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# A UNIFICATION OF BOSON EXPANSION THEORIES

# (III) APPLICATIONS

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## Abstract

A general scheme of constructing boson expansions that was proposed in earlier work is applied to a number of examples. The Fukutome expansion is obtained by considering the spinor representation of the SO(2N+1) group. Its hermitian, Holstein-Primakoff-type version is also derived. The generalized Dyson expansions for even and odd fermion systems are given in terms of two spinor representations of the SO(2N) group. For fixed fermion number systems the relevant boson expansions are obtained by considering the fundamental representations of SU(N) while for fixed seniority those of Sp(N) are concerned. The collective boson expansions corresponding to the Ginocchio model, the interacting boson model of Arima and Iachello and the Elliott model are given for the symmetric representations of SO(8) and SU(l+1) and any representation of SU(3).

## 1. INTRODUCTION

In the previous papers [1)], hereafter referred to as part I and II, a general method for constructing boson representations of fermion Hilbert spaces was given. The method allows one to relate in a definite way a boson state to every fermion state while the fermion operators may be presented in the form of boson expansions. Infinitely many boson representations can be obtained each being valid for the carrier space of an irreducible representation of a semisimple group.

The present study aims at giving a few examples of the construction of boson expansions to illustrate the general method and to provide the relation with previous approaches. The examples based on the allowed representations of the Lie groups SO(2N+1), SO(2N), SU(N) and Sp(N), symmetric representations of SU(1+1) and SO(8) and all representations of SU(3) are investigated. Such a choice is dictated by two possible ways of utilizing boson expansions when describing the collective excitations of a fermion system. The first one is to map a "large" space of fermion states onto the boson space and then to search among many bosons for the collective ones. The kinematic step, i.e. boson mapping, preceeds in this case the dynamical one which consists of reducing the number of bosons based on the properties of the Hamiltonian. The boson expansions for the chain of subalgebras SO(2N+1) ⊃ SO(2N) ⊃ SU(N) ⊃ Sp(N) provide the mapping of "large" fermion spaces which are the entire fermion space, the subspaces for even and odd fermion number and the subspaces with both fermion number and seniority fixed. The generators of algebras in the above chain are known combinations of fermion and bifermion operators. The number of individual boson excitations is large, each of them corresponding to some individual fermion or two-fermion excitation.

The second way to utilize boson expansions for the description of collective motion is to perform the dynamical step before the kinematic one by picking up the collective space first and then mapping this "small" fermion space onto the boson space. The sections dealing with the SU(l+1), SO(8) and SU(3) algebras examplify such a mapping. One obtains the boson spaces with a small number of different collective bosons.

Sect.2, concerning the SO(2N+1) algebra, contains a rather detailed discussion of the method ; the same scheme is utilized in the other sections in a more compact form. All sections are independent of one another and in particular symbols like $\underset{\sim}{\hat{E}_{\alpha}}$, $\underset{\sim}{\hat{H}}$, , $\underset{\sim}{j}$, $|\underset{\sim}{j}\rangle$, $|C\rangle$ and $w(C)$ may have a different meaning in every section.

## 2. ALGEBRA so(2N+1)

The orthogonal algebra $\mathcal{A}$ = so(2N+1), is composed of all fermion and bifermion operators

$$a^+_\mu \ , \ a_\mu \ , \ a^+_\mu a^+_\nu \ , \ a_\nu a_\mu \ , \ \tfrac{1}{2}\delta_{\mu\nu} - a^+_\mu a_\nu , \tag{2.1}$$

where the indices $\mu$, $\nu$ number N single-particle (or single-quasiparticle) states. The Cartan subalgebra is formed by the N generators

$$\hat{H}_\mu = \tfrac{1}{2} - a^+_\mu a_\mu \ , \tag{2.2}$$

and the rank $l$ of $\mathcal{A}$ is equal to N. Algebra so(2N+1) is the algebra $B_l$ of the standard classification[2-4]. The root vectors belong to N dimensional Euclidean space and can be expressed[2-4] in terms of the unit vectors $\underset{\sim}{w}_\mu$,

$$\underset{\sim}{w}_\mu = (0,0,\ldots,0,1,0,\ldots,0) \ , \tag{2.3}$$

with the unity on the $\mu$th place. The root is called positive (negative) if its first non-vanishing component is positive (negative). To every root $\underset{\sim}{\alpha}$ corresponds the generator $\hat{E}_{\underset{\sim}{\alpha}}$ and for positive roots this correspondence reads

$$\hat{E}_{\underset{\sim}{\alpha}} = a_\mu / \sqrt{2} \qquad \text{for} \quad \underset{\sim}{\alpha} = \underset{\sim}{w}_\mu \ , \tag{2.4a}$$

$$\hat{E}_{\underset{\sim}{\alpha}} = a_\mu a_\nu \ , \nu < \mu, \ \text{for } \underset{\sim}{\alpha} = \underset{\sim}{W}_\nu + \underset{\sim}{W}_\mu , \tag{2.4b}$$

$$\hat{E}_{\underset{\sim}{\alpha}} = a^+_\mu a_\nu \ , \ \nu < \mu, \ \text{for } \underset{\sim}{\alpha} = \underset{\sim}{W}_\nu - \underset{\sim}{W}_\mu . \tag{2.4c}$$

The hermitian conjugations of this generators correspond to negative roots:

$$\hat{E}_{\underset{\sim}{\alpha}} = a^+_\mu / \sqrt{2} \qquad \text{for } \underset{\sim}{\alpha} = -\underset{\sim}{W}_\mu , \tag{2.5a}$$

$$\hat{E}_{\underset{\sim}{\alpha}} = a^+_\nu a^+_\mu \ , \ \nu < \mu, \ \text{for } \underset{\sim}{\alpha} = -\underset{\sim}{W}_\nu - \underset{\sim}{W}_\mu , \tag{2.5b}$$

$$\hat{E}_{\underset{\sim}{\alpha}} = a^+_\nu a_\mu \ , \ \nu < \mu, \ \text{for } \underset{\sim}{\alpha} = -\underset{\sim}{W}_\nu + \underset{\sim}{W}_\mu . \tag{2.5c}$$

The irreducible representations of $\mathcal{A}$ are determined by the highest weight $\underset{\sim}{j}$, which is the N dimensional vector of eigenvalues of the Cartan generators $\hat{\underset{\sim}{H}}$,

$$\hat{\underset{\sim}{H}} \ | \underset{\sim}{j} > = \underset{\sim}{j} \ | \underset{\sim}{j} > , \tag{2.6}$$

while the highest weight states, $|\underset{\sim}{j}>$ , are the corresponding eigenvectors. The allowed representaions (part I) of $\mathcal{A}$ can thus be found by examining the spectra of $\hat{H}_\mu$, eq.(2.2). As $a^+_\mu a_\mu$ are the particle number operators, they can have in the fermion space only the eigenvalues 0 or 1 ; hence the highest weights $\underset{\sim}{j}$ must have components equal to $\pm\frac{1}{2}$.

The only highest weight with this property is for the classical algebra $B_l$ the weight

$$\underset{\sim}{j} = \frac{1}{2}\sum_{\mu=1}^{N} \underset{\sim}{w}_\mu = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) \tag{2.7}$$

and thus the only allowed representation is the spinor representation[5] of SO(2N+1). Evidently, the highest weight state is equal to the fermion vacuum

$$|\underset{\sim}{j}\rangle = |0\rangle . \tag{2.8}$$

Every state of this representation can be obtained by successive action on $|0\rangle$ with the generators $\hat{E}_{\underset{\sim}{\alpha}}$, eqs.(2.4-5), and thus the representation space is equal to the entire fermion space.

The roots $\underset{\sim}{w}_\mu$ and $\underset{\sim}{w}_\nu + \underset{\sim}{w}_\mu$, $\nu<\mu$, are positive and nonorthogonal with respect to the highest weight j, and the corresponding $\hat{E}_{-\underset{\sim}{\alpha}}$ generators, eqs.(2.5ab), determine the generalized coherent states (part I) :

$$|C\rangle = \exp\Big\{\sum_{\substack{\nu,\mu=1\\ \nu<\mu}}^{N} C^*_{\nu\mu}\, a^+_\nu a^+_\mu + \sum_{\mu=1}^{N} C^*_\mu\, a^+_\mu/\sqrt{2}\Big\}|0\rangle . \tag{2.9}$$

The total boson number is thus equal to M= N(N+1)/2 which includes N(N-1)/2 bosons represented by complex numbers $C_{\nu\mu}$ and N bosons represented by $C_\mu$ .

It is convenient to consider the complex numbers $C_{\nu\mu}$ as the components of an antisymmetric matrix $C^T = -C$, and $\bar{C}_\mu = C_\mu/\sqrt{2}$ as the components of a N dimensional column vector $\bar{C}$. The corresponding boson operators are denoted by $b^\dagger_{\nu\mu}$, $b^\dagger_{\nu\mu} = -b^\dagger_{\mu\nu}$, and $b^\dagger_\mu$, respectively. Alternatively, one can group the $N(N+1)/2$ complex numbers $C_{\nu\mu}$ and $\bar{C}_\mu$ into the antisymmetric matrix $\tilde{C}$ in N+1 dimensions :

$$\tilde{C}_{\nu\mu} = C_{\nu\mu} \qquad \text{for } \nu,\mu = 1,2,\ldots,N, \tag{2.10a}$$

$$\tilde{C}_{\nu,N+1} = -\tilde{C}_{N+1,\nu} = \bar{C}_\nu \quad , \tag{2.10b}$$

$$\tilde{C} = \begin{pmatrix} C & \bar{C} \\ -\bar{C}^T & 0 \end{pmatrix} \quad , \tag{2.11}$$

and the corresponding bosons are denoted by $\tilde{b}^\dagger_{\nu\mu}$, $\tilde{b}^\dagger_{\nu\mu} = -\tilde{b}^\dagger_{\mu\nu}$, $\nu,\mu = 1,2,\ldots, N+1$.

As the generators $a^+_\mu$ commute with $a^+_\nu a^+_\mu$ one has

$$|C\rangle = \left(1 + \sum_{\mu=1}^{N} \bar{C}^*_\mu a^+_\mu\right) \exp\left\{\frac{1}{2}\sum_{\nu,\mu=1}^{N} C^*_{\nu\mu} a^+_\nu a^+_\mu\right\}|0\rangle, \tag{2.12}$$

where use was made of

$$\left(\sum_{\mu=1}^{N} \bar{C}^*_\mu a^+_\mu\right)^2 |0\rangle = 0 . \tag{2.13}$$

Based on the formulae (3.6) of the next section the norm of generalized coherent states can be expressed as

$$\langle C|C\rangle = \det^{\frac{1}{2}}(I^{(N)} + CC^{+})\left(1 + \sum_{\nu,\mu=1}^{N} \bar{C}_{\nu}\,(I^{(N)} + C^{+}C)^{-1}_{\nu\mu}\,\bar{C}^{*}_{\mu}\right), \tag{2.14}$$

where $I^{(N)}$ is the unit matrix in N dimensions. Considering $\bar{C}$ to be the column vector one has

$$\langle C|C\rangle = \det(I^{(N)} + CC^{+} + \bar{C}\bar{C}^{+})\det^{-\frac{1}{2}}(I^{(N)} + CC^{+}) \tag{2.15}$$

or

$$\langle C|C\rangle = \det^{\frac{1}{2}}(I^{(N+1)} + \tilde{C}\tilde{C}^{+}) \tag{2.16}$$

for $\tilde{C}$ given by eq. (2.11).

Summing up the positive roots which are not orthogonal to $\underset{\sim}{j}$ one obtains the specific weight (part I) related to $\underset{\sim}{j}$ :

$$\underset{\sim}{\tilde{j}} = \sum_{\mu=1}^{N} \underset{\sim}{w}_{\mu} + \sum_{\substack{\nu,\mu=1 \\ \nu<\mu}}^{N} (\underset{\sim}{w}_{\nu} + \underset{\sim}{w}_{\mu}) = N\sum_{\mu=1}^{N} \underset{\sim}{w}_{\mu} = 2N\underset{\sim}{j} . \tag{2.17}$$

Thus the weight function w(C), determining the scalar product for the functional representation (part I), reads

$$w(C) = W\left[\det(I^{(N+1)} + \tilde{C}\tilde{C}^{+})\right]^{-\frac{2N+1}{2}}, \tag{2.18}$$

$$W = \left(\frac{2}{\pi}\right)^{N(N+1)/2} \prod_{\nu=1}^{N} (2\nu-1)!! \quad , \tag{2.19}$$

where the formulae (3.11) and (A.12) of part I were used and W was determined by direct integration using methods of ref.[6].

The Wick theorem allows us to calculated the functional images (part I), and thus the boson images, of the many-fermion states $|\Psi>$,

$$|\Psi> = a^+_{\nu_1} \dots a^+_{\nu_A} |0> \quad , \tag{2.20}$$

i.e.

$$|\Psi> \leftrightarrow |\Psi) = A!!^{-1} b^\dagger_{[\nu_1\nu_2} \dots b^\dagger_{\nu_{A-1}\nu_A]} |0) \; ; \; A \textit{ even}, \tag{2.21a}$$

$$|\Psi> \leftrightarrow |\Psi) = (A-1)!!^{-1} b^\dagger_{[\nu_1} b^\dagger_{\nu_2\nu_3} \dots b^\dagger_{\nu_{A-1}\nu_A]} |0) \; ; A \textit{ odd}, \tag{2.21b}$$

where the square bracket denotes the antisymmetrization of all enclosed indices.

The action of generators (2.4-5) on the generalized coherent states $|C>$ can be in terms of the identities

$$\frac{\partial}{\partial C^*_{\nu\mu}} |C> = a^+_\nu a^+_\mu |C> \; , \; \frac{\partial}{\partial \bar{C}^*_\mu} |C> = a^+_\mu \left(1 - \sum_{\nu=1}^{N} \bar{C}^*_\nu a^+_\nu\right) |C> , \tag{2.22ab}$$

$$(a_\gamma - \sum_{\delta=1}^{N} C^*_{\gamma\delta}\, a^+_\delta) \exp\left\{\frac{1}{2}\sum_{\nu,\mu=1}^{N} C^*_{\nu\mu}\, a^+_\nu a^+_\mu\right\}|0\rangle = 0 \qquad (2.22c)$$

represented by first-order differential operators ; the formulae given in part I can be used as well. In this way the Dyson-type boson expansion of generators (2.4--5) is obtained, which expressed in terms of $\tilde{b}^\dagger_{\nu\mu}$ or $b^\dagger_{\nu\mu}$ and $b^\dagger_\mu$ bosons reads

$$a_\mu \leftrightarrow \tilde{b}_{\mu N+1} - \sum_{\gamma=1}^{N+1} \tilde{b}^\dagger_{\gamma N+1}\tilde{b}_{\gamma\mu} = b_\mu - \sum_{\gamma=1}^{N} b^\dagger_\gamma b_{\gamma\mu} \quad , \qquad (2.23a)$$

$$a^+_\mu \leftrightarrow \tilde{b}^\dagger_{\mu N+1} - \sum_{\gamma=1}^{N+1} \tilde{b}^\dagger_{\gamma\mu}\tilde{b}_{\gamma N+1} - \sum_{\gamma\delta=1}^{N+1} \tilde{b}^\dagger_{\mu\gamma}\tilde{b}^\dagger_{N+1\,\delta}\tilde{b}_{\gamma\delta} = \qquad (2.23b)$$

$$= b^\dagger_\mu\left(1 - \sum_{\gamma=1}^{N} b^\dagger_\gamma b_\gamma\right) + \sum_{\gamma\delta=1}^{N} b^\dagger_\delta b^\dagger_{\mu\gamma} b_{\gamma\delta} - \sum_{\gamma=1}^{N} b^\dagger_{\gamma\mu} b_\gamma \quad ,$$

$$a^+_\mu a^+_\nu \leftrightarrow \tilde{b}^\dagger_{\mu\nu} - \sum_{\gamma\delta=1}^{N+1} \tilde{b}^\dagger_{\mu\gamma}\tilde{b}^\dagger_{\nu\delta}\tilde{b}_{\gamma\delta} = \qquad (2.23c)$$

$$= b^\dagger_{\mu\nu} - \sum_{\gamma\delta=1}^{N} b^\dagger_{\mu\gamma} b^\dagger_{\nu\delta} b_{\gamma\delta} + \sum_{\gamma=1}^{N}\left(b^\dagger_\mu b^\dagger_{\nu\gamma} - b^\dagger_\nu b^\dagger_{\mu\gamma}\right) b_\gamma \quad ,$$

$$a^+_\mu a_\nu \leftrightarrow \sum_{\gamma=1}^{N+1} \tilde{b}^\dagger_{\mu\gamma}\tilde{b}_{\nu\gamma} = \sum_{\gamma=1}^{N} b^\dagger_{\mu\gamma} b_{\nu\gamma} + b^\dagger_\mu b_\nu \quad , \qquad (2.23d)$$

$$a_\nu a_\mu \leftrightarrow \tilde{b}_{\mu\nu} = b_{\mu\nu} \quad . \qquad (2.23e)$$

This is the finite and boson-like non-hermitian expansion derived by Fukutome[7]. A slightly different expansion was given by Okubo [8] who did not consider the operators $b^\dagger_\mu$, $b_\mu$ as representing ideal bosons.

The fermion states $|\Psi\rangle$, eq.(2.20), form the orthonormal basis in the fermion space. The overlap matrix Q (Part II) can thus be determined in terms of the boson images $|\Psi)$, eq.(2.21). It turns out that Q is diagonal and

$$(\Psi|\Psi\rangle = \begin{cases} (A-1)!! & ; \ A \ \text{even} , \\ A!! & ; \ A \ \text{odd} \end{cases} \tag{2.24}$$

or

$$(\Psi|\Psi) = (2N_B - 1)!! \quad , \tag{2.25}$$

where $N_B$ denotes the number of bosons in the state $|\Psi)$, while the total boson number operator is expressed as

$$\hat{N}_B = \sum_{\substack{\nu,\mu=1 \\ \nu<\mu}}^{N} b^{\dagger}_{\nu\mu} b_{\nu\mu} + \sum_{\mu=1}^{N} b^{\dagger}_{\mu} b_{\mu} \tag{2.26}$$

(round hat denotes the operator acting in the boson space).

The orthonormalizing operators $\hat{G}_B$ and $\hat{G}_F$ defined in part II are thus diagonal in the boson basis and their eigenvalues depend only on the number of bosons . Introducing the symbolical notation $G_{nn'}$ for the matrix element of $\hat{G}$ between the physical boson states with $N_B = n$ and $N_B = n'$, one has

$$G_{Bnn'} = \delta_{nn'}\left[(2n-1)!!\right]^{-\frac{1}{2}} , \quad G_{Fnn'} = \delta_{nn'}\left[(2n-1)!!\right]^{\frac{1}{2}} . \tag{2.27ab}$$

As discussed in part II, the above relations can be fulfilled by many different forms of the $\hat{G}_F$ and $\hat{G}_B$ operators. For example, one can choose $\hat{G}_F$ as

$$\hat{G}_F = \sum_{n=0}^{\infty} \left[(2n-1)!!\right]^{\frac{1}{2}} \hat{P}_n \quad , \tag{2.28}$$

where $\hat{P}_n$ projects the boson states on the subspace with number of bosons $N_B = n$. The boson expansion of $\hat{P}_n$,

$$\hat{P}_n = :\hat{N}_B^n / n! \exp\{-\hat{N}_B\}: \quad , \tag{2.29}$$

(and therefore that of $\hat{G}_F$) is infinite and convergent .

An other form of the $\hat{G}_F$ operator which also fulfills condition (2.27b) is

$$\hat{G}_F = \sum_{n=0}^{m} \left[(2n-1)!!\right]^{\frac{1}{2}} \hat{P}_n \quad , \tag{2.30}$$

where $m = N_B^{max}$ is the maximal boson number in the physical space, which is always finite. Evidently, the two forms, eqs. (2.28) and (2.30), differ only outside the physical space. A third possible form of $\hat{G}_F$,

$$\hat{G}_F = \left[(2\hat{N}_B - 1)!!\right]^{\frac{1}{2}} \quad , \tag{2.31}$$

which makes direct use of the square root Taylor infinite expansion, is always faced with convergence problems.

As it was argued in part I, a finite expansion for $\hat{G}_F$ can always be given . This is so because one has to fulfill only a finite number of conditions for the matrix elements of $\hat{G}_F$ and $\hat{G}_B$ with respect to the physical space. As in many examples these conditions have the form of eqs.(2.27ab), the general method for finding the $\hat{G}_F$ and $\hat{G}_B$ operators can be presented as follows :

Define the operators $\hat{F}\{X_n\}$ depending on the sequence of numbers $X_n$, $n = 0,1, \ldots, m$, as

$$\hat{F}\{X_n\} = \sum_{k=0}^{m} Y_k^{(m)} \hat{N}_B^{k} \quad , \tag{2.32}$$

where the coefficients $Y_k^{(m)}$ can be determined in terms of $X_n$ from

$$Y_k^{(m)} = (-1)^k \sum_{l=k}^{m} \frac{1}{l!} \left[ \sum_{p=0}^{l} (-1)^p \binom{l}{p} X_p \right] \left\{ {l-1 \atop k-1} \right\} , \tag{2.33}$$

$$Y_0^{(m)} = X_0 \quad ,$$

and the coefficients $\left\{ {l \atop k} \right\}$ , $0 \leq k \leq l$, resembling the Newton coefficients $\binom{l}{p}$, are determined by

$$\left\{ {l+1 \atop k+1} \right\} = \left\{ {l \atop k} \right\} + (l+1) \left\{ {l \atop k+1} \right\} \quad , \tag{2.34}$$

$$\left\{ {l \atop l} \right\} = 1 \quad , \quad \left\{ {l \atop 0} \right\} = l!$$

and have the property

$$\sum_{k=0}^{m} \left\{ {m \atop k} \right\} t^k = \prod_{k=1}^{m} (t+k) \tag{2.35}$$

for every t. It is a matter of simple algebra to prove that

$$F_{nn'}\{X_n\} = \delta_{nn'} X_n , \qquad n,n' = 0,1,\ldots,m. \tag{2.36}$$

Hence conditions (2.27ab) can be fulfilled in terms of finite expansions :

$$\hat{G}_B = \hat{F}\left\{[(2n-1)!!]^{-\frac{1}{2}}\right\} , \quad \hat{G}_F = \hat{F}\left\{[(2n-1)!!]^{\frac{1}{2}}\right\} . \tag{2.37ab}$$

In view of the identities

$$\hat{P}_n b^\dagger = b^\dagger \hat{P}_{n-1} , \qquad \hat{P}_n b = b \hat{P}_{n+1} \tag{2.38}$$

one has

$$\begin{aligned} \hat{G}_B b^\dagger \hat{G}_F &= \hat{F}\left\{(2n-1)^{-\frac{1}{2}}\right\} b^\dagger , \\ \hat{G}_B b \hat{G}_F &= \hat{F}\left\{(2n+1)^{\frac{1}{2}}\right\} b \end{aligned} \tag{2.39}$$

and thus the Holstein-Primakoff-type expansion is obtained from the expansion (2.23) by multiplying the terms which increase and decrease the boson number by $\hat{F}\{(2n-1)^{-\frac{1}{2}}\}$ and $\hat{F}\{(2n+1)^{\frac{1}{2}}\}$, respectively. Hence the hermitian and finite expansion for the generators (2.4-5) reads

$$a_\mu \leftrightarrow \hat{F}\{(2n+1)^{\frac{1}{2}}\}\, b_\mu - \sum_{\gamma=1}^{N} b_\gamma^\dagger b_{\gamma\mu} \quad , \tag{2.40a}$$

$$a_\mu^\dagger \leftrightarrow \hat{F}\{(2n-1)^{-\frac{1}{2}}\}\Big[b_\mu^\dagger\Big(1-\sum_{\gamma=1}^{N} b_\gamma^\dagger b_\gamma\Big)+\sum_{\gamma\delta=1}^{N} b_\delta^\dagger b_{\mu\gamma}^\dagger b_{\gamma\delta}\Big]-\sum_{\gamma=1}^{N} b_{\gamma\mu}^\dagger b_\gamma \;, \tag{2.40b}$$

$$a_\mu^\dagger a_\nu^\dagger \leftrightarrow \hat{F}\{(2n-1)^{-\frac{1}{2}}\}\Big[b_{\mu\nu}^\dagger-\sum_{\gamma\delta=1}^{N} b_{\mu\gamma}^\dagger b_{\nu\delta}^\dagger b_{\gamma\delta}+\sum_{\gamma=1}^{N}\big(b_\mu^\dagger b_{\nu\gamma}^\dagger-b_\nu^\dagger b_{\mu\gamma}^\dagger\big)b_\gamma\Big] \;, \tag{2.40c}$$

$$a_\mu^\dagger a_\nu \leftrightarrow \sum_{\gamma=1}^{N} b_{\mu\gamma}^\dagger b_{\nu\gamma} + b_\mu^\dagger b_\nu \quad , \tag{2.40d}$$

$$a_\nu a_\mu \leftrightarrow \hat{F}\{(2n+1)^{\frac{1}{2}}\}\, b_{\mu\nu} \quad . \tag{2.40e}$$

The maximal boson number m, for which the $\hat{F}$ operators should be determined, eqs.(2.32), (2.33), is in this case equal to the integral part of N/2,

$$m = N_B^{max} = \left[\frac{N}{2}\right] \quad . \tag{2.41}$$

Notice, that expansion (2.40) is hermitian and forms the so(2N+1) algebra only with respect to the physical space. By choosing the nonsingular form of the $\hat{G}_F$ operator, eq.(2.28), and its inverse for the $\hat{G}_B$ operator,

$$\hat{G}_B = \sum_{n=0}^{\infty}\left[(2n-1)!!\right]^{-\frac{1}{2}} \hat{P}_n \quad , \tag{2.42}$$

one can obtain Holstein-Primakoff-type bosons expansion for the generators with the commutation relations fulfilled in the entire boson space . Then however, the expansion becomes infinite. The infinite and hermitian Garbaczewski-type expansion can be obtained from that of eqs.(2.40) by using the operator $\hat{P}$ projecting boson states onto the physical space as discussed in part II.

All boson annihilation operators are physical ; this is obvious for $b_{\mu\nu}$, eq.(2.23e), and can be explicitly checked for $b_{\mu}$ in terms of eq.(2.21). Hence the R projection of a boson state (part II) can be obtain by using the $\mathcal{C}_{\underset{\sim}{\alpha}}$ operators in the functional representation,

$$\mathcal{C}_{\underset{\sim}{\alpha}} = \partial^{+}_{\underset{\sim}{\alpha}} \mathcal{P} \quad , \tag{2.43}$$

(cf. eq.(5.21) of part II) or the corresponding operators $\hat{B}^{\dagger}_{\underset{\sim}{\alpha}}$,

$$\hat{B}^{\dagger}_{\underset{\sim}{\alpha}} = b^{+}_{\underset{\sim}{\alpha}} \hat{P} \quad , \tag{2.44}$$

in the boson space. Note that $b^{+}_{\underset{\sim}{\alpha}}$ is not the boson creation operator as it is the hermitian conjugation of $b_{\underset{\sim}{\alpha}}$ in the fermion sense. For the fermion hermitian conjugation given by the weight function w(C) of eq.(2.18) one obtains

$$\hat{B}^{\dagger}_{\mu\nu} = \Big( \tilde{b}^{\dagger}_{\mu\nu} - \sum_{\gamma\delta=1}^{N+1} \tilde{b}^{\dagger}_{\mu\gamma} \tilde{b}^{\dagger}_{\nu\delta} \tilde{b}_{\gamma\delta} \Big) \hat{P} \quad , \tag{2.45}$$

which can be easily expressed in terms of $b^{\dagger}_{\mu\nu}$ and $b^{\dagger}_{\mu}$ bosons. Thus the R projection of a boson state can be obtained by replacing every boson creation operator $\tilde{b}^{\dagger}_{\mu\nu}$ by the corresponding operator $\hat{B}^{\dagger}_{\mu\nu}$ .

## 3. ALGEBRA so(2N)

The bifermion operators

$$a^+_\mu a^+_\nu \quad , \quad a_\nu a_\mu \quad , \quad \frac{1}{2}\delta_{\mu\nu} - a^+_\mu a_\nu \tag{3.1}$$

form the orthogonal algebra, $\mathcal{A}$ = so(2N), which is the classical $D_l$ algebra with the rank $l$ = N. The Cartan subalgebra and the correspondence between roots and $\hat{E}_{\underset{\sim}{\alpha}}$ generators are identical as for so(2N+1), eqs.(2.2) and (2.4bc), (2.5bc). Again, the highest weights of allowed representations must have components equal to $\pm\frac{1}{2}$ . For SO(2N) there are two spinor representations[5] which fulfill this requirement, namely

$$\underset{\sim}{j} = \frac{1}{2}\sum_{\mu=1}^{N} \underset{\sim}{w}_\mu = \left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2}\right) \tag{3.2}$$

and

$$\underset{\sim}{j} = \frac{1}{2}\sum_{\mu=1}^{N-1} w_\mu - \frac{1}{2}w_N = \left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2}\right). \tag{3.3}$$

The boson expansions for them are considered separately in the subsections 3.1 and 3.2.

## 3.1 Representaion $\underset{\sim}{j} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$

The fermion vacuum $|0\rangle$ is the highest weight state for this representation. The generators (3.1) can only create pairs of fermions ; the representation space thus contains all fermion states with even particle number.

The roots $\underset{\sim}{w}_\nu + \underset{\sim}{w}_\mu$, $\nu<\mu$, are positive and nonorthogonal to the highest weight. Hence the generalized coherent states are given by

$$|C\rangle = \exp\left\{\frac{1}{2}\sum_{\nu\mu=1}^{N} C^{*}_{\nu\mu}\, a^{+}_{\nu}\, a^{+}_{\mu}\right\}|0\rangle \;, \qquad (3.4)$$

where again the complex numbers $C_{\nu\mu}$ are considered as the elements of the antisymmetric matrix, $C^T = -C$. The corresponding M bosons, $M = N(N-1)/2$, are denoted by $b^{\dagger}_{\nu\mu}$, $b^{\dagger}_{\nu\mu} = -b^{\dagger}_{\mu\nu}$ . For this representation the generalized coherent states are identical to the Thouless vacua [9]. It is worth noting that the alternative form of Thouless theorem presented by Ring and Schuck [10] examplifies the general relation, eq.(A.8) of part I, between two different parametrizations of generalized coherent states.

The norm of $|C\rangle$ reads [11,12]

$$\langle C|C\rangle = \det^{\frac{1}{2}}\left(I^{(N)} + CC^{+}\right) . \qquad (3.5)$$

Despite the square root, $\langle C|C\rangle$ is a polynomial ; this is so because above determinant is always the square of a polynomial. In terms of the identity (2.22c) one has the formula

$$\langle C|a_\mu a_\nu^+|C\rangle = \delta_{\mu\nu} - \sum_{\gamma\delta=1}^{N} C_{\mu\gamma}^* C_{\nu\delta} \langle C|a_\delta a_\gamma^+|C\rangle = \\ = (I^{(N)} + C^+C)^{-1}_{\mu\nu} \langle C|C\rangle \tag{3.6}$$

utilized in sect. 2.

The specific weight is given by

$$\tilde{\underline{j}} = \sum_{\substack{\nu,\mu=1\\ \nu<\mu}}^{N} (\underline{w}_\nu + \underline{w}_\mu) = (N-1)\sum_{\mu=1}^{N} \underline{w}_\mu = 2(N-1)\underline{j} \tag{3.7}$$

and thus the weight function $w(C)$ reads

$$w(C) = W\left[\det(I^{(N)} + CC^+)\right]^{-\frac{2N-1}{2}}, \tag{3.8}$$

$$W = \left(\frac{2}{\pi}\right)^{N(N-1)/2} \prod_{\nu=1}^{N-1} (2\nu-1)!! \ . \tag{3.9}$$

The same result has also been obtained by Suzuki [13] who proved the unity resolution using assumptions different from those given in part I.

The generalized coherent states of the present section, eq. (3.4), are equal to those of sect. 2, eq. (2.9), when setting $c_\mu = 0$. Hence in the present case the boson image $|\Psi)$ of the fermion state $|\Psi\rangle$, eq.(2.20), is given by eq.(2.21a) while the boson expansions for the generators (3.1) are given by eqs. (2.23c-e) with the bosons $b^\dagger_\mu$, $b_\mu$ disregarded. One obtains the generalized Dyson expansion derived by Janssen et al.[14] and the corresponding expressions will not be repeated. The generalized Holstein-Primakoff expansion [14] can be obtained as in sect.2. Its finite version is given by eqs.(2.40c-e) and (2.26) with the bosons $b^\dagger_\mu$, $b_\mu$ disregarded again.

### 3.2 Representation $j = (\frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2})$

The highest weight state for this representation is the single-fermion state

$$|\underset{\sim}{j}\rangle = a^+_N|0\rangle \equiv |a^+_N\rangle \; . \tag{3.10}$$

Which fermion state is occupied depends only on the convention adopted for the numbering of the fermion states from 1 to N. Each state can be called the N-th one, and the approach is independent of this choice unless approximations are made. For applications, e.g. when one tries to reduce the number of different bosons, this choice may be important.

Generators (3.1) creating pairs of fermions can transfrom $|a^+_N\rangle$ into any other odd-fermion state. Hence the representation space contains odd states and is the orthogonal complement of that considered in sect.3.1.

The roots $\underset{\sim}{w}_\nu + \underset{\sim}{w}_\mu$, $\nu<\mu<N$, and $\underset{\sim}{w}_\nu - \underset{\sim}{w}_N$, $\nu<N$, are positive and non-orthogonal to the highest weight $\underset{\sim}{j}$, eq.(3.3). The corresponding $\hat{E}_{\underset{\sim}{\alpha}}$ generators, eqs.(2.5bc), determine the generalized coherent states

$$|C\rangle = \exp\Big\{\sum_{\substack{\nu\mu=1\\ \nu<\mu}}^{N-1} C^*_{\nu\mu} a^+_\nu a^+_\mu + \sum_{\nu=1}^{N-1} C^*_\nu a^+_\nu a_N\Big\}|a^+_N\rangle \tag{3.11}$$

or

$$|C\rangle = \Big(1+\sum_{\nu=1}^{N-1}\bar{C}^*_\nu a^+_\nu a_N\Big)\exp\Big\{\frac{1}{2}\sum_{\nu\mu=1}^{N-1} C^*_{\nu\mu} a^+_\nu a^+_\mu\Big\}|a^+_N\rangle \tag{3.12}$$

in obvious analogy to eqs.(2.9) and (2.12). The antisymmetric matrix C, $C_{\nu\mu} = -C_{\mu\nu}$, in N-1 dimensions and N-1 dimensional column vector $\bar{C}$, $\bar{C}_\nu = C_\nu$, can be presented in the form of antisymmetric matrix $\tilde{C}$ in N dimensions, as in eq.(2.11). The boson operators $\tilde{b}^\dagger_{\nu\mu} = -\tilde{b}^\dagger_{\mu\nu}$ are related to the components of the matrix $\tilde{C}$, while $b^\dagger_{\nu\mu}$ and $b^\dagger_\mu$ are related to $C_{\nu\mu}$ and $\bar{C}_\mu$, respectively.

The generalized coherent states $|C\rangle$, eq.(3.11), are now the Thouless vacua with respect to the quasiparticle vacuum $|a^+_N\rangle$ for the quasiparticle annihilation operators $\tilde{a}$:

$$\tilde{a}_\nu = a_\nu \ , \quad \nu = 1,\dots, N-1 \ , \tag{3.13}$$
$$\tilde{a}_N = a^+_N \ .$$

Similarly as in sect.3.1 one has

$$\langle C | C \rangle = \det^{\frac{1}{2}}(I^{(N)} + \tilde{C}\tilde{C}^{+}) \tag{3.14}$$

and in view of the expression for the specific weight,

$$\underset{\sim}{\tilde{j}} = \sum_{\substack{\nu\mu=1\\ \nu<\mu}}^{N-1}(\underset{\sim}{w}_{\nu}+\underset{\sim}{w}_{\mu}) + \sum_{\nu=1}^{N-1}(\underset{\sim}{w}_{\nu} - \underset{\sim}{w}_{N}) = 2(N-1)\underset{\sim}{j} \;, \tag{3.15}$$

the weight function $w(C)$ is given by eq.(3.8) after the substitution $C \to \tilde{C}$.

The boson image of the fermion state $|\Psi\rangle$, eq.(2.20), for odd A reads

$$\begin{aligned} |\psi\rangle \leftrightarrow |\psi) &= (A-1)!!^{-1}\, b^{\dagger}_{[\nu_1} b^{\dagger}_{\nu_2\nu_3} \cdots b^{\dagger}_{\nu_{A-1}\nu_A]}|0) \qquad \text{for } \nu_i \neq N, \\ |\psi\rangle \leftrightarrow |\psi) &= (A-1)!!^{-1}\, b^{\dagger}_{[\nu_2\nu_3} \cdots b^{\dagger}_{\nu_{A-1}\nu_A]}|0) \qquad \text{for } \nu_1 = N. \end{aligned} \tag{3.16}$$

Hence to the A-fermion state corresponds the $(A-1)/2$ or $(A+1)/2$ boson state depending on whether the N-th single fermion state is occupied or not . The boson vacuum $|0)$ corresponds to the single-fermion state $|a_N^+\rangle$, eq.(3.10). The bosons $b^{\dagger}_{\nu\mu}$, $\nu,\mu < N$, and $b^{\dagger}_{\mu}$ , $\nu < N$, play a different role. The former add pairs of fermions while the latter transport the fermion from the N-th state to all other states.

The boson expansion for generators (3.1) can be presented in the form of the generalized Dyson expansion :

$$\tilde{a}^+_\mu \tilde{a}^+_\nu \leftrightarrow \tilde{b}^\dagger_{\mu\nu} - \sum_{\gamma\delta=1}^{N} \tilde{b}^\dagger_{\mu\gamma} \tilde{b}^\dagger_{\nu\delta} \tilde{b}_{\gamma\delta} \quad ,$$

$$\tilde{a}^+_\mu \tilde{a}_\nu \leftrightarrow \sum_{\gamma=1}^{N} \tilde{b}^\dagger_{\mu\gamma} \tilde{b}_{\nu\gamma} \quad , \tag{3.17}$$

$$\tilde{a}_\nu \tilde{a}_\mu \leftrightarrow \tilde{b}_{\mu\nu}$$

where

$$\tilde{a}^+_\nu = a^+_\nu \, , \; \tilde{a}_\nu = a_\nu \, , \; \tilde{b}^\dagger_{\nu\mu} = b^\dagger_{\nu\mu} \quad \text{for } \nu,\mu = 1,\dots,N-1 \, ,$$
$$\tilde{a}^+_N = a_N \, , \; \tilde{a}_N = a^+_N \, , \; \tilde{b}^\dagger_{\nu N} = b^\dagger_\nu \quad \text{for } \nu = 1,\dots,N-1 . \tag{3.18}$$

As the norm of the boson image $|\psi)$, eq.(3.16), reads

$$(\psi|\psi) = (2N_B - 1)!! \tag{3.19}$$

for

$$\hat{N}_B = \sum_{\substack{\nu\mu=1 \\ \nu<\mu}}^{N-1} b^\dagger_{\nu\mu} b_{\nu\mu} + \sum_{\nu=1}^{N-1} b^\dagger_\nu b_\nu \quad , \tag{3.20}$$

the Holstein-Primakoff-type expansion in the odd fermion space is identical with that in the even space when the quasiparticle operators $\tilde{a}^+_\nu$ and $\tilde{a}_\nu$ , eq.(3.18), are taken instead of the $a^+_\nu$ and $a_\nu$ operators.

## 4. ALGEBRA su(N)

The particle number conserving bifermion operators

$$a^+_\mu a_\nu \tag{4.1}$$

constitute the unitary u(N) algebra. After the particle number operator $\hat{N}_F$ ,

$$\hat{N}_F = \sum_{\mu=1}^{N} a^+_\mu a_\mu \quad , \tag{4.2}$$

is removed from the algebra, the remaining $N^2-1$ generators form the unitary unimodular algebra $\mathcal{A} =$ su (N), which is the classical $A_l$ algebra with the rank $l = N - 1$. The Cartan subalgebra is formed by N-1 linearly independent generators chosen from among N generators

$$\hat{H}_\mu = a^+_\mu a_\mu \ . \tag{4.3}$$

For simplicity, one can use all N generators $\hat{H}_\mu$ bearing in mind that the set

$$\hat{H}'_\mu = \hat{H}_\mu + \varepsilon \sum_{\mu=1}^{N} \hat{H}_\mu = \hat{H}_\mu + \varepsilon \hat{N}_F \tag{4.4}$$

can be used as well for any $\varepsilon$. This ambiguity requires the identification of every two weigth vectors differing by $\varepsilon \underset{\sim}{1}$,

$$\underset{\sim}{1} = \sum_{\mu=1}^{N} \underset{\sim}{w}_\mu = (1, 1, \ldots, 1) \ . \tag{4.5}$$

The roots $\underset{\sim}{\alpha}$ and the corresponding $\hat{E}_{\underset{\sim}{\alpha}}$ generators are given by eqs. (2.4c) and (2.5c). The components of weights related to the $\hat{H}_{\mu}$ operators, eq. (4.3), can only be equal to 0 or 1. Thus the only allowed representations are the fundamental ones [2-4], for the highest weights

$$\underset{\sim}{j} = \sum_{\mu=1}^{A} \underset{\sim}{w}_{\mu} = (1,1,\ldots,1,0,\ldots,0) \quad , A=1,\ldots,N-1 , \tag{4.6}$$

and the one-dimensional identity representation for

$$\underset{\sim}{j} = 0 \; . \tag{4.7}$$

Decomposing the spinor representation of SO(2N+1) (full fermion space) into the irreducible allowed representations of SU(N) one finds the identity representation appearing twice for the highest weight states

$$|\underset{\sim}{j}> = |0> \quad \text{and} \quad |\underset{\sim}{j}> = a_1^+ \ldots a_N^+ |0> ; \tag{4.8}$$

where the nonuniqueness with respect ot addtion of vector $\underset{\sim}{1}$, eq. (4.5), should be used. The highest weight states for the fundamental representations read

$$|\underset{\sim}{j}> = a_1^+ \ldots a_A^+ |0> \equiv |HF> \; . \tag{4.9}$$

Similarly as in sect.3.2, the choice of occupied states is immaterial for general considerations and may become important if some approximations are made.

As generators (4.1) do not change the fermion number, the representation space sonsists of all A particle states for the A-th representation. The roots $\underset{\sim}{w}_i - \underset{\sim}{w}_m$ are positive and nonorthogonal to the highest weight $\underset{\sim}{j}$, eq.(4.6), (the indices $i,j = 1, \ldots, A$ and $m,n = A+1, \ldots, N$ denote the hole and particle states, respectively) and hence the generalized coherent states read

$$|C\rangle = \exp\left\{\sum_{mi} C_i^{m\,*} a_m^+ a_i\right\} |HF\rangle . \tag{4.10}$$

Again, these are the quasiparticle vacua for the particle number preserving version of Thouless theorem. The complex variables $C_i^m$ are considered to constitute the rectangular matrix with A rows and N-A colums ; the upper index is introduced for future convenience. These variables determine M, M = A(N-A), bosons $b_i^{m\dagger}$. The well-known formula for the norm of $|C\rangle$ reads

$$\langle C|C\rangle = \det\left(I^{(A)} + CC^+\right) , \tag{4.11}$$

where the absence of the square root in comparison to eq.(3.5) should be noted.

The specific weight is given by

$$\tilde{\underset{\sim}{j}} = \sum_{i=1}^{A} \sum_{m=A+1}^{N} \left(\underset{\sim}{w}_i - \underset{\sim}{w}_m\right) = N\sum_{i=1}^{A} \underset{\sim}{w}_i - A\underset{\sim}{1} \cong N\underset{\sim}{j} , \tag{4.12}$$

where the vector $A\underset{\sim}{1}$ was disregarded. Hence the weight function w(C) reads

$$w(C) = W\left[\det\left(I^{(A)} + CC^+\right)\right]^{-N-1} , \tag{4.13}$$

$$W = \Pi^{-A(N-A)} \prod_{\nu=1}^{N-A} \frac{(\nu+A)!}{\nu!} \tag{4.14}$$

and is identical to that obtained in refs.[13,15].

All states of the considered representation are obtained by creation of particle-hole pairs in the vacuum state $|HF\rangle$, eq.(4.9), i.e.

$$|\Psi\rangle = (a^+_{m_1} a_{i_1}) \ldots (a^+_{m_k} a_{i_k}) |HF\rangle \tag{4.15}$$

and in terms of Wick theorem the boson image of $|\Psi\rangle$ is given by

$$|\Psi\rangle \leftrightarrow |\Psi) = b^{m_1\dagger}_{[i_1} b^{m_2\dagger}_{i_2} \ldots b^{m_k\dagger}_{i_k]} |0) . \tag{4.16}$$

where the square bracket denotes the antisymmetrization of the enclosed indices. The boson expansions of generators (4.1), originally derived by Rowe et al.[16], read

$$a^+_m a_i \leftrightarrow b^{m\dagger}_i - \sum_{nj} b^{m\dagger}_j b^{n\dagger}_i b^n_j \quad , \tag{4.17a}$$

$$a^+_m a_n \leftrightarrow \sum_i b^{m\dagger}_i b^n_i \quad , \tag{4.17b}$$

$$a^+_i a_j \leftrightarrow \delta_{ij} - \sum_m b^{m\dagger}_j b^m_i \quad , \tag{4.17c}$$

$$a^+_i a_m \leftrightarrow b^m_i \quad . \tag{4.17d}$$

In terms of these expansions the fermion number operator $\hat{N}_F$, eq.(4.2), is identically mapped onto the number A, as it should be for the representation in question. The boson images $|\Psi)$ of the orthonormal fermion states $|\Psi\rangle$ are orthogonal and their norms read

$$(\Psi|\Psi) = k! = N_B! \tag{4.18}$$

for

$$\hat{N}_B = \sum_{mi} b_i^{m\dagger} b_i^m \quad . \tag{4.19}$$

Hence Holstein-Primakoff-type hermitian and finite boson expansion corresponding to the Dyson-type expansion of eqs.(4.17) is obtained by multiplying from the left-hand side the images of eqs.(4.17a) and (4.17d) by $\hat{F}\{1/\sqrt{n}\}$ and $\hat{F}\{\sqrt{n+1}\}$, respectively, while leaving those of eqs.(4.17bc) unchanged, i.e.,

$$a_m^+ a_i \leftrightarrow \hat{F}\{1/\sqrt{n}\}\Big(b_i^{m\dagger} - \sum_{nj} b_j^{m\dagger} b_i^{n\dagger} b_j^n\Big) , \tag{4.20a}$$

$$a_m^+ a_n \leftrightarrow \sum_i b_i^{m\dagger} b_j^n \quad , \tag{4.20b}$$

$$a_i^+ a_j \leftrightarrow \delta_{ij} - \sum_m b_j^{m\dagger} b_i^m \quad , \tag{4.20c}$$

$$a_i^+ a_m \leftrightarrow \hat{F}\{\sqrt{n+1}\}\, b_i^m \quad . \tag{4.20d}$$

The $\hat{F}$ operators should be calculated as given by eqs.(2.32-33) with the $\hat{N}_B$ operator of eq.(4.19) and the maximal number of bosons

$$m = \min\{A, N-A\} \quad . \tag{4.21}$$

## 5. ALGEBRA sp(N)

Suppose N is an even integer and each single-fermion state $|\mu\rangle$ is in correspondence with some other state $|\bar{\mu}\rangle$, $\mu \neq \bar{\mu}$, $\bar{\bar{\mu}} = \mu$. Let the phase factor $s_\mu = \pm 1$ be defined for each $\mu$ in such a way that $s_{\bar{\mu}} = -s_\mu$. Obviously, the above conditions are fulfilled for the time-reversed fermion states which is indicated by using the usual notation. The reference to time-reversal, however, is not essential and is not used in the following.

The operators $\hat{B}_{\mu\nu}$,

$$\hat{B}_{\mu\nu} = a^+_\mu a_{\bar{\nu}} + s_\mu s_\nu a^+_\nu a_{\bar{\mu}} , \tag{5.1}$$

form the symplectic, $\mathcal{A}$ = sp(N), algebra which is the classical [2-4] algebra $C_l$ with the rank $l$ = N/2. Let the $\mu$-th state be called positive (negative), $\mu > 0$ ($\mu < 0$), for $s_\mu$ = +1(-1). The Cartan subalgebra is given by

$$\hat{H}_\mu = \hat{B}_{\mu\bar{\mu}} = a^+_\mu a_\mu - a^+_{\bar{\mu}} a_{\bar{\mu}} \quad \text{for } \mu > 0, \tag{5.2}$$

while the $\hat{E}_{\underset{\sim}{\alpha}}$ generators are related to positive roots as

$$\hat{E}_{\underset{\sim}{\alpha}} = \hat{B}_{\mu\nu}(1+\delta_{\mu\nu})^{-\frac{1}{2}} , \quad 0 < \nu \leq \mu , \quad \text{for } \underset{\sim}{\alpha} = \underset{\sim}{w}_\nu + \underset{\sim}{w}_\mu , \tag{5.3a}$$

$$\hat{E}_{\underset{\sim}{\alpha}} = \hat{B}_{\nu\bar{\mu}} , \quad 0 < \nu < \mu , \quad \text{for } \underset{\sim}{\alpha} = \underset{\sim}{w}_\nu - \underset{\sim}{w}_\mu \tag{5.3b}$$

and to negative roots as

$$\hat{E}_{\underset{\sim}{\alpha}} = \hat{B}_{\bar{\nu}\bar{\mu}}(1+\delta_{\mu\nu})^{-\frac{1}{2}}, \quad 0 < \nu \leq \mu , \quad \text{for } \underset{\sim}{\alpha} = -\underset{\sim}{w}_\nu - \underset{\sim}{w}_\mu , \tag{5.4a}$$

$$\hat{E}_{\underset{\sim}{\alpha}} = \hat{B}_{\mu\bar{\nu}} \quad , \quad 0<\nu<\mu \quad , \quad \text{for } \underset{\sim}{\alpha} = -\underset{\sim}{w}_{\nu} + \underset{\sim}{w}_{\mu} , \tag{5.4b}$$

with the relation between both sets :

$$\hat{E}^{+}_{\underset{\sim}{\alpha}} = \hat{E}_{-\underset{\sim}{\alpha}} \quad , \quad \hat{B}^{+}_{\mu\nu} = \hat{B}_{\bar{\nu}\bar{\mu}} \ . \tag{5.5}$$

The components of the weight vector of an allowed representation can only equal to $0, \pm 1$ ; hence only the fundamental and identity representations are allowed :

$$\underset{\sim}{j} = 0 \tag{5.6}$$

or

$$\underset{\sim}{j} = \sum_{\nu=1}^{\tilde{\nu}} \underset{\sim}{w}_{\nu} = (1,1,\dots,1,0,\dots,0) \quad , \tag{5.7}$$

where tilde distinguishes the seniority quantum number[5] from the single-particle index $\nu$. For the given highest weight $\underset{\sim}{j}$ many highest weight states $|\underset{\sim}{j}\rangle$ can be found in the fermion space. In the subspace with particle number A, however, the highest weight states are unique and the allowed values of $\tilde{\nu}$ are[5]

$$\tilde{\nu} = A, A-2, \dots, 1\ (0) \tag{5.8}$$

being the maximal number of decoupled particles in a given representation.

Let the positive index $d = 1,2,\dots,\tilde{\nu}$ number the states decoupled in the highest weight state, and the index $\rho = \tilde{\nu}+1, \dots, N/2$ the empty or paired states forming the seniority zero .

core, i.e.

$$\hat{H}_d \, | \underset{\sim}{j} > \; = \; | \underset{\sim}{j} > \; , \tag{5.9}$$

$$\hat{H}_p \, | \underset{\sim}{j} > \; = \; 0 \tag{5.10}$$

(cf. eq.(5.7)).

The positive and nonorthogonal to $\underset{\sim}{j}$ roots, $\underset{\sim}{w}_d - \underset{\sim}{w}_p$ and $\underset{\sim}{w}_d + \underset{\sim}{w}_\mu$ $(d \leq \mu)$, single out through eqs.(5.4) the generators which determine the generalized coherent states :

$$|C> = \exp\left\{ \sum_{pd} \bar{C}^*_{pd} \hat{B}_{p\bar{d}} + \sum_{d \leq \mu} \tilde{C}^*_{d\mu} \hat{B}_{\bar{d}\bar{\mu}} \right\} | \underset{\sim}{j} > \; , \tag{5.11}$$

where the factor $(1+\delta_{d\mu})^{-\frac{1}{2}}$ , eq.(5.4a), has been included in the definition of $\tilde{C}_{d\mu}$. Hence the boson representation involves $M = 2\tilde{\nu}(N/2 - \tilde{\nu}) + \tilde{\nu}(\tilde{\nu}+1)/2$ different bosons. The specific weight reads

$$\tilde{\underset{\sim}{j}} = \sum_{dp} (\underset{\sim}{w}_d - \underset{\sim}{w}_p) + \sum_{d \leq \mu} (\underset{\sim}{w}_d + \underset{\sim}{w}_\mu) = (N - \tilde{\nu} + 1) \underset{\sim}{j} \, , \tag{5.12}$$

and thus the weight function w(C) has the form

$$w(C) = W \, <C|C>^{-N+\tilde{\nu}-2} \; . \tag{5.13}$$

In the case $\tilde{\nu} = 1$, when the representation space contains states formed by one fermion coupled to an even core, one has with obvious notations

$$|C> = \left\{ \sum_{|\mu|>1} C^*_\mu \hat{B}_{\mu\bar{1}} + C^* \hat{B}_{\bar{1}\bar{1}} \right\} | \underset{\sim}{j} > \; , \tag{5.14}$$

$$<C|C> = 1 + \sum_{|\mu|>1} |C_\mu|^2 + 4|C|^2 \, , \tag{5.15}$$

$$w(C) = 4N!\,\pi^{1-N} \langle C|C\rangle^{-N-1} , \tag{5.16}$$

and the boson expansions for generators (5.1) read

$$\begin{aligned}
\hat{B}_{11} &\leftrightarrow b , \\
\hat{B}_{\bar{1}\bar{1}} &\leftrightarrow 4b^{\dagger}(1-\hat{N}_B) , \\
\hat{B}_{1\bar{1}} &\leftrightarrow 1 - \hat{N}_B - b^{\dagger}b , \\
\hat{B}_{1\bar{\mu}} &\leftrightarrow b_{\mu} + \tfrac{1}{2} s_{\bar{\mu}} b^{\dagger}_{\bar{\mu}} b \qquad \text{for } |\mu| > 1 , \\
\hat{B}_{\mu\bar{1}} &\leftrightarrow b^{\dagger}_{\mu}(1-\hat{N}_B) + 2 s_{\bar{\mu}} b^{\dagger} b_{\bar{\mu}} \quad \text{for } |\mu| > 1 , \\
\hat{B}_{\mu\nu} &\leftrightarrow b^{\dagger}_{\mu} b_{\bar{\nu}} + s_{\mu} s_{\nu} b^{\dagger}_{\nu} b_{\bar{\mu}} \quad \text{for } |\mu|, |\nu| > 1 ,
\end{aligned} \tag{5.17}$$

where the bosons $b^{\dagger}$ and $b^{\dagger}_{\mu}$ correspond to complex numbers C and $C_{\mu}$, respectively, and

$$\hat{N}_B = b^{\dagger}b + \sum_{|\mu|>1} b^{\dagger}_{\mu} b_{\mu} .$$

Creation of the bosons $b^{\dagger}_{\mu}$ and $b^{+}$ corresponds to the excitation of the odd fermion from the state 1 to $\mu$ and to the flip process $1 \to \bar{1}$, respectively.

## 6. ALGEBRA so(8)

The algebra so(8) has been studied by Ginocchio[17] in order to provide the microscopic justification of the interacting boson model [18]. It is composed of the monopole and quadrupole fermion pair creation operators,

$$\hat{S}^+ = \frac{1}{2}\sum_{jm}(-1)^{j-m} a^+_{jm} a^+_{j-m} \quad , \tag{6.1}$$
$$\hat{D}^+_\mu = \sum_{jj'}(-1)^{k+3/2+j}\left[(2j+1)(2j'+1)\right]^{\frac{1}{2}} \begin{Bmatrix} j & j' & 2 \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix} \left[a^+_j a^+_{j'}\right]^2_\mu \quad ,$$

the corresponding pair annihilation operators $\hat{S}$ and $\hat{D}_\mu$ and the single-particle operators $\hat{S}_0 = -\frac{1}{2}(\Omega - \hat{N}_F)$, $\hat{S}_0^+ = \hat{S}_0$ and $\hat{P}^r_\mu$, $r=1,2,3$, $\hat{P}^{r+}_\mu = (-1)^\mu \hat{P}^r_{-\mu}$, as defined in ref.[17]. The integer number k defines the j values included in the valence shell, $k+\frac{3}{2} \geq j \geq \left|k-\frac{3}{2}\right|$, while $\Omega = 2(2k+1)$ is half of the number of the valence states. The brackets $[\;]^J_M$ denote the standard angular momentum coupling and $\hat{N}_F$ is the valence fermion number operator. The commutation relations between above generators, as given in ref.[17], can be presented in the canonical form for the $\hat{H}$ and $\hat{E}_\alpha$ generators defined as :

$$\hat{H}_1 = -\hat{S}_0 \quad , \qquad \hat{H}_2 = -(2\hat{P}^1_0 - \hat{P}^3_0)/2\sqrt{5} \, , \tag{6.2ab}$$

$$\hat{H}_4 = -\frac{1}{2}\hat{P}^2_0 \quad , \qquad \hat{H}_3 = -(2\hat{P}^1_0 + \hat{P}^3_0)/2\sqrt{5} \, , \tag{6.2cd}$$

$$\hat{E}_{(-1,\pm1,0,0)} = \hat{D}^+_{\mp2}/\sqrt{2} \quad , \quad \hat{E}_{(-1,0,\pm1,0)} = \hat{D}^+_{\mp1}/\sqrt{2} \, , \tag{6.2ef}$$

$$\hat{E}_{(-1,0,0,\pm1)} = \frac{1}{2}(\hat{S}^+ \mp \hat{D}^+_0) \quad , \tag{6.2g}$$

$$\hat{E}_{(0,-1,-1,0)} = \tfrac{1}{2}\hat{P}_3^3\ , \quad \hat{E}_{(0,-1,1,0)} = \left(\sqrt{2}\,\hat{P}_1^1 - \sqrt{3}\,\hat{P}_1^3\right)/\sqrt{20}\ , \tag{6.2hi}$$

$$\hat{E}_{(0,-1,0,\pm 1)} = \left(\hat{P}_2^3 \mp \hat{P}_2^2\right)/\sqrt{2}\ , \quad \hat{E}_{(0,0,-1,\pm 1)} = \left(\sqrt{3}\,\hat{P}_1^1 + \sqrt{2}\,\hat{P}_1^3 \mp \sqrt{5}\,\hat{P}_1^2\right)/\sqrt{40}\ , \tag{6.2jk}$$

where the four-dimensional root vectors $\underset{\sim}{\alpha}$ are written explicitly. The $\hat{E}_{\underset{\sim}{\alpha}}$ generators for positive roots can be obtained by hermitian conjugation. Due to the high symmetry of the Dynkin diagram[19] corresponding to SO(8), five other equivalent solutions for the $\hat{\underset{\sim}{H}}$ and $\hat{E}_{\underset{\sim}{\alpha}}$ generators can be found by means of rotations in the root space which do not mix the positive and negative roots. The state $|\mathrm{core}\rangle$, for which all valence single-particle states are unoccupied, is the highest weight state with the highest weight $\underset{\sim}{j}$ :

$$\underset{\sim}{j} = \left(\frac{\Omega}{2}, 0, 0, 0\right) \qquad , \qquad \hat{\underset{\sim}{H}}\,|\mathrm{core}\rangle = \underset{\sim}{j}\,|\mathrm{core}\rangle\ . \tag{6.3ab}$$

Only the generators $\hat{S}^+$ and $\hat{D}_\mu^+$ , eq.(6.2e-g), correspond to negative roots nonorthogonal to $\underset{\sim}{j}$ and thus the generalized coherent states read

$$|C\rangle = \exp\left\{C^*\hat{S}^+ + \sum_{\mu=-2}^{2} C_\mu^*\,\hat{D}_\mu^+\right\}|\mathrm{core}\rangle\ . \tag{6.4}$$

The complex variables C and $C_\mu$ correspond to six boson creaction operators $s^\dagger$ and $d_\mu^\dagger$ , respectively. For the eight-dimensional fundamental representation with $\underset{\sim}{j} = (1, 0, 0, 0)$, i.e. $\Omega = 2$, the overlap $\langle C | C \rangle$ can be calculated by explicit expansion of the exponential function in eq.(6.4). Alternatively, one can rotate the root space so as to transform the weight $\underset{\sim}{m}_1 - \dot{\underset{\sim}{j}}$ into the spinor fundamental weight $\underset{\sim}{m}_4 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and then use eq.(3.5) for N=4 by identifying the variables C and $C_\mu$ with the six independent components of the 4×4 antisymmetric matrix $C_{\mu\nu}$.

Based on eq.(A.12) of part I one obtains the overlap $\langle C|C\rangle$ for arbitrary $\Omega$:

$$\langle C|C\rangle = \Big(1+2|C|^2+2\sum_{\mu=-2}^{2}|C_\mu|^2+\Big|\sum_{\mu=-2}^{2}(-1)^\mu C_{-\mu}C_\mu - C^2\Big|^2\Big)^{\frac{\Omega}{2}}, \tag{6.5}$$

while knowing the specific weight $\tilde{\underset{\sim}{j}}$,

$$\tilde{\underset{\sim}{j}} = (6,0,0,0), \tag{6.6}$$

the weight function w(C) can be presented as :

$$w(C) = W\Big(1+2|C|^2+2\sum_{\mu=-2}^{2}|C_\mu|^2+\Big|\sum_{\mu=-2}^{2}(-1)^\mu C_{-\mu}C_\mu - C^2\Big|^2\Big)^{-\frac{\Omega}{2}-6}, \tag{6.7}$$

$$W = \Big(\frac{2}{\pi}\Big)^6 \frac{(\Omega/2+5)!}{2\,(\Omega/2)!}(\Omega+6) \quad . \tag{6.8}$$

The Dyson-type boson expansions for generators can be obtained by expressing their action on the generalized coherent states in terms of differential operators. As the operators $\hat{S}^+$ and $\hat{D}^+_\mu$ commute, one has

$$\hat{S}^+|C\rangle = \partial^*|C\rangle, \quad \hat{D}^+_\mu|C\rangle = \partial^*_\mu|C\rangle, \tag{6.9ab}$$

where $\partial \equiv \partial/\partial C$, $\partial_\mu \equiv \partial/\partial C_\mu$ . Calculating the commutators of $\hat{S}$ with

$$\hat{C} = C^{*}\hat{S}^{+} + \sum_{\mu=-2}^{2} C_{\mu}^{*}\ \hat{D}_{\mu}^{+} \quad ; \tag{6.10}$$

$$[\hat{C},\hat{S}] = 2C^{*}\hat{S}_{o} + C_{\mu}^{*}\,\hat{P}_{\mu}^{2} \quad , \tag{6.11a}$$

$$[\hat{C},[\hat{C},\hat{S}]] = -4C^{*}\hat{C} - 2\Big(\sum_{\mu=-2}^{2}\tilde{C}_{\mu}^{*}C_{\mu}^{*} - C^{*2}\Big)\hat{S}^{+} , \tag{6.11b}$$

one obtains in terms of eq.(C.1) of part I

$$\begin{aligned} \hat{S}|C> &= e^{\hat{C}}\Big[C^{*}(\Omega-2\hat{C}) - \Big(\sum_{\mu=-2}^{2}\tilde{C}_{\mu}^{*}C_{\mu}^{*} - C^{*2}\Big)\hat{S}^{+}\Big]|core> = \\ &= \Big[C^{*}(\Omega-2C\partial^{*}-2\sum_{\mu=-2}^{2}C_{\mu}\partial_{\mu}^{*}) - \Big(\sum_{\mu=-2}^{2}\tilde{C}_{\mu}C_{\mu}^{*} - C^{*2}\Big)\partial^{*}\Big]|C> , \end{aligned} \tag{6.12}$$

where the abbreviation $\tilde{C}_{\mu} = (-1)^{\mu}C_{-\mu}$ was used. After similar considerations for the generators $\hat{D}_{\mu}$ , $\hat{S}_{o}$ and $\hat{P}_{\mu}^{r}$ the Dyson-type boson expansion can be expressed in the form

$$\hat{S} \leftrightarrow s \ , \ \hat{S}^{+} \leftrightarrow s^{\dagger}(\Omega-2\hat{N}_{B}) - \Big(\sum_{\mu=-2}^{2}\tilde{d}_{\mu}^{\dagger}d_{\mu}^{\dagger} - s^{\dagger}s^{\dagger}\Big)s \ , \tag{6.12ab}$$

$$\hat{D}_{\mu} \leftrightarrow d_{\mu} \ , \ \hat{D}_{\mu}^{+} \leftrightarrow d_{\mu}^{\dagger}(\Omega-2\hat{N}_{B}) + \Big(\sum_{\mu=-2}^{2}\tilde{d}_{\mu}^{\dagger}d_{\mu}^{\dagger} - s^{\dagger}s^{\dagger}\Big)\tilde{d}_{\mu} \ , \tag{6.12cd}$$

$$\hat{S}_{o} \leftrightarrow -\tfrac{1}{2}(\Omega-2\hat{N}_{B}) \ , \tag{6.12e}$$

$$\hat{P}_{\mu}^{r} \leftrightarrow 2\delta_{r,2}(s^{\dagger}\tilde{d}_{\mu}+d_{\mu}^{\dagger}s) + 20\begin{Bmatrix} 2 & 2 & r \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\end{Bmatrix}[d^{\dagger}\tilde{d}]_{\mu}^{r} \ , \ r=1,2,3, \tag{6.12f}$$

where $\tilde{d}_{\mu} = (-1)^{\mu}d_{-\mu}$ and $\hat{N}_{B}$ is the boson number operator

$$\hat{N}_{B} = s^{\dagger}s + \sum_{\mu=-2}^{2} d_{\mu}^{\dagger}d_{\mu} \ . \tag{6.13}$$

In eqs.(6.12bd) one should note the appearance of the boson pair creation operator $\sum_{\mu}\tilde{d}_{\mu}^{\dagger}d_{\mu}^{\dagger} - s^{\dagger}s^{\dagger}$ corresponding to the similar expression in the weight function w(C), eq.(6.7).

The above expansion is identical (up to an unimportant factor) to that derived by Geyer and Hahne [20]. They used the generalized Dyson expansion (sect.3) for the generators of the so(8) algebra, which requires the introduction of collective as well as non-collective bosons, and then truncated the boson space by retaining only its collective part. Our derivation is based on the properties of the so(8) sublagebra only and does not makes use of the non-collective bosons in the intermediate steps. The equivalence of both methods hinges on the fact that the so(8) generators $\hat{E}_{-\underset{\sim}{\alpha}}$, eq.(6.1), for $\underset{\sim}{\alpha}\cdot\underset{\sim}{j} > 0$ and $\underset{\sim}{j}$ given by eq.(6.3a) are at the same time the generators $\hat{E}_{-\underset{\sim}{\alpha}}$ of so(2N), eq.(3.1), for $\underset{\sim}{\alpha}\cdot\underset{\sim}{j} > 0$ and $\underset{\sim}{j}$ being the spinor highest weight, eq.(3.2). Hence the so(8) generalized coherent states, eq.(6.4), for the discussed representation can be obtained from those of so(2N), eq.(3.4), by setting some complex variables $C_{\mu\nu}$ equal to zero and expressing C and $C_\mu$ by the others. On the level of boson expansion this corresponds to removing the non-collective bosons.

## 7. ALGEBRA su(l+1)

Suppose that one can find $(l+1)^2$ linear combinations of of fermion and bifermion operators which fulfill the commutation relations of the unitary, $\mathcal{A} = u(l+1)$, algebra :

$$[\hat{B}_{ij}, \hat{B}_{mn}] = \delta_{jm}\hat{B}_{in} - \delta_{in}\hat{B}_{mj} , \tag{7.1}$$
$$\hat{B}^{+}_{ij} = \hat{B}_{ji} , \qquad i,j,m,n = 0,1,\ldots,l .$$

Such a supposition is essential for the interacting boson model[18) which makes use of u(6) algebra (l=5) expressed explicitly by Schwinger representation in terms of six boson operators $s^\dagger$ and $d^\dagger_\mu$, $\mu = -2, \ldots, 2$. One can relate the index i = 0 of eq.(7.1) to the monopole $s^\dagger$ boson and $i > 0$ to the other multipolarities.

Similarly as in sect.4, the linear Casimir operator

$$\hat{B} = \sum_{i=0}^{l} \hat{B}_{ii} \tag{7.2}$$

will be kept throughout with the convention allowing the addition of the vector

$$\underset{\sim}{1} = \sum_{i=0}^{l} \underset{\sim}{w}_i \tag{7.3}$$

to every weight vector with an arbitrary factor. The Cartan subalgebra is given by

$$\hat{H}_i = \hat{B}_{ii} \tag{7.4}$$

and the correspondence between $\hat{E}_{\underset{\sim}{\alpha}}$ generators and roots $\underset{\sim}{\alpha}$ reads

$$\hat{E}_{\underset{\sim}{\alpha}} = \hat{B}_{ij} , \quad i \neq j , \qquad \text{for } \underset{\sim}{\alpha} = \underset{\sim}{w}_i - \underset{\sim}{w}_j . \tag{7.5}$$

Examining the spectra of $\hat{B}_i$ operators diagonalized in the full fermion space one could pick out all allowed representations. Let us assume that the symmetric representations with the highest weights

$$\underset{\sim}{j} = N \underset{\sim}{w}_0 = (N, 0, \ldots, 0) \tag{7.6}$$

are the allowed ones. Then the positive and nonorthogonal to $\underset{\sim}{j}$ roots, $\underset{\sim}{w}_0 - \underset{\sim}{w}_i$, $i = 1,2,\ldots,l$, determine the generalized coherent states

$$|C\rangle = \exp\left\{ \sum_{i=1}^{l} C_i^* \hat{B}_{i0} \right\} |\underset{\sim}{j}\rangle , \tag{7.7}$$

and thus the boson expansion involves $l$ different bosons, $b_i^\dagger \leftrightarrow C_i$ ; e.g. five bosons for the SU(6) group of the interacting boson model[17)]

The norm of $|C\rangle$ reads

$$\langle C|C\rangle = \left(1 + \sum_{i=1}^{l} |C_i|^2\right)^N \tag{7.8}$$

and after calculating the specific weight $\tilde{\underset{\sim}{j}}$ ,

$$\tilde{\underset{\sim}{j}} = \sum_{i=1}^{l} (\underset{\sim}{w}_0 - \underset{\sim}{w}_i) = (l+1)\underset{\sim}{w}_0 - \underset{\sim}{1} \cong \frac{l+1}{N} \underset{\sim}{j} , \tag{7.9}$$

one obtains the weight function $w(C)$

$$w(C) = W \left(1 + \sum_{i=1}^{l} |C_i|^2\right)^{-N-l-1} , \tag{7.10}$$

$$W = \pi^{-l} (N+l)!/N! \tag{7.11}$$

The boson expansion of the generators $\hat{B}_{ij}$ has the form :

$$\begin{aligned} \hat{B}_{00} &\leftrightarrow N - \hat{N}_B , & & \\ \hat{B}_{0i} &\leftrightarrow b_i & \text{for } & i>0, \\ \hat{B}_{i0} &\leftrightarrow b_i^\dagger (N - \hat{N}_B) & \text{for } & i>0, \\ \hat{B}_{ij} &\leftrightarrow b_i^\dagger b_j & \text{for } & i,j>0, \end{aligned} \tag{7.12}$$

where the total boson number operator $\hat{N}_B$ is given by

$$\hat{N}_B = \sum_{i=1}^{l} b_i^\dagger b_i \quad . \tag{7.13}$$

In order to transform Dyson-type expansion (7.12) into the Holstein-Primakoff type the normalizing operators will be found by a method alternative to that used in the previous sections. It is easy to check that all holomorphic functions normalizable with the weight $w(C)$, eq.(7.10), must be polynomials of $C_1, \dots, C_l$ of the order not greater than N. Hence the physical boson space $S_B$ is represented by functions

$$f(C) = (n_1! \dots n_l!)^{-\frac{1}{2}} C_1^{n_1} \cdot \dots \cdot C_l^{n_l} , \tag{7.14}$$

$$N_B = n_1 + \dots + n_l \leq N$$

which form an orthonormal basis (in the boson sense). These functions are also orthogonal in the fermion sense while their fermion norms obtained by direct integration read

$$< f | f > = \int dC \, w(C) \, |f|^2 = \frac{(N - N_B)!}{N!} . \tag{7.15}$$

Thus the orthonormalizing operators are given by

$$\hat{G}_B = \hat{F}\left\{\left[(N-n)!/N!\right]^{\frac{1}{2}}\right\} \quad , \quad \hat{G}_F = \left\{\left[N!/(N-n)!\right]^{\frac{1}{2}}\right\} \tag{7.16ab}$$

for the $\hat{F}$ operators calculated with the use of eqs.(2.32-33) and (7.13) for the maximal boson number $m = N$. Using formulae similar to eqs.(2.39) one obtains the finite and hermitian Holstein-Primakoff-type expansion :

$$\begin{aligned} \hat{B}_{00} &\leftrightarrow N - \hat{N}_B \ , \\ \hat{B}_{0i} &\leftrightarrow \hat{F}\left\{(N-n)^{\frac{1}{2}}\right\} b_i \qquad \text{for } i > 0 \ , \\ \hat{B}_{i0} &\leftrightarrow b_i^{\dagger} \hat{F}\left\{(N-n)^{\frac{1}{2}}\right\} \qquad \text{for } i > 0 \ , \\ \hat{B}_{ij} &\leftrightarrow b_i^{\dagger} b_j \qquad \text{for } i,j > 0 . \end{aligned} \tag{7.17}$$

If in place of the $\hat{F}$ operators the infinite expansion

$$\sqrt{N - \hat{N}_B} = \sum_{k=0}^{\infty} \tilde{Y}_k^{(N)} \hat{N}_B^k \ , \tag{7.18}$$

$$\tilde{Y}_0^{(N)} = \sqrt{N} \ , \quad \tilde{Y}_k^{(N)} = -\frac{(2k-3)!!}{k!\,(2N)^k}\sqrt{N} \quad \text{for } k > 0$$

is inserted, then expansion (7.17) becomes infinite and identical (for $l = 5$, i.e. for SU(6) group) to that proposed by Janssen et al.[21]. Evidently, the series in eq.(7.18) diverges for $N_B > N$.

In view of the expression for $\langle c|c\rangle$, eq.(7.8), the $\hat{R}$ projection operator ( the boson counterpart of the differential operator $\mathcal{R}$ defined in part II) reads

$$\hat{R} = :(1+\hat{N}_B)^N \exp\{-\hat{N}_B\}: , \qquad (7.19)$$

or in terms of boson number projection operators, eq.(2.29) :

$$\hat{R} = \sum_{n=0}^{N} N!/(N-n)!\,\hat{P}_n \quad . \qquad (7.20)$$

In this way the operator $\hat{P}$ projecting the boson states onto the physical space can be calculated in terms of $\hat{G}_B$, eq.(7.16a),

$$\hat{P} = \hat{G}_B^2\,\hat{R} = \sum_{n=0}^{N} \hat{P}_n \quad , \qquad (7.21)$$

which is consistent with the previous observation concerning the normalizability of polynomials with the weight w(C) of eq.(7.10). The Garbaczewski-type expansion (part II) is obtained by multiplying expansion (7.17) by $\hat{P}$ from the right-hand side ,

$$\begin{aligned}
\hat{B}_{00} &\leftrightarrow (N-\hat{N}_B)\sum_{n=0}^{N}\hat{P}_n \quad , && \\
\hat{B}_{0i} &\leftrightarrow \Big[\sum_{n=0}^{N}(N-n)^{\frac{1}{2}}\hat{P}_n\Big]\, b_i && \text{for } i>0, \\
\hat{B}_{i0} &\leftrightarrow b_i^\dagger\Big[\sum_{n=0}^{N}(N-n)^{\frac{1}{2}}\hat{P}_n\Big] && \text{for } i>0, \\
\hat{B}_{ij} &\leftrightarrow b_i^\dagger b_j\sum_{n=0}^{N}\hat{P}_n && \text{for } i,j>0
\end{aligned} \qquad (7.22)$$

with the boson expansion for $\hat{P}_n$ as in eq.(2.29).

Boson expansions for spin operators $\hat{J}_0$, $\hat{J}_\pm$ can be obtained from eqs.(7.12), (7.17) and (7.22) by making use of the isomorphism su(2) $\cong$ so(3).

## 8. ALGEBRA su(3)

In this section the unitary algebra of sect.7 is studied in more detail for $l = 2$, while the allowed representations are not restricted to the symmetric ones. The well-known example of the su(3) algebra consturcted in terms of bifermion operators is given by the Elliott model [22].

The two fundamental weights of SU(3) are [2-4]

$$\underset{\sim}{m}_1 = (1,0,0) \qquad , \qquad \underset{\sim}{m}_2 = (1,1,0) \tag{8.1}$$

and thus the highest weights are determined by two integers, usually denoted by $\lambda$ and $\mu$ :

$$\underset{\sim}{j} = \lambda \underset{\sim}{m}_1 + \mu \underset{\sim}{m}_2 = (\lambda+\mu, \mu, 0) . \tag{8.2}$$

Depending on whether $\lambda$ or $\mu$ or both are nonzero integers the highest weight is related to one of the three specific weights :

$$\tilde{\underset{\sim}{j}} = 2\underset{\sim}{m}_1 + 2\underset{\sim}{m}_2 = (4,2,0) \qquad \text{for } \lambda \neq 0, \mu \neq 0, \tag{8.3a}$$
$$\tilde{\underset{\sim}{j}} = 3\underset{\sim}{m}_1 \qquad = (3,0,0) \qquad \text{for } \lambda \neq 0, \mu = 0, \tag{8.3b}$$
$$\tilde{\underset{\sim}{j}} = 3\underset{\sim}{m}_2 \qquad = (3,3,0) \qquad \text{for } \lambda = 0, \mu \neq 0. \tag{8.3c}$$

Three types of boson expansions are thus possible for su(3) each of them being valid for the class of representations related to the common specific representation.

For $\lambda \neq 0$, $\mu \neq 0$ all roots are nonorthogonal to $\underset{\sim}{j}$ and the generalized coherent state reads

$$|C\rangle = \exp\{C_0^* \hat{B}_{21} + C_1^* \hat{B}_{10} + C_2^* \hat{B}_{20}\} |\underset{\sim}{j}\rangle , \tag{8.4}$$

where the notation of sect. 7 is preserved for the $\hat{B}_{ij}$ generators. Hence the boson expansion obtained makes use of three different boson operators $b_0^\dagger$, $b_1^\dagger$, $b_2^\dagger$ related to three complex numbers $C_0$, $C_1$, $C_2$. The norm of the state $|C\rangle$ can be obtained from eq. (A.12) of part I in terms of overlaps calculated for the two fundamental representations. Both of them have only three dimensions and after a simple derivation one obtains the weight function

$$w(C)=W\left(1+|C_1|^2+|C_2+\tfrac{1}{2}C_0C_1|^2\right)^{-\lambda-2}\left(1+|C_0|^2+|C_2-\tfrac{1}{2}C_0C_1|^2\right)^{-\mu-2}, \tag{8.5}$$

$$W=\pi^{-3}(\lambda+1)(\mu+1)(\lambda+\mu+2)\quad, \tag{8.6}$$

where eq. (8.3a) was used. The boson expansion for the $\hat{B}_{ij}$ generators reads

$$\hat{B}_{00}\leftrightarrow\lambda+\mu-b_1^\dagger b_1-b_2^\dagger b_2\quad,\quad\hat{B}_{11}\leftrightarrow\mu+b_1^\dagger b_1-b_0^\dagger b_0\ , \tag{8.7}$$

$$\hat{B}_{22}\leftrightarrow b_2^\dagger b_2+b_0^\dagger b_0\quad,$$

$$\hat{B}_{02}\leftrightarrow b_2\ ,\quad\hat{B}_{01}\leftrightarrow b_1-\tfrac{1}{2}b_0^\dagger b_2\ ,\quad\hat{B}_{12}\leftrightarrow b_0+\tfrac{1}{2}b_1^\dagger b_2\ ,$$

$$\hat{B}_{20}\leftrightarrow(\lambda+\mu)b_2^\dagger+(\lambda-\mu)\tfrac{1}{2}b_0^\dagger b_1^\dagger-\left(b_2^\dagger+\tfrac{1}{2}b_0^\dagger b_1^\dagger\right)b_1^\dagger b_1+$$

$$-\left(b_2^\dagger-\tfrac{1}{2}b_0^\dagger b_1^\dagger\right)b_0^\dagger b_0-\left(b_2^{\dagger 2}+\tfrac{1}{4}b_0^{\dagger 2}b_1^{\dagger 2}\right)b_2\quad,$$

$$\hat{B}_{10}\leftrightarrow\lambda b_1^\dagger-\left(b_2^\dagger-\tfrac{1}{2}b_0^\dagger b_1^\dagger\right)b_0-\tfrac{1}{2}\left(b_2^\dagger+\tfrac{1}{2}b_0^\dagger b_1^\dagger\right)b_1^\dagger b_2-b_1^{\dagger 2}b_1\ ,$$

$$\hat{B}_{21}\leftrightarrow\mu b_0^\dagger+\left(b_2^\dagger+\tfrac{1}{2}b_0^\dagger b_1^\dagger\right)b_1-\tfrac{1}{2}\left(b_2^\dagger-\tfrac{1}{2}b_0^\dagger b_1^\dagger\right)b_0^\dagger b_2-b_0^{\dagger 2}b_0\ .$$

The boson images of the $\hat{B}_{ij}$ generators contain the single-boson terms which annihilate and create bosons for $i<j$ and $i>j$, respectively. The complicated additional multi-boson terms ensure the proper symmetry conditions.

For $\lambda \neq 0$, $\mu = 0$ the positive root $\underset{\sim}{\alpha} = \underset{\sim}{w}_1 - \underset{\sim}{w}_2$ becomes orthogonal to the highest weight $\underset{\sim}{j}$ and thus the generator $\hat{B}_{21}$ does not enter the generalized coherent state, eq.(8.4), any more. This can be formally achieved by setting $C_0$ complex variable equal to zero. Consequently, the boson expansion in the present case can be obtained by setting

$$b_0^\dagger \equiv 0 \quad , \quad b_0 \equiv \frac{1}{2} b_1^\dagger b_2 \qquad (8.8ab)$$

in eqs.(8.7). Condition (8.8b) results from the fact that $C_0$ should be set equal to zero after all differentiations $b_0 \leftrightarrow \partial/\partial C_0$ are completed.

Evidently, as $\lambda \neq 0, \mu = 0$ represents the symmetric representations, the resulting boson expansion is identical to that of sect. 7 for $l = 2$. The weight functions $w(C)$ cannot be obtained from that of eq.(8.5) by setting $C_0=0$. This is so because by removing the $b_0^\dagger$ boson one removes the integration over $C_0$ from the scalar product in the functional space. The standard way of determining $w(C)$ in terms of $\underset{\sim}{\tilde{j}}$, eq.(8.3b), gives again the result already presented in sect.7.

For $\lambda = 0, \mu \neq 0$ the root $\underset{\sim}{\alpha} = \underset{\sim}{w}_0 - \underset{\sim}{w}_1$ is orthogonal to $\underset{\sim}{j}$ and the above considerations can be repeated in order to remove the $b_1^\dagger$ boson from eqs.(8.7) by setting

$$b_1^\dagger \equiv 0 \quad , \quad b_1 \equiv \tfrac{1}{2} b_0^\dagger b_2 \ . \tag{8.9}$$

After changing the numbering of complex numbers $C_0' = C_2$, $C_1' = C_0$, and that of the corresponding boson operators, the boson expansion can be presented as

$$\hat{B}_{22} \leftrightarrow \hat{N}_B \ ; \tag{8.10}$$

$$\hat{B}_{i2} \leftrightarrow b_i' \qquad \text{for} \quad i<2 \ ,$$

$$\hat{B}_{2i} \leftrightarrow b_i'^\dagger(\mu - \hat{N}_B) \qquad \text{for} \quad i<2 \ ,$$

$$\hat{B}_{ij} \leftrightarrow \mu\delta_{ij} - b_j'^\dagger b_i' \qquad \text{for} \quad i,j<2$$

for

$$\hat{N}_B = b_0'^\dagger b_0' + b_1'^\dagger b_1' \ , \tag{8.11}$$

while the weight function reads

$$w(C) = \bar{W}\left(1 + |C_0'|^2 + |C_1'|^2\right)^{-\mu-3} , \tag{8.12}$$

$$\bar{W} = \pi^{-2}(\mu+1)(\mu+2) \ . \tag{8.13}$$

## 9. CONCLUSION

It was shown how the most previously known and also some new boson expansions can be derived by using a general method presented in the first two parts of this work. Clearly, besides the applications presented here for the Lie groups SO(2N+1), SO(2N), SU(N), Sp(N), SU(1+1), SO(8) and SU(3) many others can be deduced for different choices of the underlying group. Thus every fermion problem can be transposed into a boson space provided the corresponding fermion space is an irreducible representation space for some compact semisimple Lie group.

As indicated by Onofri[23)], the holomorphic functional representations, and hence the boson representations, exist for noncompact semisimple or solvable Lie groups as well. If the corresponding invariant measures were explicitly known it would have been possible to construct the boson expansion for a wider class of fermion spaces. In the recent paper by Suzuki[13)] the invariant measures were derived without assuming semisimplicity of the Lie algebra while strongly and explicitly restricting the commutation relations. Hence many semisimple algebras and their representations cannot be dealt with his method. On the other hand it includes the non-semisimple Heisenberg-Weyl algebra of boson operators. The finding of an extended approach for obtaining invariant measures appears then as a natural goal for future investigations.

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