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THE DIRECT-INTERACTION APPROXIMATION AND STATISTICALLY STEADY STATES of NONLINEARLY COUPLED MODES

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The direct-interaction approximation and statistically steady states of three nonlinearly coupled modes

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Abstract

The direct-interaction approximation is used to find statistically steady states of a system of three modes, with complex frequencies, coupled by a quadratic nonlinearity. These states are compared to the exact predictions of an ensemble of realizations with Gaussianly distributed initial conditions. The direct-interaction approximation is shown to be reasonably successful in this context.

Many aspects of the nonlinear behavior of low frequency (e.g., drift) fluctuations in plasma as well as in many other dynamical systems can be described by the set of quadratically coupled equations[1]

$$\frac{d}{dt}\phi_{\underset{\sim}{k}}(t) + i\Omega_{\underset{\sim}{k}}(t)\phi_{\underset{\sim}{k}} = \tfrac{1}{2}\sum_{\underset{\sim}{k}+\underset{\sim}{p}+\underset{\sim}{q}=0} M_{\underset{\sim}{k}|\underset{\sim}{p},\underset{\sim}{q}}\phi^{*}_{\underset{\sim}{p}}\,\phi^{*}_{\underset{\sim}{q}} \tag{1}$$

for certain (in general, complex) $\Omega_{\underset{\sim}{k}}$ and $M_{\underset{\sim}{k}|\underset{\sim}{p},\underset{\sim}{q}}$. Because it is well-known that such equations often exhibit stochastic behavior,[2,3] one is led to study the statistical dynamics of Eq. (1). Traditionally, analytic work has been mostly concerned with developing evolution equations[4,5] for low order statistical functions like the covariance $C_{\underset{\sim}{k}}(t,t') \equiv \langle\delta\phi_{\underset{\sim}{k}}(t)\delta\phi^{*}_{\underset{\sim}{k}}(t')\rangle$, where the angular brackets denote an average over an appropriate (generally Gaussian) ensemble of initial conditions. Although such an approach has the significant deficiency of giving little insight into the detailed underlying phase space dynamics (e.g., the nature of the associated strange attractors or unstable fixed points), it remains an important tool. The experimentally accessible fluctuation spectrum is just the Fourier transform of C. Furthermore, transport coefficients can be related to the mean infinitesimal response function $R_{\underset{\sim}{k}}(t;t') \equiv \langle\delta\phi_{\underset{\sim}{k}}(t)/\delta\eta_{\underset{\sim}{k}}(t')\rangle$, where $\eta_{\underset{\sim}{k}}$ is an arbitrary source added to the right-hand side of Eq. (1), and $\delta/\delta\eta$ denotes the functional derivative. The procedure of ensemble averaging is the appropriate one for theoretical discussions of scaling laws.

In the present Note I comment on the use of the direct-interaction approximation[1,5,6] to find "nonlinearly saturated," statistically steady states of Eq. (1) for the model case in which the wavenumber summation is truncated so that precisely three modes labelled by $\underset{\sim}{K}$, $\underset{\sim}{P}$, and $\underset{\sim}{Q}$ interact. This work was directly motivated by the recent works of Terry[3] and of Molvig et al.[7] Terry considered the three mode truncation just described. In addition to studying the exact dynamics and demonstrating that certain choices of Ω and M led to intrinsically stochastic behavior, he discussed the "random phase approximation." He noted that this approximation is inadequate for the three mode problem, since it fails to predict the essential resonance broadening and correlation damping associated with the nonlinear, stochastic interactions. In view of the extensive literature on statistical descriptions of the Navier-Stokes and related equations,[5] this conclusion is not surprising, and one is led to study renormalized theories, of which the direct-interaction approximation is the outstanding representative.[1,4-6]

Molvig et al. attempted to argue that the direct-interaction and similar Eulerian-based approximations are grossly inadequate when the underlying dynamics are stochastic, and argued for a Lagrangian description. Unfortunately, their conclusion that the direct-interaction approximation was wrong by "orders of magnitude in a measured exponent" was based on a misinterpretation of a key equation. While it is true that Lagrangian schemes[8] can be superior to Eulerian ones in various ways, it does not follow that the Eulerian direct-interaction approximation is useless. Note that since the arguments of Molvig et al. were based on the properties of stochastic instability, their criticisms can be applied as well to theories of the three mode system studied by Terry as to the system with a broad spectrum of modes which they studied. Fortunately, the three mode model is sufficiently simple that the direct-interaction approximation for it can be readily formulated and (numerically) solved, with no further analytic approximations except those controllable ones intrinsic to a numerical approach. As I will discuss, the results of such a program support the assertion that the direct-interaction approximation is reasonable in this context.

Indeed, a very similar analysis was performed by Kraichnan[9] many years ago. Kraichnan studied the three mode version of the inviscid two-dimensional Navier-Stokes equation [which also has the form (1), with $\Omega_{\underset{\sim}{k}} \equiv 0$], and found quite reasonable agreement between the predictions of the direct-interaction approximation and the "exact" statistics of an ensemble with Gaussian initial conditions. Because of this work and many later studies of Navier-Stokes turbulence,[10,11] the direct-interaction approximation is well-understood. One can argue quite generally that it should provide a reasonable description of the low-order statistical behavior of Eq. (1) in stochastic regimes with moderate turbulence levels, except possibly for pathological wavevector triads. There are, however, two important differences between the fluid and the plasma versions of Eq. (1) which motivate further study.

The first concerns the way energy is injected and dissipated. Consider a Navier-Stokes problem with viscous dissipation, so that $\mathrm{Re}(\Omega_{\underset{\sim}{k}}) = 0$, $\mathrm{Im}(\Omega_{\underset{\sim}{k}}) = \mu k^2 > 0$. In this case, steady states are obtained by injecting energy with a (usually random) forcing function added to the right-hand side of Eq. (1). By contrast, in the plasma case a forcing function is generally absent, but since $\mathrm{Im}(\Omega) \equiv \gamma \neq 0$, the rate of energy increase in the k-th mode is initially (in linear theory) proportional to $2\gamma_{\underset{\sim}{k}}\langle|\phi_{\underset{\sim}{k}}|^2\rangle$. Since $\gamma_{\underset{\sim}{k}}$ can have either sign, steady states can arise when,[2] say, $\gamma_{\underset{\sim}{K}} > 0$, $\gamma_{\underset{\sim}{P}} < 0$, $\gamma_{\underset{\sim}{Q}} < 0$. If an effective Reynolds' number is defined by the ratio of the nonlinear term to the linear term in Eq. (1), the resulting turbulence has a Reynolds' number of order unity[12] and is thus inherently weaker than that of the Navier-Stokes problem, where the Reynolds' number can be arbitrarily large.

Another potentially important difference between the three mode models is that the inviscid case studied by Kraichnan is integrable, so that the statistical aspects are extrinsic, while the case with complex frequencies is, in general, nonintegrable and can display intrinsic stochasticity in the form of a strange attractor.[2] In view of the arguments of Molvig et al., the latter case is of particular interest. Thus, I have performed numerical integrations of the direct-interaction approximation to Eq. (1) with $\Omega_{\underset{\sim}{k}} \neq 0$, using the reasonable (predictor-corrector) algorithm first employed in this context by Kraichnan.[9,10] A complete presentation and discussion of the results will be given elsewhere. In this report of preliminary results, I will emphasize the energetics and demonstrate that the direct-interaction approximation predicts saturated steady states in reasonable agreement with the "exact" predictions for an ensemble with Gaussian initial conditions, not only for the integrable case, but also for one which involves intrinsic stochasticity.

The mode-coupling coefficient had the form

$$M_{\underset{\sim}{k}|\underset{\sim}{p},\underset{\sim}{q}} = A(\chi_q^* - \chi_p^*)/(\iota+\chi_k),$$

where $A = \hat{\underset{\sim}{z}}\cdot(\underset{\sim}{p}\times\underset{\sim}{q})$ for Fig. 1 and $A = 1$ for Fig. 2. For comparison (e.g., Fig. 1) with Kraichnan's work,[9] $\chi_{\underset{\sim}{k}} = k^2$ and $\iota = 0$. Otherwise, $\iota = 1$ (the signature of the compressible, adiabatic electron response[13]). The energy-like quantity associated with Eq. (1) is $W = \sum_{\underset{\sim}{k}} (\iota+\chi_{\underset{\sim}{k}})|\phi_{\underset{\sim}{k}}|^2$; this, as well as its average, is conserved by the nonlinear term of Eq. (1). The quantity <W> is also conserved by the nonlinear terms of the direct-interaction approximation, both exactly and by the numerical algorithm. (There is also an enstrophy-like quantity, which I do not discuss here.)

Figure 1 represents an inviscid, two-dimensional Navier-Stokes case where $\Omega_{\underset{\sim}{k}} \equiv 0$ and $M_{\underset{\sim}{k}|\underset{\sim}{p},\underset{\sim}{q}} = \hat{\underset{\sim}{z}}\cdot\underset{\sim}{p}\times\underset{\sim}{q}(q^2-p^2)/k^2$, with $K = 2$, $P = 2$, and $Q = \sqrt{2}$. This case corresponds to Kraichnan's Fig. 5; it is exactly soluble[9] and can be reduced to the well-known stochastic oscillator.[1,14] The ensemble consisted of 5000 realizations; its predictions are indistinguishable from the exact results. (Kraichnan's result for the exact solution appears to be in error by absence of a Jacobian factor. I have succeeded in reproducing the essential features of each of Kraichnan's other figures.)

Figure 2 describes a situation similar to one studied by Terry. The frequencies had the values $\Omega_{\underset{\sim}{K}} = (0.8349,\ 0.1600)$, $\Omega_{\underset{\sim}{P}} = (-1.230,\ -0.2500)$, and $\Omega_{\underset{\sim}{Q}} = (0.4989,\ -0.0191)$; also $\chi_{\underset{\sim}{K}} = (0.4870,\ 0.2776)$, $\chi_{\underset{\sim}{P}} = (0.5470,\ 0.0829)$, and $\chi_{\underset{\sim}{Q}} = (0.2500,\ 0.0479)$. The corresponding mode-coupling coefficients were $M_{\underset{\sim}{K}} = (-0.1888,\ 0.0588)$, $M_{\underset{\sim}{P}} = (0.1448,\ -0.1562)$, and $M_{\underset{\sim}{Q}} = (0.0539,\ 0.1537)$, where $M_{\underset{\sim}{K}} \equiv M_{\underset{\sim}{K}|\underset{\sim}{P},\underset{\sim}{Q}}$. These parameters correspond to a regime in which intrinsic stochasticity is expected[2]; this is verified by examination of individual realizations. The ensemble had 5000 realizations. Large initial conditions were used to avoid an uninteresting linear regime.

One can see that in each case shown (and in other cases I have studied) the final energy states predicted by the direct-interaction approximation are in reasonable agreement with the exact statistical results. (Typically, agreement in final energy levels is of order 5% to 30%.) Of course, "reasonable" is subjective. In the present context, the most relevant comparison is to the predictions of the random phase approximation. When $\Omega_{\underset{\sim}{k}} \neq 0$, this involves in the nonlinear term the factor $\delta(\Delta\omega)$, where $\Delta\omega \equiv \sum_{\underset{\sim}{k}} \mathrm{Re}(\Omega_{\underset{\sim}{k}})$ is the frequency mismatch. It thus predicts no nonlinear effects at all, and thus no saturation, unless the frequency mismatch vanishes. The direct-interaction approximation clearly represents a substantial improvement. Of course, quantitative inaccuracies of order unity are to be expected in a first-principles theory of this kind. Alternative approximations[1,4,5,8,11] may increase the accuracy. This, however, is not the point and does not vitiate the conclusion that, as expected, the direct-interaction approximation provides a reasonable description of the statistical dynamics of three interacting modes (e.g., drift or shear), at least at the level of energetics, even when intrinsic stochasticity is present. (The two-time information furnished by the direct-interaction approximation is also of interest, and will be discussed elsewhere.) One can infer that its applicability extends to the more general and important case of a broad spectrum of interacting drift waves.[15,16] Although in the latter case further approximations may have to be made in order to produce a computationally tractable problem,[15] it is comforting that at least one reasonably solid starting point exists.

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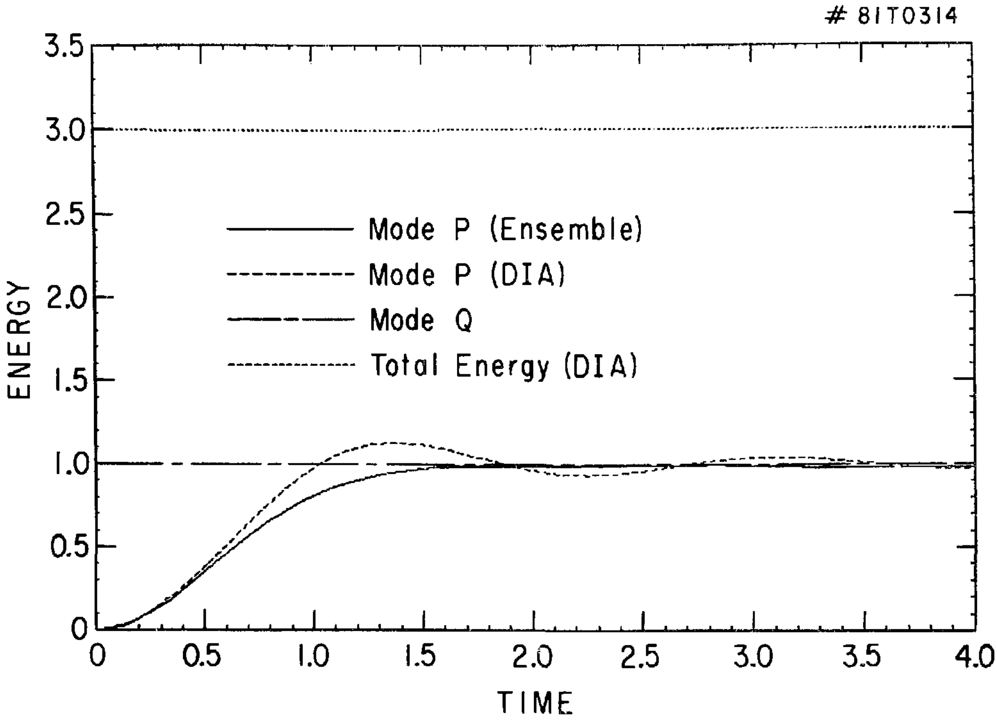
Figure captions

Fig. 1: Comparison of the direct-interaction approximation with a Gaussian ensemble for an inviscid Navier-Stokes case. See text for parameters.

Fig. 2: Comparison of the direct-interaction approximation with a Gaussian ensemble for a case with intrinsic stochasticity. See text for parameters.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Mode P (Ensemble)
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Total Energy (DIA)
ENERGY
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Fig. 1

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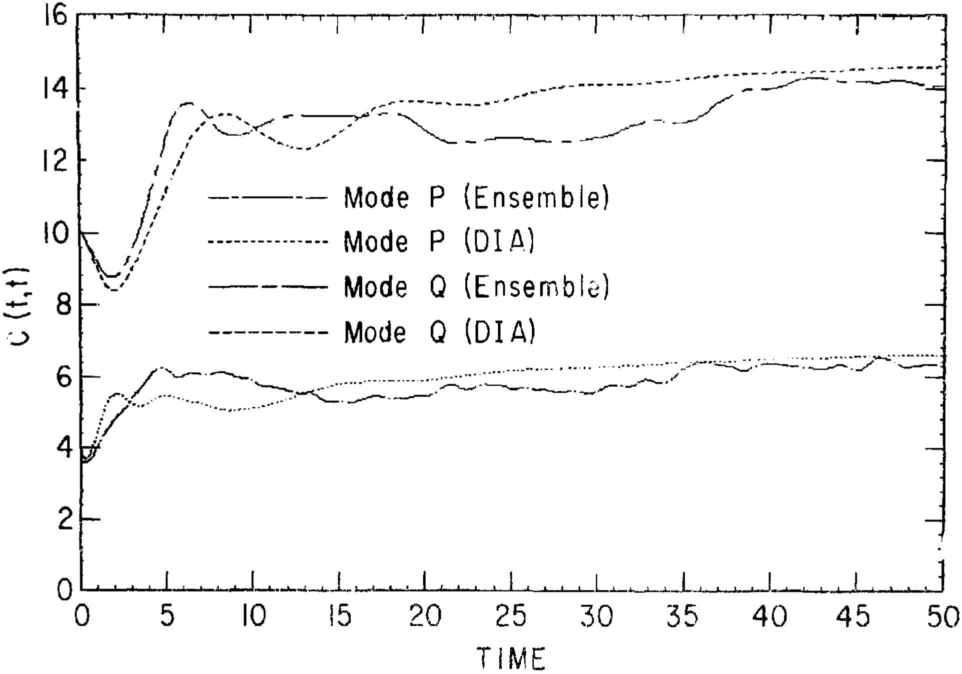
Mode P (Ensemble)
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