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Abstract

~~We have found~~ A simple relation between the Classical Wilson Loop and the angular parallel displacement along a closed path (circle) for the case of the instanton-like potentials

Encontramos uma relação simples entre o "loop" de Wilson clássico e o deslocamento paralelo angular ao longo de curvas fechadas (círculos) para potenciais tipo instanton.

Lately, the Wilson loop[1,2] has aroused considerable interest in Gauge Theories because it is a gauge invariant quantity and it is thought that it can act as a dynamical variable containing all information about the Gauge Fields[3,7].

In a recent paper, Bollini, Giambiagi and Tiomno[8] computed the classical Wilson loop for the instanton potential by using the spinor representation of the rotation group. In the first part of this paper we repeat that calculation for the instanton-like potential using the vector representation of the rotation group. This calculation is justified by the fact that in this case we have real rotations and this allows us to bear out the effect of the parallel transport operator. in the internal space, for instanton-like potentials, on vectors that have vanishing temporal components.

The motivation for these calculations is that they can help us to understand an eventual classical theory based on Wilson loops.

Now, let us proceed to the computation of the classical Wilson loop in the vector representation of the rotation group.

Consider the instanton-like potentials

1)
$$A_\mu = ian_{\mu\nu} \frac{X_\nu - X^0_\nu}{X-X^0)^2 + \lambda}$$

For each α (real number) we have a instanton-like potential. For $\alpha=1$, e.g., we have the instanton in the vector representation.

The matrices $\eta_{\mu\nu}$ satisfies the following relations,

2)
$$\eta_{ij} = \epsilon_{ijk}\eta_k \quad \text{and} \quad \eta_{i4} = \eta_i$$

with

3)
$$\eta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} ; \quad \eta_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} ; \quad \eta_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

In the specific case of instantons-like potentials with the center at $X_1^0 = X_2^0 = 0$, we have explicitly for X on the (X_1, X_2) - plane,

4)
$$A_\mu = i\alpha \, \frac{\eta_{\mu 1} X_1 + \eta_{\mu 2} X_2 - \eta_{\mu 3} X_3^0 - \eta_{\mu 4} X_4^0}{X^2 + \lambda^2}$$

5)
$$X^2 = X_1^2 + X_2^2 + (X_3^0)^2 + (X_4^0)^2$$

If we consider now polar coordinates in the (X_1, X_2) - plane and use (2), we find

6)
$$A_\mu dx^\mu = B(0,\hbar) \, d\theta$$

where

7)
$$B(0,\hbar) = -iA(0,\hbar) = -\frac{\alpha\hbar^2}{X^2+\lambda^2} \left[\eta_1 (cos\theta \, \frac{X_3^0}{\hbar} - sen\theta \, \frac{X_4^0}{\hbar}) + \eta_2 (sen\theta \, \frac{X_3^0}{\hbar} + cos\theta \, \frac{X_4^0}{\hbar}) + \eta_3 \right] .$$

In analogy with Bollini, Giambiagi and Tiomno[8], we

consider an angle interval $\theta_i \leq \theta \leq \theta_b$, **split** it into N equal parts $(\theta_b - \theta_i)/N$ and **use the relation**

8) $\quad exp\left[iB(\theta)d\theta\right] = exp\left(-i\theta n_3\right) exp\left[iB(0)d\theta\right] exp\left(i0n_3\right)$

to express the value of $exp\left[iB(\theta)d\theta\right]$ **at the** corresponding points $\theta_n = \frac{n}{N}(\theta_b - \theta_i)$ $(n = 1, 2, \ldots, N)$. **Then**, taking the limit $N \to \infty$, the Wilson operator

9) $\qquad W_{\theta_b \theta_i}(\hbar) = P \, exp\left[\int_{\theta_i}^{\theta_b} B(\theta) \, d\theta\right]$

is given by

10) $\quad W_{\theta_b \theta_i}(\hbar) = exp(-i\theta_b n_3) \, exp\left[i(B(0) + n_3)(\theta_b - \theta_i)\right] \, exp(i\theta_i n_3).$

By making use of the relation

11) $\qquad\qquad n_i^3 = n_i \quad (i = 1, 2, 3) \qquad$ (Cf. eq.(3))

we find

$\quad W_{\theta_b \theta_i}(\hbar) = exp(-i\theta_b n_3)\left[1+M \, sen(\theta_b - \theta_i)L + M^2(cos(\theta_b - \theta_i)L - 1)\right] exp(i\theta_i n_3),$

where

13) $\quad L^2 = B_1^2 + B_2^2 + (B_3 + 1)^2 \, , \, B(0) = \vec{B}(0).\vec{n}$

and

14) $\quad M = \vec{M}.\vec{n} \, , \, M_1 = \dfrac{B_1}{L} \, , \, M_2 = \dfrac{B_2}{L} \, , \, M_3 = \dfrac{B_3 + 1}{L}$

Eq. (12) represents the value of the "integral" (9) for any arc of radius $\hbar$ with $\theta_i \leq \theta \leq \theta_b$.

In particular, for a complete circle $(\theta_i = 0$ and $\theta_f = 2\pi)$, Eq. (12) reduces to

15) $\qquad W_{2\pi,0}(\hbar) = 1 + iM \, sen(2\pi L) + M^2 \left[\cos(2\pi L) - 1 \right]$.

If we take the trace of Eq. (15) we get the corresponding Wilson Loop, namely,

16) $\qquad W(\hbar) = Tr W_{2\pi,0}(\hbar) = 1 + 2\cos(2\pi L)$.

Let us now parallel displace three vectors and calculate the cossines of the angles between them and the displaced ones along circles.

Consider the following normalized vectors.

17) $\qquad \phi_{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $\phi_{(2)} = \begin{pmatrix} 0 \\ \cdot \\ 0 \end{pmatrix}$; $\phi_{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

If we parallel displace each of the vectors along a circle, the final vectors are given by

18) $\qquad \phi'_{(i)} = W_{2\pi,0} \, (i)$

where

19) $\qquad W_{2\pi,0} = P \, exp \left(\int_0^{2\pi} B(\theta) d\theta \right)$.

is the parallel displacement operator[9] (along circles).

As the vectors $\phi_{(i)}$ and $\phi'_{(i)}$ have vanishing temporal components, we can define real angles[9] $\alpha_{(i)}$ by means of

20) $$\cos \alpha_{(i)} = \tilde{\phi}_{(i)} \, W_{2\pi,0} \, \phi_{(i)}$$

In Eq. (20) we have used the facts that $\phi_{(i)}$ and $\phi_{(i)}$ are normalized to one and that the metric has signature $(+,+,+)$.

In our notation $\tilde{\phi}_{(i)}$ is the transpose of $\phi_{(i)}$.

By using Eqs. (15) and (20) we get

21) $$\cos \alpha_{(i)} = \tilde{\phi}_{(i)} \left[1 + iM \, \text{sen}(2\pi\mu) + M^2 \, (\cos(2\pi\mu) - 1) \right] \phi_{(i)}$$

A calculation of $\cos \alpha_{(i)}$ $(i = 1,2,3)$ gives

22) $$\cos \alpha_{(1)} = 1 + \frac{1}{L^2} \left[B_2^2 + (B_3 + 1)^2 \right] (\cos(2\pi\mu) - 1)$$

23) $$\cos \alpha_{(2)} = 1 + \frac{1}{L^2} \left[B_1^2 + (B_3 + 1)^2 \right] (\cos(2\pi\mu) - 1)$$

24) $$\cos \alpha_{(3)} = 1 + \frac{1}{L^2} (B_1^2 + B_2^2)(\cos 2\pi\mu - 1)$$

Adding (22), (23) and (24) we find

25) $$\sum_{i=1}^{3} \cos \alpha_{(i)} = 1 + \cos(2\pi\mu)$$

Now, comparing (25) and (16), we conclude that

26) $$W = Tr \, W_{2\pi,0} = \sum_{i=1}^{3} \cos \alpha_{(i)}$$

Eq. (26) is the announced relation between the classical Wilson Loop and the angles of parallel displacement.

Notice that Eq. (26) is valid independently of the value of a, i.e., for the potentials of the instanton, meron and general instanton-like potentials. Observe, however, that the values of the angles are dependent on a. We know that the values

of these angles are a measure of the mean curvature. Thus the field of the instanton, for example, "bends" space in a way different from that of the field of merons and other instanton-like configurations.

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