u_3^o\|_{L^2(\omega)} = 0$$

and this completes the proof of Theorem 7.1.

8. - CONCLUSION

An answer is given in this paper to the following three questions which arise in thin shell theory :

(i) what are the assumptions needed for deriving the Budiansky-Sanders' model from the three dimensional one ?

(ii) what is the behaviour of the Budiansky-Sanders' model (in solution), for a very thin shell, (membrane Theory) ?

(iii) what is a variational formulation of the membrane model, and how does one prove the existence and uniqueness of a solution ?

The answer to the first question is given in section 3 and 4. We showed that Kirchhoff.Love kinematical' assumptions are sufficient for deducing the Budiansky-Sanders model from the three dimensional one.

The asymptotic behaviour of the Budiansky-Sanders's model lead us to the membrane Theory in section 5 and 7 where a mathematical justification is given. The variational formulation was particularly efficient for the analysis of the membrane model done in section 6.

An existence and uniqueness Theorem is given for a uniformly convex shell. But the procedure given can be extended to other shapes of shell, as for instance shells of revolution. This will be done in a forth coming paper. Finally let us outline the advantage of the membrane. Theory. Instead of solving Budiansky-Sanders' model which involves the bending effects and therefore complicated finite elements [9]. It is prefered to solve the membrane model which involves less variables and a lower order of derivatives.

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