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GRAND UNIFICATION AND GROUP THEORY : THE HIGGS PROBLEM

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0. INTRODUCTION

When Maxwell succeeded to unify electricity and magnetism he presented a set of differential equations. One century later, when trying to unify fundamental interactions physicists are questing for a gauge group. The success of renormalizable gauge theories provided by $SU(2) \times U(1)$ and $SU(3)$ in describing electroweak and strong interactions respectively is an encouragement to look further and try larger unification, including ultimately gravitation. Gauge theories give a central role to group theory since the group not only classifies the particles but also fixes their interactions. In this talk we will try to examine how deeply group theory is involved in building grand unified theories (GUT's) which embody electroweak and strong interactions. We leave aside the SuperGUT's which aim at unification with gravity[1].

After a brief survey of grand unification theories we shall concentrate on a particular aspect: the Higgs problem. Since the only renormalizable gauge theories we know of are those where the symmetry is spontaneously broken we need to introduce Higgs scalar particles in the theory, be they elementary or composite. Because of the many free parameters appearing in the Higgs Lagrangian (and some bad renormalization properties) fundamental Higgses are disliked and many physicists prefer to see them as composite states of some new gauge interaction. This approach is termed as dynamical symmetry breaking, technicolour,...[2], and will not be touched upon because of the lack of space-time. Anyway, symmetry breaking is achieved using scalar multiplets added to the fermionic matter multiplets and it is this aspect we want to focuse on.

Section II will be devoted to the physical implication of the choice of Higgs representations for particle masses (charged fermions and neutrinos). We shall say a few words on two other items: the strong CP problem and the hierarchy puzzle. In section III we shall consider the mathematical aspects of symmetry breaking having in mind the, so far unsuccessful, quest for a natural way of breaking symmetries with Higgses in a gauge theory.

I. GRAND UNIFIED THEORIES FOR NON SPECIALISTS[3]

Weak and electromagnetic interactions seem today very well described by the "unifying" gauge group $SU(2) \times U(1)$. Colliding protons and antiprotons at a center of mass energy of 540 GeV will confirm (or disable) very soon the existence of the predicted weak gauge bosons, $W^{\pm}$ and Z. In such a theory quarks and leptons are classified in doublets and singlets of $SU(2)$, the weak isospin group. Namely for the first fermion family with quarks u, d and leptons e^-, ν_e :

$$\begin{pmatrix} u \\ d \end{pmatrix}_L , \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \quad u_R , \quad d_R , \quad e^-_R$$

with the notation $f_{L,R} = \frac{1}{2}(1 \mp \gamma_5)f$.

The second family of fermions (c, s, μ, ν_μ) and the third one (including the so far unseen t quark (t, b, τ, ν_τ) are classified in a similar manner under this group.

The gauge group $SU(2) \times U(1)$ is spontaneously broken down to the electromagnetic group $U(1)_{em}$, generated by the generator $T_3 + Y$ if T_1, T_2, T_3 and Y are the generators of $SU(2)$ and $U(1)_Y$ respectively. This breaking is triggered by a doublet of Higgs scalars. To the remaining $U(1)_{em}$ symmetry is associated the massless photon while to the three broken generators are associated the massive gauge bosons $W^\pm$, Z. These particles are expected to get masses of order 80 to 90 GeV, given by the vacuum expectation value acquired by the Higgs doublet (2_H). The coupling constants of $SU(2)$ (g_2) and $U(1)$ (g_1) are related to the electric charge $e = g_1 g_2/\sqrt{g_1^2 + g_2^2}$ but are not truly unified as this theory contains a new "constant", the weak mixing angle θ_W : $tg\theta_W = g_1/g_2$.

Now strong interactions appear to be described by quantum chromodynamics (QCD) based on the gauge group of color $SU(3)_c$. Under this group each quark transforms as a triplet and each lepton as a singlet. Therefore, using a gauge theory based on the non-simple group:

$$SU(3)_c \times SU(2) \times U(1)$$

we can obtain a fair description of the interaction (but gravitation) of elementary particles. So far so good, but we have a direct product of 3 groups and then as many independent coupling constants; this is no unification. Unifying interactions with so different strengths (at low energies) is in fact possible due to the extraordinary property of asymptotic freedom. Coupling constants of non-abelian gauge theories decrease with increasing energy, at a rate fixed by the group and the particle multiplets we have. In the above case, it just happens that unification of the coupling constants, i.e.

$$g_1 = g_2 = g_3 = g_o$$

can be realized at an energy scale below the Planck mass which allows us not to worry about gravity in the scheme. In group theory language we look for a simple group G embedding $SU(3) \times SU(2) \times U(1)$ which can be a symmetry group for the Lagrangian of the unified theory. Of course such a group has to have suitable representations to classify quarks and leptons. An archetypal grand unified theory (GUT) is based on the SU(5) group[4], which is actually the smallest simple (compact) Lie group whose Lie algebra contains the Lie algebra of $SU(3) \times SU(2) \times U(1)$.[*] The particle states of each family are classified with 2 irreducible representations (IR) $\overline{5} + 10$.

$$\overline{5} = (d_1^c\ d_2^c\ d_3^c\ e^-\ \nu_e)_L \quad 10 = \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ & 0 & u_1^c & u_2 & d_2 \\ & & 0 & u_3 & d_3 \\ & & & 0 & e^+ \\ & & & & 0 \end{pmatrix}_L$$

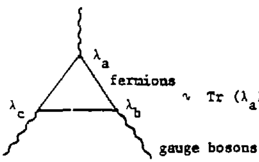
where 1,2,3 are the color indices, the subscript L means that all states are left handed fields and the superscript c refers to charge conjugation, defined as follows

$$(u^c)_L = \frac{1}{2}(1 - \gamma_5)\, i\gamma_2 u^* = i\gamma_2 u_R^*$$

When looking at the reduction of $\overline{5}$ and 10 under $SU(3) \times SU(2)$

$$\overline{5} = (\overline{3},1) + (1,2)\ ; \quad 10 = (\overline{3},1) + (3,2) + (1,1)$$

we check that we have all desired particles and no room for a right handed neutrino. In order to preserve renormalizability, anomalies proportional, for a given IR of fermions, to the third rank symmetric tensor (see below) have to vanish. For instance



$$\sim \ \mathrm{Tr}\,(\lambda_a\lambda_b\lambda_c + \lambda_a\lambda_c\lambda_b)$$

where the λ are the matrices coupling all fermions to gauge fields. The condition of anomaly freedom means that anomalies of the IR of fermions must add to zero a condition fulfilled by $\overline{5} + 10$.

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[*] Actually, the group SU(5) contains a subgroup $S[U(3) \times U(2)] = SU(3) \times SU(2) \times U(1) / Z_3 \times Z_2$ which has the same Lie algebra as $SU(3) \times SU(2) \times U(1)$. In the following, when considering subgroups of a group, we will forget in our notations the discrete part, as usually done in the related literature. This will be intrinsecally incorrect from a mathematical point of view, but physics will stay safe as long as problems like monopoles are not considered.

The gauge fields of this theory belong to the adjoint representation 24 whose $SU(3) \times SU(2)$ content is

$$24 = (3,2) + (\bar{3},2) + (8,1) + (1,3) + (1,1)$$

In addition to the gluons of QCD (8,1) and the electroweak force carreers $W^{\pm}$, Z, γ (1,3) + (1,1) we have 12 new gauge bosons $(3,2) + (\bar{3},2)$. These carry color and fractional electric charges ±4/3, ±1/3, they are named X(4/3), Y(1/3) and their antiparticles $\bar{X}$, $\bar{Y}$.

Using the renormalization group equations[3] (which govern the evolution rate of the coupling constants) it has been possible to evaluate rather precisely the scale M_{GU} at which unification occurs

$$M_{GU} \sim 8.3\Lambda \times 10^{14}\ \text{GeV} \quad (M_{Planck} \sim 10^{19}\ \text{GeV})$$

Λ being the QCD scale, $\Lambda \sim 0.1\text{-}1$ GeV that is $M_{GU} \sim 10^{14-15}$ GeV.

All this approach relies on the belief (somewhat criticized) that nothing happens between 10^2 GeV, the unification scale of $SU(2) \times U(1)$ and 10^{14} GeV: we have crossed the grand desert!

Another grand unified scheme is based on the group SO(10) [5] which contains U(5) as a subgroup and has some attractive features: all representations are anomaly free, property which is general for SO(n) groups $n > 6$, and the fermions can be accomodated into a single IR, the basic spinorial one, 16 dimensional. Its SU(5) content is

$$16 = 10 + \bar{5} + 1$$

If the singlet part is attributed to a possible (?) right handed neutrino ν_R we can generate naturally neutrino masses. As far as the fermion representations are concerned we can draw the following conclusion for a consistent scenario:

i) The representations of the fermions must be anomalies free in order to preserve renormalizability. These anomalies are only present in the complex representations of the SU(n) groups.

ii) The fermions must appear in complex representations and then the relevant groups are SU(n), SO(4n+2), E_6, E_7 and E_8. The reason for this requirement is to avoid the appearance of mass terms invariant under the group because the Higgs singlet which gives the mass would have a vacuum expectation value of the order of M_{GU}. This is the survival hypothesis[6]: the states which can get masses invariant under $SU(3) \times SU(2) \times U(1)$ become superheavy and do not survive in the seeable spectrum.

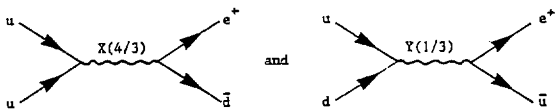
Now that we have plausible unification scenarios we would like to understand how the correct breaking can be made as we go down from M_{GU} to today energies. For this purpose we shall make extensive use of Higgs multiplets, which can also give fermions masses.

In the case of $G = SU(5)$ the first breaking $SU(5) \to SU(3) \times SU(2) \times U(1)$ can be made using a 24-plet of Higgses (24_H), in this occasion, the 12 gauge bosons $X, Y, \bar{X}, \bar{Y}$ outside of $SU(3) \times SU(2) \times U(1)$ acquire a mass of order M_{GU}. It is enough to add a 5_H to achieve the second breaking down to $SU(3) \times U(1)_{em}$ and give usual masses to $W^{\pm}$ and Z, while the gluons and the photons associated with unbroken generators remain massless. By the way, let us note that we dictated here that gauge bosons get considerably different masses $M_X/M_W \sim 10^{12}$, that is that the vacuum expectation values of the 24_H and the 5_H are in this ratio. This is far from natural and it is usually thought of as one major problem in this approach: the hierarchy problem (see next section).

The fact that quarks and leptons belong to the same representation of G has a dramatic consequence: the baryonic number (B) can be violated and the proton can decay. Indeed in the effective $SU(5)$ invariant Lagrangian we have a term

$$\bar{\psi}_L \gamma^\mu (\partial_\mu - ig\, T_i V^i_\mu)\, \psi_L + (L \leftrightarrow R)$$

where $\psi_{L,R}$ are left (right) handed spinor fields, T_i the generators of $SU(5)$ and V^i_μ the corresponding gauge fields. This generates the following diagrams



Adding a spectator quark line we just have the transition

$$p \to e^+ \pi^0 \;; \quad n \to e^+ \pi^-$$

Calculations of the proton lifetime τ_p indicate that[3),7)]

$$\tau_p = D^2 \left(\frac{M_X^4}{M_p^5} \right) = 8.10^{30\pm2} \text{ years}$$

where D^2 depends on the details of the decay model. This proton decay is actually one of the few direct tests of the scenario of grand unification and much work to see it is being done.

In the minimal SU(5) model (with $24_H \oplus 5_H$) the Lagrangian is invariant under a global U(1) symmetry which corresponds to the conservation of (B-L). In SO(10) model (B-L) is a generator of the group and when the breaking occurs, it is broken. Therefore we expect (beyond the tree level) to see proton decays which violate (B-L) if SO(10) or another version of SU(5) is the relevant theory.

From this general (and quick) outlook one realizes how Lie groups or more precisely Lie algebra techniques play an important role in the elaboration of GUT's. Also if such a scenario exists we have so far not unified the families: there is no mechanism which explains the threefold replication of $\overline{5} + 10$ in SU(5) and of 16 in SO(10). In words SO(10) and SU(5) appear to be not big enough. A related question is that, since quarks and leptons appear in the same representation, they might just be bound states of the same subconstituents, in the same way hadrons are made up of quarks[8)].

Attempts to solve the family problem do not yield, up to now, a satisfactory description of our world[9)]. From what has been achieved with SO(10) and SU(5), a natural direction is to try with the groups SU(2n+1) or SO(4n+2) $\supset$ SU(2n+1) $\times$ U(1) and use for the fermions the fundamental spinor representation - or its SU(2n+1) reduction - which is $2^{(2n-1)}$ dimensional and anomaly free. We can also consider the exceptional groups $E_8 \supset E_7 \supset E_6 \supset$ SO(10) $\supset$ SU(5), a plausibility argument being that the Dynkin diagrams of SU(5) and SO(10) appear in the same chain as those of E_6, E_7, E_8. The fact that this chain stops at E_9 can be an indication that grand unification must be solved (if ever) before or just with E_8 [10)].

Of course, as increasing the group G the knowledge of the subgroups is of first necessity[11)]. Depending on the choice of G, there are several chains of subgroups which lead down to SU(3) $\times$ $U(1)_{em}$. (In the rest of this talk subgroups are considered up to conjugation of G.) The explicit realizations of the representation of G are necessary as it is necessary to know the reduction of the Kronecker product of the representations of G [12)] to study the Lagrangian, or the reduction of a representation of G with respect to one of its subgroups[11)].

Another related interesting problem is the symmetry breaking using Higgs multiplets: what are the necessary representations in order to break G down to S and can we insure that we were indeed at a minimum of the potential? This last question raises a very difficult problem about which little is known in general and further study is certainly worth, as will be discussed in section III. A related physical question is the Higgs scalars problem. It does not seem possible to obtain "naturally" mass relations. The Higgs potential introduces a large number of parameters and one needs an incredibly accurate tuning of these to obtain $M_X/M_W \sim 10^{12}$ (hierarchy problem). Actually high energy physicists feel uncomfortable with Higgs scalars because of the inherent freedom in choosing the multiplets to realize the correct breaking and also some bad high energy behaviour. Scalar particle mass corrections have quadratic divergences whereas for fermions only logarithms occur. In general one tries to use the smallest possible multiplets (minimal $SU(5)$ for instance) but at some stage one needs more Higgses to obtain acceptable masses for the fermions. An alternative to elementary Higgses is to see them as bound states of a new kind of fermions with a new gauge interaction, whereby, a priori, their properties would be calculable. However this approach (Technicolor, Extended technicolor theories) do not appear to be conclusive up to now[2]. The attitude of model builders is then to consider Higgses as a necessary evil and to use as many multiplets as needed to make the model consistent. The hope is that ultimately nature will tell us that we were right.

We shall discuss, too briefly because of the lack of room, some physical aspects of the Higgs problem in the next section. The mathematical question will be touched upon in the third section, where is emphasized the necessity of thinking to a mathematical criterion which could select the Higgs multiplets for achieving our physical purposes.

II. PHYSICAL USE OF HIGGSES IN GUT'S

In this section we want to describe some topics which exemplify how Higgs field representations can be used to attack physical problems. Grand unified models are a giant step on the way to the unification of fundamental interactions but still they do not provide a satisfactory spectrum for the many fermions appearing, neither do they explain the family replication. As far as the fermion spectrum is concerned, it is well known that massless fermions acquire their masses through their coupling to the Higgs fields when these acquire non zero vacuum expectation values. Quite generally mass terms for the fermions can be written as

$$m\bar{\psi}\psi + m'\bar{\psi}\psi^c = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) + m'(\bar{\psi}_L\psi_L^c + \bar{\psi}_R\psi_R^c)$$

m is called the Dirac mass while m' is the Majorana mass. Indeed $\bar{\psi}\psi^c$ entails a fermion-antifermion transition which is often forbidden by quantum number conservation; however for Majorana spinors, $\psi = \psi^c$, which carry no quantum numbers such a mass term is perfectly respectable. In view of electric charge conservation Majorana mass terms can only be relevant for neutrinos provided lepton number is not conserved. Therefore we shall study separately the charged fermion masses (Dirac) and the neutrino masses (Majorana). To be complete we shall say a few words on two hot subjects: the axion problem and the hierarchy problem which seems to be "solved" in GUT's using supersymmetry[1].

Charged fermion masses

Dirac mass terms originate from the coupling $\lambda\bar{\psi}_R.H.\psi_L$ present in the invariant Lagrangian. It is then easy to single out the Higgs multiplets suitable to give fermions masses.

In the SU(5) model since

$$\bar{5} \times 10 = 5 + \overline{45} \; ; \quad 10 \times 10 = \bar{5} + 45 + 50$$

a 5_H and/or a 45_H of Higgs meet our requirements. Let us just take, for economy, the 5_H of SU(5). Then due to the SU(4) symmetry of its vacuum expectation value one obtains, at the grand unification scale M_X, the following mass relation .

$$m_e = m_d \; ; \quad m_\mu = m_s \; ; \quad m_\tau = m_b$$

These relations get modified by renormalization effects as we go down to lower energy, for instance

$$\frac{m_b(Q)}{m_\tau} = \left[\frac{\alpha_s(Q)}{\alpha_s(M_X)}\right]^{12/(33-2f)}$$

where $\alpha_s(Q)$ is the strong coupling constant at scale Q and f is the number of quark flavours. This f dependence is crucial: indeed with $f = 6$ (3 families) and $Q = 2m_b \sim m_Y$ one obtains $m_b \sim (5-5.5)$ GeV whereas $f > 6$ would increase m_b in disagreement with experimental observation. Therefore grand unification tells us at once that quarks are heavier than leptons and that 3 families is a favoured scheme. However this brilliant result is dulled by the bad, scale independent, relation

$$\frac{m_d}{m_s} = \frac{m_e}{m_\mu} = \frac{1}{200}$$

in violent disagreement with current algebra estimate $m_d/m_s \sim 1/20$. This failure may be indicative of post SU(5) interactions[13] or of a more complicated Higgs structure[14]. Indeed using a combination of 5_H and 45_H one obtains

$$3m_e = m_d \; ; \quad m_\mu = 3m_s \; ; \quad m_\tau = m_b$$

which respects the successful relation $m_\tau = m_b$ and gives $m_d/m_s = 9\, m_e/m_\mu \sim 1/20$ an acceptable ratio. But this clever solution is not very "natural" in the sense that the 45_H does not act in the same way on the 3 families.

In the case of SO(10) model where the fermions of one family occur into a single irreducible representation (16) we have to face with the same problem. Indeed

$$16 \times 16 = 10 + 126 + 120$$

which under SU(5) reduce to:

$$10 = 5 + \bar{5}$$
$$120 = 45 + \overline{45} + 10 + \overline{10} + 5 + \bar{5}$$
$$126 = 50 + 45 + \overline{15} + 10 + \bar{5} + 1$$

then using the 10 of Higgs one recovers the good (and the bad!) mass relations given by the 5_H in SU(5) model. The remedy is then to introduce 120_H and/or 126_H (which contain the 45_H of SU(5)!). By the way let us note that the 126_H also breaks the (B-L) generator of SO(10) which then allows Majorana mass terms for neutrinos if desired.

Another tantalizing problem is the "hierarchy" observed between the masses of the 3 fermion families. Even if we can obtain satisfactory mass relations in GUT's, we have no explanation of why 2 families appear light and a third one heavy. In fact the mass hierarchy suggests that masses of the different families are generated radiatively at different orders of the perturbation expansion. This is the $(\alpha^2, \alpha, 1)$ scheme[15] which means that:

the third family gets a tree level mass $(F_3) = (\tau, b, t)$

the second family gets a one loop level mass $(F_2) = (\mu, s, c)$

the first family gets a two loop level mass $(F_1) = (e, d, u)$.

This $(\alpha^2, \alpha, 1)$ can be achieved in a SU(5) model in which F_3 gets a direct mass by a 5_H and where the Yukawa Lagrangian possesses a global symmetry such that F_2 has only a one loop radiative mass and F_1 gets only a two loop mass. This necessitates the introduction of other Higgs multiplets like $50_H, 75_H$ and 10_H which may render the scheme unattractive, however in addition to the local SU(5) the Yukawa part of the Lagrangian exhibits a global $U(1)^4$ symmetry which protects the light fermions from getting masses and ensures the B-L conservation. Out of this global $U(1)^4$ symmetry after the breaking, emerges a remnant U(1) which plays the role of the Peccei-Quinn U(1) so useful for getting rid of strong CP violation (more on this item later!).

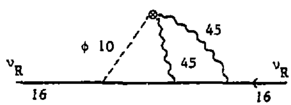
Neutrino masses

The masses of neutrinos are clearly very special. Indeed there is no experimental evidence for a right handed neutrino which excludes a Dirac mass term, and a Majorana mass term would rely on L violation. In the simplest SU(5) model the neutrino is expected to be massless, this is not true for more complicated GUT's which may contain ν_R's or Higgs multiplets with I = 1. In fact L conservation is not a dogma since no gauge principle is associated with it, so even in SU(5) one can imagine generating a ν mass.

The O(10) model contains a right-handed neutrino [the SU(5) singlet in the reduction $16 = 10 + \bar{5} + 1$] and then a Dirac mass of the same order as that of quarks and leptons can be generated. This can be avoided if the ν_R receives a large mass, following the "survival" hypothesis which states that the states which can get masses by Higgs singlets under the GUT group get superheavy. In this case we have a mass term for the neutrino

$$(\nu_L, \bar{\nu}_R) \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \begin{pmatrix} \nu_L \\ \bar{\nu}_R \end{pmatrix}$$

$M \geq 10^{15}$ GeV and m is a conventional Dirac mass, then the eigenvalues of the mass matrix are M and $m^2/M = m_{\nu_L}$ as wanted[9]. This mechanism can be actually realized using a 126-plet of Higgs (it contains an SU(5) singlet) but the vacuum expectation value is a free parameter. Witten[16] has shown that without introducing the 126_H, one can generate a two loop mass in a simple SO(10) model. In this model one has 3 multiplets of Higgses, a vector 10_H giving masses to the fermions (16_f), a spinor 16_H which breaks O(10) down to SU(5), and the adjoint 45_H which realizes the breaking of SU(5) to $SU(3) \times SU(2) \times U(1)$. Since ν_R can only get mass via an effective 126 interaction, we need to look to the simplest way to obtain a 126. Using 10_H, 16_f and 45_H one obtains $10_H \times 45_G \times 45_G \supset 126$ which corresponds to a two loop diagram, where 45_G stands for the 45-plet of gauge bosons of SO(10)



+ crossed.

The net outcome is that for each generation

$$m_{\nu_L} \sim 10^{-7}\, m_q \quad \text{that is}$$

$$m_{\nu_e} \sim 1 \text{ eV}, \quad m_{\nu_\mu} \sim 100 \text{ eV} \quad \text{and} \quad m_{\nu_\tau} \sim 1\text{-}10 \text{ KeV}.$$

These estimates are consistent with experimental bounds stating that

$$m_{\nu_e} < 35 \text{ eV}, \quad m_{\nu_\mu} \leq 500 \text{ KeV} \quad \text{and} \quad m_{\nu_\tau} < 200 \text{ MeV}.$$

However cosmological bounds are much more stringent and may imply to take as grand unification scale a larger scale than 10^{15} GeV as usually understood in all the calculations.

Hierarchy

As seen in the previous section GUT's seem to provide a very appealing scheme for the unification of all interactions in a natural way due to the evolution of the coupling "constants". However all the constructions are unable to explain why $M_X/M_W \sim 10^{12}$ in a natural way. Let us see on the simple SU(5) model how the problem comes in. In SU(5) the breaking of the symmetry is achieved by 2 Higgs multiplets 24_H and 5_H. The 24-plets breaks $SU(5) \to SU(3) \times SU(2) \times U(1)$ and gives a mass to the X and Y gauge bosons because they couple SU(3) and SU(2) indices. X and Y are very peculiar objects in that they mediate proton decay and in order to give a proton lifetime consistent with what is known they better be superheavy

$$M_{X,Y} \sim 10^{15} \text{ GeV}.$$

This means that the vacuum expectation value of the 24_H $v_{24} \sim 10^{15}$ GeV. On the other hand the 5-plet of Higgs makes the second step of breaking

$$SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1)_{em}$$

and gives a mass to the weak interaction bosons $W^{\pm}$, Z. This imposes that $v_5 \sim 10^2$ GeV an extremely small number as compared with the scale of unification. The situation is even worse if one considers that radiative corrections will couple "effectively" the 24_H and the 5_H, and the only way to prevent the 5_H to get enormous contributions at this level is to impose "unnaturally" that some combination of parameters in the potential is of order $\mathcal{O}(10^{-24})$. Of course such a miracle has to occur at each order of the perturbation expansion. A possible way out is to introduce supersymmetry (SUSY) [17] in the scheme, a symmetry which relates bosons and fermions. In this way quadratic divergences of the scalars may disappear and only logarithmic ones (like for fermions) survive; one also protects scalars from getting huge masses by the chiral symmetries of their fermionic partners until SUSY is broken. Experimental observations do not exclude a breaking scale of 10^{2-3} GeV,this is comfortable to obtain $M_W \sim 10^2$ GeV. Of course new generation accelerators will be able to see if supersymmetry is present at such a low energy.

Strong CP and the Axion[18]

This subsection is intended to say a few words on a question which has been revived recently and in which GUT's seem to give a natural explanation provided one uses "Higgs tricks".

In quantum chromodynamics instanton solutions generate an extra term $\mathcal{L}_\theta$ in the Lagrangian which violates P and CP.

$$\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} \frac{1}{2} \vec{G}^{\mu\nu} \epsilon_{\mu\nu\rho\sigma} \vec{G}^{\rho\sigma}$$

where $\vec{G}^{\mu\nu}$ is the non-abelian gluon field tensor and θ is a free parameter. The electric dipole moment of the neutron gives a bound on CP violation by strong interactions, which implies $\theta < 10^{-8}$. Why should θ be so small or could it be zero? Peccei and Quinn[19] proposed a mechanism which solves the problem at the expense of introducing a new light particle: the axion[20]. Let us consider the $SU(2) \times U(1)$ model of weak interactions with the u and d quarks; the chain of arguments leading to the axion can be summarized as follows:

- Suppose u and d quarks are not massless (as strongly suggested by current algebra) and that we give them masses with 2 different Higgs doublets ϕ_1, ϕ_2 (a crucial ingredient), then the Lagrangian is invariant under a global chiral $U(1)$ Peccei-Quinn

$$u \to e^{\gamma_5\alpha} u \; ; \quad d \to e^{\gamma_5\alpha} d \; ; \quad \phi_1 \to e^{+2i\alpha} \phi_1 \; ; \quad \phi_2 \to e^{-2i\alpha} \phi_2$$

This allows us to rotate away the parameter θ and get rid of the annoying CP violation. However this is not all the story.

When ϕ_1 and ϕ_2 acquire non zero vacuum expectation values (vev), gauged $SU(2) \times U(1)$ and global $U(1)_{PQ}$ get broken: hence 4 Goldstone bosons. Three of them are eaten by the $W^\pm$ and the Z_o to acquire masses, the remaining one is the axion which would be massless if instantons were not present. Because of instantons $U(1)_{PQ}$ is an approximate symmetry and the axion has a small mass. The physical parameters of the axion can be calculated and one finds that its mass and its couplings to matter are inversely proportional to the vev of the Higgses, in our case $\sim 10^2$ GeV :

$$m_{axion} \sim 100 \text{ KeV}.$$

The axion was searched and not found (so far), more, as it is, it rises enormous problems for star evolution!

A clever solution has been proposed which makes it invisible[21]. The trick is the following: one builds a GUT[20),21)] and adds the suitable Higgs multiplets so as to obtain an extra chiral symmetry $U(1)_{PQ}$. When the "grand" breaking occurs, $U(1)_{PQ}$ is also broken but now what enters in the mass and couplings of the grand unified axion is $M_X \sim 10^{14}$ GeV which reduces these by 14 orders of magnitude e.g. $m_{GUa} \sim 10^{-8}$ eV.

We do not see the axion just because we cannot see it! An example of possible SU(5) model is the one proposed by Wise, Georgi and Glashow[22] who use two 5_H's and $24_H + i24_H'$. Of course the "unnatural" small θ parameter has been replaced by a "suitable" addition of Higgs multiplets.

III. THE HIGGS POTENTIAL MINIMUM PROBLEM

Let G be a (compact) gauge group which is spontaneously broken down to its subgroup S. We have to solve the following problems:

i) Find the representations R_G^S of G containing a vector ϕ_o associated with the vacuum expectation value $\langle\phi\rangle$ invariant under (and only under) the subgroup S. S is the stabilizer or little group of ϕ_o :

$$S = \{g \in G \mid g(\phi_o) = \phi_o\} = \mathrm{Stab}\,(\phi_o) = G_{\phi_o}.$$

ii) Find the invariants of the representations R_G^S in order to obtain the most general G invariant polynomial of degree 4 from R_G^S, which will be the Higgs potential $V(\phi)$ (the restriction to degree 4 being imposed by renormalizability).

iii) Then select the representations R_G^S such that the absolute minimum of the corresponding Higgs potential admits exactly S as little group.

Informations for part i) can be found in Refs (24) and (25) for G = SU(n) and SO(n). Indeed general theorems are given there characterizing the G irreducible representations which admit a vector invariant under a subgroup S, for a large class of subgroups S.

Actually this problem can he seen as a first step for the classification of the orbits under G of a G representation - such a program is quite huge, except for special cases as SU(3) (see Ref. 24). It is worth to mention that the results obtained in Refs (24, 25) are specially simple for a direct application. As an example let us mention the following ones: any SU(n) representation $R(\lambda_1 \ldots \lambda_{n-1})$ (associated with the Young tableau with λ_1 boxes in the first row, ... λ_{n-1} boxes in the $(n-1)^{th}$ row) contains a vector the stabilizer of which is $S[U(n-p) \times U(p)]$ $n-p \geq p > 1$ if and only if

i) $\lambda_i + \lambda_{n-i+1} = 2\lambda \quad i = 1,2,\dots,n-1$

$\lambda_1 = 2\lambda \qquad$ and λ positive number

ii) $\lambda_i \geq \lambda$.

Note that condition i) implies $R(\lambda_1 \dots \lambda_{n-1})$ to be self conjugate and such that $\sum_{i=1}^{n-1} \lambda_i = n\lambda$.

In the same way $R(\lambda_1 \dots \lambda_{n-1})$ will contain a vector the stabilizer of which is $U(n-1)$ if and only if: $\lambda_2 = \dots = \lambda_{n-1} = \frac{1}{2}\lambda_1$, and a vector stabilized under $SU(n-1)$ if and only if $\lambda_2 = \dots = \lambda_{n-1} \neq \frac{1}{2}\lambda_1$.

Techniques developed to prove theorems of this kind uses extensively the Gel'fand Zetlin basis for $SU(n)$ and $SO(n)$, allowing an explicit realization of the representations and suitable to deduce the eigenvalues of the $U(1)$ factors in the decomposition of a representation of G with respect to subgroups containing these $U(1)$ factors, and other results such as anomaly formula for $SU(n)$ representations[25].

- The second problem concerning the construction of the invariants of the representation R_G is a difficult problem in the general case. The representation R_G can always be considered as a real representation on which G acts linearly and orthogonally. Then what is known from invariant theory is that (i) the polynomial invariants separate the orbits; (ii) every G invariant polynomial (C^∞-function) F can be written as a polynomial (C^∞-function) $\hat{F}$ with:

$$F(\phi) = \hat{F}(\theta_1(\phi),\dots,\theta_q(\phi)); \phi \in R^n \equiv R$$

where $\theta_1 \dots \theta_q$ form a minimal set of homogeneous G-invariant polynomials $\theta_\alpha(\phi)$ (i.e. an integrity basis).

Let us remind[26),27)] that the orbit of $m \in R_G$ is the set $G(m) = \{g(\phi) \mid g \in G\}$ if one denotes by $g(\phi)$ the transformed of ϕ by g. If two points m and m' are on the same orbit, their little groups (or stabilizers) G_m and $G_{m'}$ are conjugate. However two points m and m' need not to be on the same orbit to have conjugate little groups. By definition, they are on the same stratum; in other words the stratum $S(m)$ in the union of all orbits such that the little groups of their points are all conjugated. One has a partition of R_G into orbits, as well as partition of R_G into strata.

Therefore the decomposition of R_G into orbits and strata is equivalent to the classification of little groups for R_G. The number of little groups for a given representation is finite and a partial ordering (defined by inclusion) exists among them (that is: S_1 is smaller than S_2 if S_1 is included into S_2 up to a conjugation).

Then with respect to this ordering, the minimal little group H_o is unique - what is not the case in general for the maximal little group (if we exclude in R_G the 0) - and the associated stratum has the property to be open dense: it is also called the generic stratum.

The number μ of functionally and algebraically independent invariants for the representation R_G is:

$$\mu_{R_G} = \dim R_G - \dim G + \dim (S_o)$$

where S_o is the little group of the generic stratum - or minimal little group.

Let us consider, as an example, the case $G = SU(2)$ acting on the fundamental two dimensional (real on the quaternionic field) representation. Then $\dim R_G = 4$ (on the real) and the only possible little group is the identity. Therefore there exists only $\mu = 4-3+0 = 1$ invariant, i.e. the scalar product $\vec{X}^2$ if $\vec{X} \in R_G$.

If $G = SU(3)$ and R_G is the three dimensional representation again there is only one little group which is the subgroup $SU(2)$ and one invariant ($\mu = 6-8+3$). Now looking at the 8-dimensional (adjoint and real) representation of $SU(3)$, two different little groups[26] can be found, i.e. $SU(2) \times U(1)$ and $U(1) \times U(1)$. Indeed considering $8_{SU(3)}$ as the set of 3×3 hermitean traceless matrices h on which $SU(3)$ acts as follows:

$$U \in SU(3) \qquad h \to UhU^{-1}$$

We see that any h can be put by $SU(3)$ action on a diagonal form

$$h = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & -(\alpha+\beta) \end{pmatrix}$$

There exist therefore two kinds of orbits - or two strata - following the 3 eigenvalues are all different (little group = $U(1) \times U(1)$) or two of them are equal (little group = $SU(2) \times U(1)$). In this last case the orbit can be parametrized by

$$h_o = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

We can notice that in $R^8/R-\{0\}$ (i.e. if we forget about the multiplicative parameter ξ) there exists only one orbit in the stratum associated with the little group $SU(2) \times U(1)$: the orbit $G(h_o)$ is said isolated in its stratum (see a complete definition later).

In our example we see immediately that for any $\epsilon > 0$

$$h_1 = \begin{pmatrix} 1+\epsilon & & \\ & 1+\epsilon & \\ & & -2-2\epsilon \end{pmatrix}$$

is no more stabilized by $SU(2) \times U(1)$ but by $U(1) \times U(1)$.

The two independent invariants can be chosen to be

$$I_2 = \mathrm{Tr}\, h^2 \quad \text{and} \quad I_3 = \mathrm{Tr}\, h^3 = 3 \det h$$

which satisfy:

$$I_2^3 \geq 6\, I_3^2$$

The strict inequality: $I_2^3 > 6\, I_3^2$ on the generic stratum is replaced by the equality $I_2^3 = 6\, I_3^2$ on the isolated orbit. We see explicitly on this simple example the property above mentioned that the polynomial invariants separate the orbits (and strata).

Now comes the last problem to be solved, i.e. the Higgs potential $V(\phi)$ with $\phi \in R_G$ must have as absolute minimum ϕ_o such that its little group $G_{\phi_o} = S$ is fixed in advance.

A detailed study of this problem has been made in some special cases. In the case of R_G irreducible, a detailed study has been first given by Li[28] for $G = SU(N)$ or $SO(n)$ and R_G of the type vector, second rank symmetric and anti-symmetric and adjoint representation. For exceptional groups, Higgs scalars in the adjoint representations are discussed in Ref. (29).

In the interesting case of $G = SU(5)$, the minimal breaking can be achieved with the adjoint 24 breaking $SU(5) \to SU(3) \times SU(2) \times U(1)$ and a vector one 5, producing the second breaking up to $SU(3) \times U(1)_{em}$. Let us denote by ϕ_i^j the 5×5 traceless hermitean matrix and H_i $(i,j = 1\ldots5)$ the complex vector which transform respectively as the adjoint and fundamental representation of $SU(5)$. If the discrete symmetry $\phi_i^j \to -\phi_i^j$ is imposed one gets for the Higgs potential

$$V(\phi,H) = -\frac{1}{2}\mu^2 \mathrm{Tr}\phi^2 + \frac{1}{4} a(\mathrm{Tr}\phi^2)^2 - \frac{1}{2}\nu^2 H^+H$$
$$+ \frac{1}{4}\lambda(H^+H)^2 + \alpha(H^+H)\,\mathrm{Tr}\phi^2 + \frac{1}{2} b\, \mathrm{Tr}\phi^4 + \beta\, H^+\phi^2 H$$

It has been shown[30),31)] that the absolute minimum of this potential can be on an orbit with the little group $SU(3) \times U(1)_{em}$ for values of the coefficients $b > 0$ and $\beta < 0$. Then the "naturalness" of the symmetry is preserved, that is the range of the coefficients in $V(\phi,H)$ is not limited to specific values (which make the minimum "unstable"). The hierarchy is preserved (see section II) by choosing ("by hand") the ratio $|x|^2/\delta^2 \ll 1$ if H and ϕ are parametrized as follows:

$$H = (x,0,0,0,0) \quad \text{and} \quad \phi = \text{diag}\,(\delta,\delta,\delta,\delta(-3/2+\varepsilon),\, \delta(-3/2+\varepsilon))$$

In the $G = SO(10)$ case, the breaking can occur with the help of a 16, 45 and 10 as follows:

$$SO(10) \xrightarrow[16]{} SU(5) \xrightarrow[45]{} SU(3) \times SU(2) \times U(1) \xrightarrow[10]{} SU(3) \times U(1)$$

A detailed study of the Higgs potential with a $45 + 16 + \overline{16}$ representation can be found in Ref. (32).

Is it possible to find a mathematical criterion for the minimum of the Higgs potential, such that the breaking appears more natural? Although one cannot answer completely to this question, a way of thinking seems particularly attractive. About ten years ago, Michel and Radicati[33)], studying various examples in elementary particle physics, remarked that the directions of breaking appear on isolated orbits in the representations on which the symmetry group acts. From this property they conjectured the following theorem, which was proved by Michel[34)]:

Theorem: Let G be a compact Lie group acting smoothly (i.e. infinitely differentiable mapping) on the real manifold M, and let $m \in M$. Then the properties (a) and (b) are equivalent:

(a) the orbit $G(m)$ is critical, i.e. the differential $d\,f_{m'}$ of any smooth real G-invariant function f on M vanishes for $m' \in G(m)$.

(b) the orbit $G(m)$ is isolated in its stratum, i.e. there exists a neighbourhood V_m of m such thet if $p \in G(m)$, $p \in V(m)$, then the little group G_p is not conjugated to G_m.

We realize the importance of this theorem for the Higgs problem but at the same time its limitations. The above property allows to replace very elegantly an analysis problem: search for the extrema of a G-invariant function by a geometrical one: classification of the isolated orbits. The theorem is general in the sense that the property is valid for any G-invariant function.

However, it does not fit completely with our problem, since we are interested not in any extremum, but in the minimum of our potential. One may think that the restriction of G-invariant functions to fourth degree polynomial could be of some simplification. It has been possible to show (only) on some examples that the minimum of the Higgs potential lies on a critical orbit: that is the case in $SU(5) \to SU(3) \times U(1)$ with a $24 + 5$ [31)35)] as well as in $SU(n)$ when only the adjoint representation is used[36)].

Michel,[37)] studying in detail Landau theory of second order phase transitions, has conjectured the following property: if the Higgs potential depends only of one irreducible (on the real) representation, its minima have maximal little groups. Indeed if isolated orbits (or critical orbits) have little groups which are maximal in the set of little groups, the converse is not true in general, i.e. it can exist maximal little groups to which are not associated isolated orbits.

- Let us bring a precision on the above conjecture: if the minimum of a Higgs potential might be on an orbit associated with a maximal little group, this does not mean that any orbit associated with a maximal little group works. For example, if one considers the spinor representation 64 of $O(14)$, one finds three maximal little groups, actually $SU(7)$, $G(2) \times G(2)$ and $SU(3) \times O(7)$, but only the two first correspond to absolute minima of the Higgs potential[38)].

It is clear that more work is needed in this appealing direction, where the geometry of the orbit space plays a predominant rôle. In that spirit, it is worth to mention the recent study given in Ref. (39). The authors state the following theorem, which have quite interesting consequences:

<u>Theorem</u>: The vector space spanned by the gradients at ϕ of the polynomials invariants of an integrity basis coincides with the invariant slice through ϕ.

First this theorem calls for some definitions. If we denote T_ϕ the subspace of the representation $(\equiv R^n)$ spanned by the elements $t\phi$ with $t \in \mathfrak{G}$ Lie algebra of G (tangent space) and S_ϕ its orthogonal complement in R^n (global slice), the invariant slice S_{ϕ_0} will be the subspace of S_ϕ spanned by the G_ϕ-invariant vectors of S_ϕ (G_ϕ being the little group of ϕ). One recovers Michel's theorem as a consequence of this statement, but one can also deduce from it several properties. In particular, equations of strata can be obtained by studying the principal minors of the symmetric matrix:

$$P_{\alpha\beta}(\phi) = \sum_{i=1}^{n} \partial_i \, \theta_\alpha(\phi) \, \partial_i \, \theta_\beta(\phi) \qquad \alpha,\beta = 1,\ldots,q$$

where the θ_α are the q homogeneous G-invariant polynomial of an integrity basis. Their method allows also to state conditions for the minima to exist and to be natural (stable) and to clarify the origin of a class of pseudogoldstone bosons (see the paper for more details!).

CONCLUSION

As a conclusion it appears clearly that the choice of Higgs scalars is of first importance in the gauge theories of unification. They are a very convenient way to describe some general features of physics like symmetry breaking or mass generation, however they do not provide quantitative explanation for the observed phenomena. To say the worse they, in some sense, parametrize our ignorance of the dynamical behaviour of gauge theories. We referred often to the concept of naturalness, which is at the same level as aesthetics: beautiful ideas should work! This is the reason why we think that a more mathematical approach for selecting Higgs representations could yield a natural solution. In this line of thought superunified theories[1)] which use supersymmetry and can describe gravity could also, as a bonus, provides natural answers to the question that scalar particles raise. Grand unified theories are appealing, superunified theories could be great!

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