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ATOMIC ENERGY OF CANADA LIMITED

L'ÉNERGIE ATOMIQUE DU CANADA LIMITÉE

THE SYMMETRIC GROUP AND ITS RELEVANCE TO FERMION PHYSICS

Le groupe symétrique et ses rapports avec la physique des fermions

M. HARVEY

A set of lectures given at TRIUMF (TRI-UNIVERSITY MESON FACILITY), British Columbia, 20-24 Oct. 1980.

Chalk River Nuclear Laboratories Laboratoires nucléaires de Chalk River

Chalk River, Ontario

April 1981 avril

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par

M. Harvey

On donne des notes relatives à une série de cours présentés au centre TRIUMF (Vancouver) au cours de la semaine écoulée du 17 au 24 octobre 1980. Les cours et les notes s'y rapportant ont été conçus pour donner aux étudiants une connaissance pratique de la classification et de la construction d'ensembles d'états de particules-n se transformant selon une représentation irréductible et définie du groupe symétrique S_n. On donne des applications pour la classification des états quark des baryons et des multibaryons.

- série de cours donnés au centre TRIUMF (TRI-University Meson Facility) en Colombie-Britannique du 20 au 24 octobre 1980.

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Notes are given of a series of lectures presented at TRIUMF (Vancouver) during the week of October 17-24, 1980. The lectures, and accompanying notes were designed to give the student a *working* knowledge of the classification and construction of sets of n-particle states transforming according to a definite irreducible representation of the symmetric group S_n. Applications are given for the classification of quark states of baryons and multibaryons.

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PREFACE

The purpose of this set of lectures and notes is to give the student a working knowledge of the classification and construction of sets of n-particle states transforming according to definite irreducible representations (irreps) of the symmetric group S_n. I have tried to answer the obvious question of why such states have any relevance to the physics of collection of fermions be they electrons in atoms, nucleons in nuclei, or quarks in baryons or multi-baryonic states. Since the application in this latter area is perhaps the least well documented, I give my examples here. There is no claim to originality in these notes except perhaps in the order and method of presentation.

The verbal presentation of the lectures was given during three sessions at TRIUMF (Vancouver) in the week of Oct. 17-24, 1980. Not all aspects of these notes could be covered in such a short period, but this expanded set is given here for the benefit of the student. It is hoped that through the notes the student will be encouraged to further reading. The short bibliography is meant as a stepping stone in this direction and is certainly not meant to be exhaustive. Further references can of course be found in the reference lists of my bibliography.

Many aspects of representation theory are not unique to the symmetric group. For this reason PART I reviews some of the general ideas and leads up to proofs of five major

theorems for finite groups, the latter two of which the practitioner will find to be particularly useful. PART II moves on to the symmetric group proper and, through example, explains the ingenuity of the Young tableau and Yamanouchi label notation to describe all properties of the irreps of S_n. I show here also simple proofs of how products of functions belonging to definite irreps of S_n can yield totally symmetric or antisymmetric functions. These are but simple examples of the Clebsch-Gordan coupling scheme for S_n that is explained, but with the general method for constructing the coefficients relegated to Appendix B. I have included sections on the meaning of fractional parentage, the construction of coefficients in factored form, and how such ideas are of practical use in the calculation of matrix elements.

In PART III, I briefly explain why the Young tableau notation can also be used to describe irreps of U_m and SU_m - and is indeed preferable to the particle physicist's habit of using the dimension to describe the irreps. I discuss the classification of states according to product spaces and end with examples of the classification of baryon resonances, assuming these to arise from orbital excitation of three quarks in a confining potential.

Probably the greatest difficulty in understanding representation theory is comprehending the notation. It must be admitted that this is often the fault of authors not explaining their notations and definitions perhaps (shudder!) with the

arrogant assumption that their usage is so standard it does not need to be explained. I attempt to help the reader understand my notation through the collection of definitions in Appendix A.

This preface was written whilst flying at 37,000 feet over the Rocky Mountains on the BOEING-747 returning me from Vancouver to Eastern Canada. This combination of viewing some of Nature's most magnificent structures and man's ingenuity in placing me in such a position perhaps best describes the physicist's almost unique position in being able to view - again with machines of man's ingenuity - the fascinating world of sub-atomic physics. I hope the notes will provide part of the "boarding pass" for the students who wish to 'fly' into this unique viewing position.

Finally I would like to thank my hosts at TRIUMF, University of British Columbia and Simon Fraser University, for a most interesting and enjoyable week.

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PART I

WHAT IS A GROUP?

Formally a group is defined (ref. 1) to be a set G of entities g_i defined under a law of combination (we use the multiplication symbol × or nothing at all) such that

(a) if g_1 and g_2 are in G, then so is $g_1 \times g_2$;

(b) the associative law applies, i.e. $g_1 \times (g_2 \times g_3) \equiv (g_1 \times g_2) \times g_3$;

(c) a "unit" element E exists such that $E \times g_i = g_i \times E = g_i$ for all g_i;

(d) to every element g_i there is an inverse element g_i^{-1} such that $g_i \times g_i^{-1} = g_i^{-1} \times g_i = E$.

We restrict our attention to groups of transformations on a Hilbert space of functions (ψ_α) considered to form a linearly independent set. The action of the group transformation can thus be written

$$g_i \psi_\alpha = \sum_\beta \Gamma_{\beta\alpha}(g_i)\, \psi_\beta$$

where $\Gamma_{\beta\alpha}(g_i)$ are the set of expansion coefficients for the particular group element g_i. We now prove that the matrices $\Gamma(g_i)$ form a group isomorphic to the group of transformations g_i.

Clearly if $g_j \times g_i \equiv g_j g_i = g_k$ then

$$g_j g_i \psi_\alpha = \sum_\beta \Gamma_{\beta\alpha}(g_i) g_j \psi_\beta$$

$$= \sum_{\beta\gamma} \Gamma_{\beta\alpha}(g_i)\Gamma_{\gamma\beta}(g_j)\psi_\gamma$$

$$= \sum_\gamma \Gamma_{\gamma\alpha}(g_k)\psi_\gamma$$

(-we use henceforth the summation convention).

Thus since ψ_α forms a linearly independent set

$$\Gamma_{\gamma\alpha}(g_k) = \Gamma_{\gamma\beta}(g_j)\Gamma_{\beta\alpha}(g_i)$$

or in matrix language $\Gamma(g_k) = \Gamma(g_j)\Gamma(g_i)$. Thus the law of combination for matrices mirrors the law of combination for the transformation operators. Clearly the associative law holds for matrix multiplication and the unit matrix I will correspond to the "unit" element E of the group. Since $g_i \times g_i^{-1} = E$ then

$$\Gamma(g_i)\cdot\Gamma(g_i^{-1}) = \Gamma(E) \equiv I$$

and hence the matrices Γ are non-singular. Thus the set of matrices $\Gamma(g_i)$ forms a group that exactly mirrors the group of transformations g_i; they thus form a matrix representation (or simply a representation) of the group G.

The set of matrices $\Gamma(g_i)$ are defined with respect to a particular set of functions ψ_α. Suppose we define a new set of functions $\phi_\beta = S_{\alpha\beta}\psi_\alpha$ where S is a

non-singular matrix. Since $g_i\psi_\alpha = \Gamma_{\gamma\alpha}(g_i)\psi_\gamma$ then

$$g_i\phi_\beta = S_{\alpha\beta}\Gamma_{\gamma\alpha}(g_i)\psi_\gamma = S_{\alpha\beta}\Gamma_{\gamma\alpha}(g_i)S^{-1}_{\delta\gamma}\phi_\delta$$

$$= (S^{-1}\Gamma(g_i)S)_{\delta\beta}\phi_\delta$$

The set of matrices $\Gamma'(g_i) = S^{-1}\Gamma S$ also form a representation of the group which is said to be equivalent. Thus matrix representations are not unique - in fact there are an infinite number of them (-as many as there are non-singular matrices S).

So far we have not said anything about the dimension of the matrices $\Gamma(g_i)$. In general, if ψ_α form an infinite complete set then the matrices will be infinite dimensional. For certain groups (we call them compact groups - and the only ones concerning us in these lectures) we can find transformations S such that all the group elements g_i transform only over a finite set of functions ϕ_α. Furthermore a transformation S' cannot be found that forms a smaller set. This finite set of functions ϕ_α is said to carry an irreducible representation of the group G. If there are k functions in the set ϕ_α, then the matrices $\Gamma(g_i)$ are k-dimensional and we are said to have a k-dimensional irreducible representation or k-dimensional irrep.

Example:

Consider the set of permutations (S_2) of two objects 1 and 2.

There are two elements of the groups E and P_{12}. Let the numbering be carried by three objects α, β and γ. Thus we have a six-dimensional space $\alpha_1\beta_2$, $\alpha_2\beta_1$, $\alpha_1\gamma_2$, $\alpha_2\gamma_1$, $\beta_1\gamma_2$, $\beta_2\gamma_1$. The matrix representations are

$$I \to \left[\begin{array}{cc:cc:cc} 1 & 0 & & & & \\ 0 & 1 & & 0 & & 0 \\ \hdashline & & 1 & 0 & & \\ & 0 & 0 & 1 & & 0 \\ \hdashline & & & & 1 & 0 \\ & 0 & & 0 & 0 & 1 \end{array}\right] \qquad P_{12} \to \left[\begin{array}{cc:cc:cc} 0 & 1 & & & & \\ 1 & 0 & & 0 & & 0 \\ \hdashline & & 0 & 1 & & \\ & 0 & 1 & 0 & & 0 \\ \hdashline & & & & 0 & 1 \\ & 0 & & 0 & 1 & 0 \end{array}\right]$$

Already we see that this six-dimensional representation is composed of three two-dimensional representations carried by the three sets of functions $(\alpha_1\beta_2, \alpha_2\beta_1)$, $(\alpha_1\gamma_2, \alpha_2\gamma_1)$, $(\beta_1\gamma_2, \beta_2\gamma_1)$. But none of these two-dimensional representations is irreducible. In the first case we can, for example, define two new functions

$$\phi_+ = \sqrt{1/2}\ (\alpha_1\beta_2 + \alpha_2\beta_1)$$

$$\phi_- = \sqrt{1/2}\ (\alpha_1\beta_2 - \alpha_2\beta_1)$$

With respect to these basis states the group operators E and P_{12} have matrix representations

$$E \to \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \qquad P_{12} \to \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$

Clearly we have reduced the original six-dimensional representation to involve two basic irreducible matrix representations of S_2 namely

$$E \to [1] \qquad P_{12} \to [1]$$

and

$$E \to [1] \qquad P_{12} \to [-1]$$

These two representations are one-dimensional and they are different. One of our problems is to find a way of characterising the representations that in some way tells us what the matrix representations are. One could of course call the first representation #1 and the second representation #2, but, unless one has a good memory, it would be hard to remember which was which. We shall return to this problem later, but for now we shall use the letter [f] to indicate a characterisation of a particular representation.

Note incidentally that in this example we have two different one-dimensional representations and so it makes no sense to use the dimension as a characterisation of the representation. This is true for the characterisation of the representations of many groups. The habit of high energy physicists of using the dimension to characterise the representations of the special unitary transformation group (SU_n) is to be discouraged.

RELEVANCE TO PHYSICS

In all branches of physics we are concerned with sets of stationary states which characterise the physical system under study (molecule, atom, nucleus, baryon) in certain energy states. These states arise as the eigensolutions of a Hamiltonian H

$$H\psi_\alpha = E_\alpha \psi_\alpha \tag{1.1}$$

Suppose H to be invariant with respect to all the transformations of a group G, i.e. $g_i H g_i^{-1} \equiv H$ for all $g_i \varepsilon G$.

Then by operating by g_i on the left of eq. 1.1 we find

$$g_i\ H\ \psi_\alpha \equiv g_i\ H\ g_i^{-1} g_i \psi_\alpha = E_\alpha g_i \psi_\alpha$$

$$\text{i.e.}\quad H(g_i\psi_\alpha) = E_\alpha (g_i \psi_\alpha)$$

Thus, if ψ_α is an eigensolution with energy E_α, $(g_i\psi_\alpha)$ is also an eigensolution with the same energy. Thus the eigensolutions of H come in sets of degenerate states. Each set carries a representation of the group G. From each set, functions can be defined which carry an irreducible representation label f. Thus the eigensolution of H can be written $\psi(fy)$ where

$$H\psi(fy) = E_f \psi(fy)$$

where f characterises the representation and y characterises the state within the representation (- there will be as many y labels as the dimension of f). Note that the "energy" depends on the representation label f but not y. [Actually it is possible for the degenerate states to belong to more than one irrep. The point is that all members of an irrep are degenerate. Degeneracy of some irreps usually signifies a higher symmetry group.]

A simple example of this is the invariance of H with respect to the generalised group of rotations (in orbital and intrinsic space) which is isomorphic to the group SU_2 (ref. 17). Each eigensolution of H can therefore be characterised with respect to the representation of the SU_2 group - usually written J (instead of f), i.e. the angular momentum label. The degenerate states within the representation J are usually characterised in this case by the symbol M (instead of y) - i.e. the magnetic quantum number. We will return later in Section 3 to discuss more about SU_2 (or in general SU_m) and the characterisation of the representations. With the harmonic oscillator hamiltonian states with different angular momenta can be degenerate, e.g. a J=0 and 2 for the 2-quantum states or J=1, and 3 for the 3-quantum. This degeneracy arises from the invariance of the oscillator Hamiltonian with respect to SU_3 transformations among the oscillator quanta in the three spatial directions.[15)]

SOME USEFUL THEOREMS

1. Any representation of a finite group is equivalent to a representation by Unitary matrices.
2. Schurs Lemma - the only matrix that commutes with all matrices of an irreducible representation is the unit matrix I multiplied by a constant (d).
3. If $\Gamma^{f}(g_i)$ and $\Gamma^{f'}(g_i)$ are two irreducible representations of dimension ℓ_f and $\ell_{f'}$, respectively, and M is a rectangular matrix such that $\Gamma^{f}(g_i)M = M\Gamma^{f'}(g_i)$ for all g_i then
   1. if $\ell_f \neq \ell_{f'}$ $M \equiv 0$
   2. if $\ell_f = \ell_{f'}$ either $M \equiv 0$ or $f \equiv f'$.
4. The Orthogonality Theorem

$$\sum_i \Gamma^{f*}_{yy'}(g_i)\, \Gamma^{\bar{f}}_{\bar{y}\,\bar{y}'}(g_i) = \delta_{f\bar{f}}\, \delta_{y\bar{y}}\, \delta_{y'\bar{y}'}\, \frac{h}{\ell_f}$$

   where h is the dimension of the group (i.e. # of group elements); ℓ_f is the dimension of the representation (i.e. size of matrices).
5. Basis functions belonging to different representations, or different rows of the same representation, are orthogonal

$$\langle \psi^{f'}_{y'} | \psi^{f}_{y} \rangle = \delta_{ff'}\, \delta_{yy'}\, N$$

Theorem #1

Any representation of a finite group is equivalent to a representation by a unitary matrix.

We prove that given any representation we can construct from it a representation with unitary matrices.

Proof. Let $\Gamma(g_i)$ be the matrix representation of the group element g_i. Construct the matrix

$$H = \sum_i \Gamma(g_i)\Gamma^+(g_i) \tag{1.2}$$

[Γ^+ is the adjoint of Γ, i.e. the complex conjugate transpose.]

The matrix H is Hermitian, i.e. $H_{\alpha\beta} = H^*_{\beta\alpha}$

$$[H^*_{\beta\alpha} = \sum_i (\Gamma_{\beta\gamma}(g_i)\Gamma^*_{\alpha\gamma}(g_i))^* = \sum_i \Gamma_{\alpha\gamma}(g_i)\Gamma^*_{\beta\gamma}(g_i) \equiv H_{\alpha\beta}]$$

But any Hermitian matrix can be diagonalised by a unitary matrix made up from the orthogonal vectors of the secular equation ($H\hat{V}_\alpha = d_\alpha \hat{V}_\alpha$).

Thus $D = U^{-1}HU$

with $$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \qquad U = \begin{bmatrix} V_1 & V_2 & \cdots\cdots & V_n \end{bmatrix}$$

$$D = \sum_i U^{-1}\,\Gamma(g_i)\Gamma^{+}(g_i)U$$

$$= \sum_i U^{-1}\,\Gamma(g_i)U\;U^{-1}\,\Gamma^{+}(g_i)\,U$$

$$= \sum_i \Gamma'(g_i)\,\Gamma'^{+}(g_i) \quad \text{with} \quad \Gamma' = U^{-1}\Gamma U \tag{1.3}$$

Now all the eigenvalues $d_i > 0$

[From eq. 1.3 $\quad d_\alpha \delta_{\alpha\beta} = \sum_i \Gamma'_{\alpha\gamma}(g_i)\,\Gamma'^{*}_{\beta\gamma}(g_i)$

$$\text{i.e. } d_\alpha = \sum_i \Gamma'_{\alpha\gamma}(g_i)\,\Gamma'^{*}_{\alpha\gamma}(g_i) > 0]$$

Thus we can construct the unit matrix

$$I = D^{-\frac{1}{2}} \sum_i \Gamma'(g_i)\,\Gamma'^{+}\,D^{-\frac{1}{2}}$$

where

$$D^{-\frac{1}{2}} = \begin{bmatrix} d_1^{-\frac{1}{2}} & & & \\ & d_2^{-\frac{1}{2}} & & \\ & & \ddots & \\ & & & d_n^{-\frac{1}{2}} \end{bmatrix}$$

Consider now the matrix $\Gamma''(g_i) = D^{-\frac{1}{2}}\,\Gamma'(g_i)D^{+\frac{1}{2}}$.

This matrix is unitary because

$$\Gamma''(g_j)\Gamma''^{+}(g_j) = D^{-\frac{1}{2}}\,\Gamma'(g_j)\,D^{+\frac{1}{2}}[I]D^{\frac{1}{2}}\Gamma'^{+}(g_j)D^{-\frac{1}{2}}$$

$$\equiv D^{-\frac{1}{2}}\Gamma'(g_j)D^{+\frac{1}{2}}[D^{-\frac{1}{2}}\sum_i \Gamma'(g_i)\Gamma'^{+}(g_i)D^{-\frac{1}{2}}]D^{\frac{1}{2}}\Gamma'^{+}(g_j)D^{-\frac{1}{2}}$$

$$= D^{-\frac{1}{2}}\sum_i \Gamma'(g_j)\Gamma'(g_i)(\Gamma'(g_j)\Gamma'(g_i))^{+}\,D^{-\frac{1}{2}} \qquad (1.4a)$$

$$\equiv D^{-\frac{1}{2}}\sum_k \Gamma'(g_k)\,\Gamma'^{+}(g_k)D^{-\frac{1}{2}} \qquad (1.4b)$$

$$= E$$

In making the change from eq. 1.4a to 1.4b we use the fact that if $g_j g_i = g_k$ then $\Gamma(g_j)\Gamma(g_i) = \Gamma(g_k)$ and that $\sum_i g_j g_i \equiv \sum_k g_k$.

The matrices $\Gamma''(g_i)$ form a unitary representation of the group. They were defined in terms of the original representation by

$$\Gamma''(g_i) = D^{-\frac{1}{2}}U^{-1}\,\Gamma\,U\,D^{+\frac{1}{2}} = (UD^{+\frac{1}{2}})^{-1}\,\Gamma(UD^{+\frac{1}{2}})$$

Thus $S = UD^{+\frac{1}{2}}$ is the transformation from the original basis defining Γ to the new basis defining Γ''.

From here on we shall always assume a representation to be unitary.

Theorem #2

Schur's Lemma - the only matrix that commutes with all matrices of an irreducible representation is the unit matrix possibly multiplied by a constant (d).

Corollary - if a non-constant matrix can be found that commutes with all matrices of a representation then the representation is reducible.

Proof - Let M commute with all $\Gamma(g_i)$

$$\text{i.e.} \quad M\Gamma(g_i) = \Gamma(g_i)M \quad \text{for all } g_i.$$

Hence

$$\Gamma^+(g_i)\, M^+ = M^+\, \Gamma^+(g_i) \tag{1.5}$$

Because of Theorem #1 we can assume all Γ to be unitary without loss of generality: $\Gamma^+ \equiv \Gamma^{-1}$. Hence multiplying eq. 1.5 on left and right by $\Gamma(g_i)$ we prove

$$M^+\, \Gamma(g_i) = \Gamma(g_i) M^+$$

Thus if M commutes with Γ, so does M^+ and hence so do the Hermitian matrices $H_+ = M+M^+$ and $H_- = i(M-M^+)$, but the matrices H_p $(p = \pm)$ must be constant because

$$H_p \Gamma = \Gamma\, H_p$$

$$U\, H_p \Gamma\, U^{-1} = U\, \Gamma\, H_p\, U^{-1}$$

with U the unitary matrix of eigenvectors of H_p, then

$$U H_p U^{-1} U \Gamma U^{-1} = U \Gamma U^{-1} U H_p U^{-1}$$

i.e. $$D_p U \Gamma U^{-1} = U \Gamma U^{-1} D_p$$

with D_p the diagonal matrix of eigenvalues of H_p, i.e.

$$D_p \Gamma' = \Gamma' D_p \qquad \Gamma' = U \Gamma U^{-1}$$

or

$$\sum_\beta (D_p)_{\alpha\alpha} \delta_{\alpha\beta} \Gamma'_{\beta\gamma} = \sum_\beta \Gamma'_{\alpha\beta} \delta_{\beta\gamma} (D_p)_{\gamma\gamma}$$

$$(D_p)_{\alpha\alpha} \Gamma'_{\alpha\gamma} = \Gamma'_{\alpha\gamma} (D_p)_{\gamma\gamma} \quad \text{(no summation over } \alpha)$$

$$\therefore [(D_p)_{\alpha\alpha} - (D_p)_{\gamma\gamma}] \Gamma'_{\alpha\gamma} = 0$$

If $(D_p)_{\alpha\alpha} \neq (D_p)_{\gamma\gamma}$ then $\Gamma'_{\alpha\gamma}(g_i) = 0$ for <u>all</u> g_i in the group, i.e. Γ' is reducible (and hence Γ is reducible). If Γ (and hence Γ') is irreducible then $(D_p)_{\alpha\alpha} = (D_p)_{\gamma\gamma}$ for <u>all</u> α,γ, i.e. $D_p = d_p I$, with d_p a single constant. Hence

$$U H_p U^{-1} = D_p = d_p I$$

i.e. $$H_p = d_p I \quad \text{and hence} \quad M = \tfrac{1}{2}(d_+ - i d_-) I.$$

<u>Corollary</u>: if a non-constant matrix M can be found that commutes with all Γ then the representation can be reduced by the transformations $U \Gamma U^{-1}$ with U the matrix of eigenvectors of $H_+ = (M + M^+)$.

Theorem #3

If $\Gamma^1(g_i)$ and $\Gamma^2(g_i)$ are two irreducible representations of dimension ℓ_1 and ℓ_2, respectively, and if M is a rectangular matrix such that

$$\Gamma^1(g_i)M = M\Gamma^2(g_i) \quad \text{for all } g_i \tag{1.6}$$

1. if $\ell_1 \neq \ell_2$ then $M = 0$
2. if $\ell_1 = \ell_2$ either $M = 0$ or $\Gamma^{(1)} \equiv \Gamma^{(2)}$.

Proof:

Again because of Theorem #1 assume Γ^f to be unitary matrices. Then

$$M^+ \Gamma^{1+}(g_i) = \Gamma^{2+}(g_i)M^+$$

but

$$\Gamma^+(g_i) = \Gamma(g_i^{-1})$$

$[\Gamma^+(g_i) = \Gamma(g_i)^{-1}$ - unitary property of Γ

$= \Gamma(g_i^{-1})$ - since $I = \Gamma(E) = \Gamma(g_i g_i^{-1})$

$= \Gamma(g_i)\,\Gamma(g_i^{-1})$

and hence $\Gamma(g_i^{-1}) \equiv (\Gamma(g_i))^{-1}$.]

Thus

$$M^+ \Gamma^1(g_i^{-1}) = \Gamma^2(g_i^{-1})\, M^+$$

Multiplying both sides by M and using eq. 1.6 we get

$$MM^{+}\Gamma^{1}(g_i^{-1}) = \Gamma^{1}(g_i^{-1})MM^{+}$$

Thus MM^{+} is a matrix that commutes with the representation matrices Γ^{1} and if these are irreducible then MM^{+} is a constant (Theorem #2)

$$MM^{+} = d\ I$$

if $\ell_1 = \ell_2$ then the determinant $|MM^{+}| = |M|^2 = d^{\ell_1}$. If $d \neq 0$ then $|M| \neq 0$ and M has an inverse. Therefore

$$\Gamma^{1}(g_i) = M\Gamma^{2}(g_i)M^{-1}$$

and the representations are <u>equivalent</u>.

If $d = 0$, then $M \equiv 0$ for $\ell_1 = \ell_2$.

If $\ell_1 \neq \ell_2$ [let us suppose $\ell_2 < \ell_1$] then M has ℓ_1 rows and ℓ_2 columns and MM^{+} is an $\ell_1 \times \ell_1$ square matrix. Construct the square matrix N from M by adding $(\ell_1 - \ell_2)$ columns of zeros. Clearly the determinant of N is zero and hence the determinant of $NN^{+} = 0$. But $NN^{+} \equiv MM^{+}$ and hence $MM^{+} = 0$. Since $MM^{+} = dI$ this proves that $d = 0$.

Theorem #4

The Orthogonality Theorem.

[Note: this is probably the most important theorem of representation theory and must be remembered].

$$\sum_i \Gamma^f_{yy'}(g_i)\, \Gamma^{\bar{f}*}_{\bar{y}\bar{y}'}(g_i) = \delta_{f\bar{f}}\, \delta_{y\bar{y}}\, \delta_{y'\bar{y}'}\, h/\ell_f$$

where h is the dimension of the group (i.e. # of group elements)

ℓ_f " " " irrep f.

Proof: with an arbitrary $\ell_{\bar{f}} \times \ell_f$ rectangular matrix X construct

$$M = \sum_i \Gamma^f(g_i)\, X\, \Gamma^{\bar{f}}(g_i^{-1})$$

Now

$$\Gamma^f(g_j)\, M = \sum_i \Gamma^f(g_j)\, \Gamma^f(g_i)\, X\, \Gamma^{\bar{f}}(g_i^{-1})\, \Gamma^{\bar{f}}(g_j^{-1})\, \Gamma^{\bar{f}}(g_j)$$

$$= \Big[\sum_i \Gamma^f(g_j g_i)\, X\, \Gamma^{\bar{f}}(g_j g_i)^{-1}\Big]\, \Gamma^{\bar{f}}(g_j)$$

$$= M\, \Gamma^{\bar{f}}(g_j)$$

Hence from Theorem #3 if $f \neq \bar{f}$ then $M \equiv 0$

i.e. $$M_{\alpha\beta} = \sum_i \Gamma^f_{\alpha\gamma}(g_i) X_{\gamma\delta}\, \Gamma^{\bar{f}}_{\delta\beta}(g_i^{-1}) = 0$$

Since X is arbitrary it can take for a particular γ,δ $X_{\gamma\delta} = 1$, but zero otherwise. Hence

$$\sum_i \Gamma^f_{\alpha\gamma}(g_i)\ \Gamma^{\bar{f}}_{\delta\beta}(g_i^{-1}) \equiv 0$$

i.e.
$$\sum_i \Gamma^f_{\alpha\gamma}(g_i)\ \Gamma^{\bar{f}*}_{\beta\delta}(g_i) = 0$$

If $f = \bar{f}$ then $M = d.I.$ (if $\bar{f}$ is irrep, by #2)

$$\therefore \sum_i \Gamma^f_{\alpha\gamma}(g_i)\ X_{\gamma\delta}\ \Gamma^f_{\delta\beta}(g_i^{-1}) = d\ \delta_{\alpha\beta}$$

Again since $X_{\gamma\delta}$ is arbitrary choose it such that all $X_{\gamma\delta} = 0$ except for one element, e.g. $X_{\nu\omega} = 1$. Then

$$\sum_i \Gamma^f_{\alpha\nu}(g_i)\ \Gamma^f_{\omega\beta}(g_i^{-1}) = d\delta_{\alpha\beta}$$

choose $\alpha = \beta$ and sum over α

$$\sum_i \Gamma^f_{\omega\alpha}(g_i^{-1})\ \Gamma^f_{\alpha\nu}(g_i) = d\ \ell_f$$

$$= \sum_i \Gamma^f_{\omega\nu}(g_i^{-1}g_i) = \sum_i \Gamma^f_{\omega\nu}(E) = h\delta_{\omega\nu}$$

$$\therefore d = h\delta_{\omega\nu}/\ell_f$$

Thus

$$\sum_i \Gamma^f_{\alpha\nu}(g_i)\ \Gamma^f_{\omega\beta}(g_i^{-1}) = \delta_{\alpha\beta}\delta_{\nu\omega}\ h/\ell_f$$

or
$$\sum_i \Gamma^f_{\alpha\nu}(g_i)\ \Gamma^{\bar{f}*}_{\beta\delta}(g_i) = \delta_{f\bar{f}}\ \delta_{\alpha\beta}\ \delta_{\nu\omega}\ h/\ell_f.$$

Projection Operator

Consider the operator

$$T^{f}_{y',y} = \frac{\ell_f}{h} \sum_i \Gamma^{f}_{y\,y',}(g_i)\; g_i^{-1} \equiv \frac{\ell_f}{h} \sum_i \Gamma^{f}_{yy',}(g_i^{-1}) g_i \equiv \frac{\ell_f}{h} \sum_i \Gamma^{f^*}_{y',y}(g_i) g_i$$

Now

$$T^{f}_{y',y}\; \psi^{\overline{f}}_{\overline{y}} = \frac{\ell_f}{h} \sum_i \Gamma^{f}_{yy',}(g_{\_})\; \Gamma^{\overline{f}}_{\overline{y},\overline{y}}(g_i^{-1})\; \psi^{\overline{f}}_{\overline{y}'}$$

$$= \frac{\ell_f}{h} \sum_i \Gamma^{f}_{yy',}(g_i)\; \Gamma^{\overline{f}^*}_{\overline{y}\,\overline{y}',}(g_i)\; \psi^{\overline{f}}_{\overline{y}'}$$

$$= \delta_{f\overline{f}}\; \delta_{y\overline{y}}\; \delta_{y',\overline{y}'}\; \psi^{\overline{f}}_{\overline{y}'} = \delta_{f\overline{f}}\; \delta_{y\overline{y}}\; \psi^{\overline{f}}_{y'}$$

from the orthogonality theorem. Thus $T^{f}_{y',y}$ operating on the $\overline{y}$ function $\psi^{\overline{f}}_{\overline{y}}$ of the $\overline{f}$ representation gives zero unless $f = \overline{f}$ and $y = \overline{y}$ in which case it yields the y' function of the f representation.

If $T^{f}_{y',y}$ operates on any function ψ the answer is zero unless ψ contains a part transforming like the y function of the f representation: if it does the resulting function is the y' state f representation. Note that the operators $T^{f}_{y',y}$ are <u>idempotent</u>, i.e.

$$T^{f}_{y',y}\; T^{\overline{f}}_{\overline{y}',\overline{y}} = \delta_{f\overline{f}}\; \delta_{y\overline{y}'}\; T^{f}_{y',\overline{y}}$$

[Proof: $T^{f}_{y',y}\, T^{\bar{f}}_{\bar{y}',\bar{y}} = \frac{\ell_f \ell_{\bar{f}}}{h^2} \sum_{ij} \Gamma^{f}_{yy',}(g_i)\, \Gamma^{\bar{f}}_{\bar{y}\,\bar{y}',}(g_j)\, g_i^{-1}\, g_j^{-1}$

writing

$$g_j g_i = g_k \quad \text{and} \quad g_j = g_k g_i^{-1}$$

we have

$$T^{f}_{y',y}\, T^{\bar{f}}_{\bar{y}',\bar{y}} = \frac{\ell_f \ell_{\bar{f}}}{h^2} \sum_{k} \sum_{i} \Gamma^{f}_{yy',}(g_i)\, \Gamma^{\bar{f}}_{\bar{y}\,\bar{y}',}(g_k g_i^{-1})\, g_k^{-1}$$

$$= \frac{\ell_f \ell_{\bar{f}}}{h^2} \sum_{k} \Big[\sum_{i} \Gamma^{f}_{yy',}(g_i)\, \Gamma^{\bar{f}^*}_{\bar{y}\,\alpha}(g_i)\Big]\, \Gamma^{\bar{f}}_{\bar{y}\alpha}(g_k)\, g_k^{-1}$$

$$= \frac{\ell_f}{h}\, \delta_{f\bar{f}}\, \delta_{y\bar{y}'} \sum_{k} \Gamma^{\bar{f}}_{yy',}(g_k)\, g_k^{-1}$$

- using the orthogonality theorem

$$= \delta_{f\bar{f}}\, \delta_{y\bar{y}'}\, T^{\bar{f}}_{y',\bar{y}}$$

Theorem #5

Functions, belonging to different representations or to different rows of the same representation are orthogonal, i.e.

$$\langle \psi_y^f | \phi_{\bar{y}}^{\bar{f}} \rangle = c\, \delta_{f\bar{f}}\, \delta_{y\bar{y}}$$

Proof

$$g_i\, \psi_y^f = \Gamma^f_{y',y}(g_i)\psi_{y'}^f,$$

and $$g_i\, \psi_{\bar{y}}^{\bar{f}} = \Gamma^{\bar{f}}_{\bar{y}',\bar{y}}(g_i)\, \psi_{\bar{y}'}^{\bar{f}},$$

$$\sum_i \langle g_i\, \psi_{\bar{y}}^{\bar{f}} \mid g_i\, \psi_y^f \rangle = \sum_i \langle \psi_{\bar{y}}^{\bar{f}} \mid g_i^+ g_i\, \psi_y^f \rangle$$

$$= \sum_i \langle \psi_{\bar{y}}^{\bar{f}} \mid g_i^{-1} g_i\, \psi_y^f \rangle$$

$$= h \langle \psi_{\bar{y}}^{\bar{f}} \mid \psi_y^f \rangle$$

$$= \sum_i \langle \Gamma^{\bar{f}}_{\bar{y}',\bar{y}}(g_i)\, \psi_{\bar{y}'}^{\bar{f}} \mid \Gamma^f_{y',y}(g_i)\, \psi_{y'}^f \rangle$$

$$= \sum_i \langle \psi_{\bar{y}'}^{\bar{f}} \mid \Gamma^{\bar{f}*}_{\bar{y}',\bar{y}}(g_i)\, \Gamma^f_{y',y}(g_i) \mid \psi_{y'}^f \rangle$$

$$= \delta_{f\bar{f}}\, \delta_{y',\bar{y}}\, \delta_{y\bar{y}}\, \frac{h}{\ell_f} \langle \psi_{\bar{y}'}^{\bar{f}} \mid \psi_{y'}^f \rangle$$

$$\langle \psi_{\bar{y}}^{\bar{f}} | \psi_y^f \rangle = \delta_{f\bar{f}}\, \delta_{y\bar{y}} \sum_{y'} \langle \psi_{y'}^f \mid \psi_{y'}^f \rangle / \ell_f$$

Clearly if all states of a representation are also normalized the latter equation reduces to

$$\langle \psi^{\bar{f}}_{\bar{y}} | \psi^{f}_{y} \rangle = \delta_{f\bar{f}} \, \delta_{y\bar{y}}$$

- another important result that should be remembered.

PART II

THE SYMMETRIC GROUP S_n

The symmetric group is the set of all permutations of n objects (say α^j, j=1, ···n). The general operator for this group has the form of products like $P_{i_1 i_2 \cdots i_k}$ where

$$P_{i_1 i_2 \cdots i_k} |\alpha^1_{i_1} \alpha^2_{i_2} \cdots \alpha^k_{i_k} \; \alpha^{k+1}_{i_{k+1}} \cdots \alpha^n_{i_n}\rangle$$

$$= |\alpha^1_{i_2} \; {}^2_{i_3} \cdots \alpha^k_{i_1} \; \alpha^{k+1}_{i_{k+1}} \cdots \alpha^n_{i_n}\rangle \qquad i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_n$$
-selected from the set of numbers (1,2,···n)

Note that any such operator can be written as products of transpositions

$$P_{i_1 i_2 \cdots i_k} = P_{i_1 i_2} P_{i_2 i_3} \cdots P_{i_{k-1} i_k}$$

Note also that any transposition can be written as products of <u>local</u> transpositions of adjacent numbers, i.e.

$$\text{if } i < j, \quad P_{ij} = P_{i,i+1} \, P_{i+1,i+2} \cdots P_{j-2,j-1} \, P_{j-1,j} \, P_{j-2,j-1} \cdots P_{i,i+1}$$

Thus if we know the matrix representation of the local transposition operators we can construct, by matrix multiplication, the representation matrices for all other operators.

For S_n therefore we need to know how to construct irreducible matrix representations for I, P_{12}, $P_{23}, \cdots P_{n-2,n-1}, P_{n-1,n}$. Note that for S_{n-1} we need the matrix representation for I, P_{12}, $P_{23} \cdots P_{n-2,n-1}$. The standard Young representation for the matrix representations of S_n is such that they are made up of irreducible representations of $S_2, S_3, \cdots S_{n-1}$. Only the matrix representation for $P_{n-1,n}$ has to be declared in going from S_{n-1} to S_n.

Before giving the method of classifying representations of S_n, or of giving the standard representation matrices for the local transpositions, let us first look at some examples which will give the somewhat abstract presentation up to now some reality.

In Part I (page 5) we have already deduced, for S_2, two irreps which are carried, for example, by basis states $\phi_+ = \sqrt{1/2}\ (\alpha_1\beta_2 + \beta_1\alpha_2)$ and $\phi_- = \sqrt{1/2}\ (\alpha_1\beta_2 - \beta_1\alpha_2)$. Assuming α and β are orthogonal single particle states, the matrices for E and P_{12} are:

for $\phi_+ = \sqrt{1/2}(\alpha\beta+\beta\alpha)$: $\Gamma(E) = [1]$ $\Gamma(P_{12}) = [1]$ - the s-irrep

for $\phi_- = \sqrt{1/2}(\alpha\beta-\beta\alpha)$: $\Gamma(E) = [1]$ $\Gamma(P_{12}) = [-1]$ - the a-irrep

Note that ϕ_+ and ϕ_- are orthogonal which they must be according to our Theorem #5 (page 20). This statement is true even if α and β are <u>not</u> orthogonal, i.e. $\langle\alpha|\beta\rangle \neq 0$. In this latter

case however the functions $\phi_\pm$ should be defined

$$\phi_\pm = \sqrt{\frac{1}{2(1\pm\langle\alpha|\beta\rangle}}\ (\alpha\beta \pm \beta\alpha)$$

in order that the representation matrices have the Unitary character shown. Thus single-particle orbits do not have to be orthogonal to classify many-particle states according to the irreps of S_n (cf. ref. 13,14). We shall assume orthogonality of single-particles states in what follows, however, since it is then less cumbersome in the writing of many-particle functions.

Let us now consider three-particle states in which two of the particles are in the α-orbit and one in the β-orbit. Clearly we can immediately write down a symmetric function

$$\psi = \sqrt{1/3}\ (\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha) \qquad (2.1a)$$

$$\equiv \sqrt{1/3}\ \alpha\alpha\beta + \sqrt{2/3}\ \sqrt{1/2}\ (\alpha\beta+\beta\alpha)\alpha \qquad (2.1b)$$

$$\equiv \sqrt{2/3}\ \alpha\ \sqrt{1/2}\ (\alpha\beta+\beta\alpha) + \sqrt{1/3}\ \beta\alpha\alpha \qquad (2.1c)$$

This state carries the one-dimension irrep of S_3 for which the local transposition matrices and identity matrices are

$$\Gamma(E) = [1] \qquad \Gamma(P_{12}) = [1] \qquad \Gamma(P_{23}) = [1]$$

There are three components to the state ψ and therefore we can write down two orthogonal states to ψ. These states will belong to a different irrep (or irreps) of S_3 since they are uncoupled to ψ by the local transposition operators (Theorem #5 - page 20). How are these states to be defined?

The standard method of Young is to define the states such that they belong to an irrep of S_2. In the form of writing ψ in eq. 2.1b the symmetry of ψ with respect to S_2 is made obvious - since $\alpha\alpha$ and $\sqrt{1/2}(\alpha\beta+\beta\alpha)$ both transform according to the s-representation of S_2. Clearly we can write down an orthogonal state to ψ that also transforms according to the s representation of S_2 namely

$$\Omega_1 = \sqrt{2/3}\ \alpha\alpha\beta - \sqrt{1/3}\ \sqrt{1/2}(\alpha\beta+\beta\alpha)\alpha$$

This state does not define a one-dimensional representation of S_3 because, although $P_{12}\Omega_1 = +1\Omega_1$, we find

$$P_{23}\Omega_1 = -\frac{1}{2}\,\Omega_1 \pm \frac{1}{2}\sqrt{3}\,\Omega_2$$

with

$$\Omega_2 = \pm\sqrt{1/2}(\alpha\beta-\beta\alpha)\alpha$$

The two functions Ω_1 and Ω_2 carry a two-dimensional representation of S_3 since $P_{12}\Omega_2 = -1\Omega_2$ and $P_{23}\Omega_2 = +\frac{1}{2}\Omega_2 \pm \frac{1}{2}\sqrt{3}\,\Omega_1$. The representation matrices for the transpositions are thus

$$\Gamma(P_{12}) = \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} \qquad \Gamma(P_{23}) = \begin{bmatrix} -\frac{1}{2} & \frac{\pm\sqrt{3}}{2} \\ \frac{\pm\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

The ambiguity in sign for the off-diagonal matrix element of $\Gamma(P_{23})$ is a result of the freedom of choice of relative phase between Ω_1 and Ω_2. Note incidentally that Ω_2 transforms like the a-representation of S_2.

YOUNG TABLEAUS AND YAMANOUCHI SYMBOLS

Clearly we would like a better way of classifying the representations of S_n which tell us more about the structure of the representation than the arbitrary symbols s, a, ψ, Ω that we used in the previous section to characterise irreps for S_2 and S_3. The Young Tableau is such a classification. In this scheme a particle is denoted by a square $\square$. A two-particle symmetric state is characterised by two squares in a row $\square\square$. A two-particle antisymmetric state is characterised by two squares in a column $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. A particular state is characterised by putting (in general) the numbers from 1 to n in the tableau such that they increase in both rows and columns. Thus in our examples

$$s \equiv \square\square \qquad \text{and} \qquad \phi_+ = \boxed{1}\boxed{2}$$

$$a \equiv \begin{smallmatrix}\square\\\square\end{smallmatrix} \qquad \text{and} \qquad \phi_- = \begin{smallmatrix}1\\2\end{smallmatrix}$$

The three-particle state ψ is symmetric with respect to both P_{12} and P_{23} and therefore is characterised by the symbol $\boxed{1}\boxed{2}\boxed{3}$. The state Ω_1 is symmetric w.r.t. P_{12} and therefore the numbers 1 and 2 must appear in the tableau that characterises the Ω rep in a symmetric way. The state Ω_2 is antisymmetric w.r.t. P_{12} and therefore the numbers 1 and 2 must appear in the tableau that characterises the Ω rep in an antisymmetric

way. We see that the tableau  satisfies both these criteria with

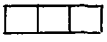
$\Omega_1 \equiv$ 

and

$\Omega_2 =$ 

The shape of the tableau characterises the representation. The numbers within the tableau characterise the state of the representation. There are only two ways of putting the numbers 1, 2 and 3 in the  tableau with numbers increasing in rows and columns and therefore this correctly characterises the two dimensional irrep.

Rather than always draw the tableau we can simply state the number of squares in each row thus

$[3] =$ 

$[21] =$ 

$[111] = [1^3] =$ 

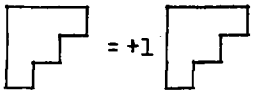
The particular state of a representation can be classified by a numbered tableau, e.g.  , or by declaring (from right to left) in which row the numbers 1, 2, ••• n (in general) appear.

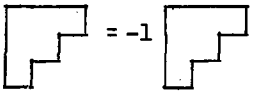
Thus $(211) \equiv$ 

$(121) \equiv$ 

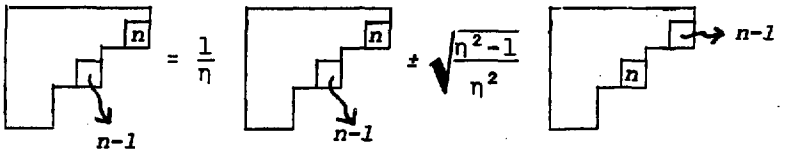
The latter are referred to as the Yamanouchi symbols of the [21] Young Tableau.

Given the Yamanouchi labelling of a Young Tableau we can easily construct the matrix representation for the local transposition $P_{n-1,n}$. The general rules are as follows.[1,6)]

1. If n-1 and n are in the same row $P_{n-1,n}$  (2.2a)

2. If n-1 and n are in the same column $P_{n-1,n}$  (2.2b)

3. If n-1 and n are in different rows and columns of a tableau then

$P_{n-1,n}$  (2.2c)

where $|\eta|$ is the number of lines crossed in the tableau (moving along rows and columns) in going from n-1 to n and the sign of η is +(-) if the move from n-1 to n is in a (anti-) clockwise direction. One can immediately verify that

this general rule yields in the particular case of the [21] irrep of S_3 for P_{23}

$$\begin{matrix} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \\ \\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \end{matrix} \begin{bmatrix} -\frac{1}{2} & \pm\frac{\sqrt{3}}{2} \\ \\ \pm\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{bmatrix}.$$

as deduced on page 26. The standard relative phase of functions in a representation is chosen such that the + sign appears in the definition eq. 2.2c.

Problems

1. Write down and classify according to S_3 the six states formed by putting three particles in three orbits α, β and γ.
2. Write down all the Young Tableau for all the irreps of S_4 and state their dimensions.
3. Construct the matrix representations for P_{12}, P_{23} and P_{34} for the irrep [31] of S_4.

Notation (See also Appendix B)

I always try to use the same notation so let us review this:

f - designation of a Young Tableau for S_n;

Y - a Yamanouchi label for the Young Symmetry f;

Y=(pqy) - a Yamanouchi label in which particle number n is in row p, particle n-1 is in row q and the remaining particles 1. 2, ... n-2 have a Yamanouchi distribution y.

The results of eq. 2.2 will be expressed in general by the expressions

$$P_{n\,n-1}|f(pqy)> = \alpha^f_{pq}|f(pqy)> + \beta^f_{pq}|f(qpy)> \qquad (2.3)$$

with α^f_{pq} and β^f_{pq} taking the values given in eq. 2.2 according to the case.

Note: $(\alpha^f_{pq})^2 + (\beta^f_{pq})^2 = 1$

and

$$\alpha^f_{pq} = -\alpha^f_{qp} \qquad \text{if } \beta^f_{pq} \neq 0.$$

THE ADJOINT REPRESENTATION $\bar{f}$

There is some ambiguity in the literature as to what exactly is meant by the Adjoint representation. I define this to mean the representation in which the β^f_{pq} coefficients in eq. 2.3 have negative sign. This representation is carried by the same set of basis states as for the standard representation but with some of the relative phases among the states of the representation changed.

THE DUAL REPRESENTATION

Given a Young Symmetry f, the Dual Symmetry is the one in which rows and columns have been interchanged. Thus

Symmetry f		Dual Symmetry $\tilde{f}$	
[2]		[11]	
[11]		[2] =	
[21] =		[21] =	

In the latter case we call the [21] symmetry self-dual. Note however that the dual of the state [211] = $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ of the [21] representation is the different state

(121) = $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$.

The Dual Symmetry can be declared in the standard representation or the adjoint representation. (N.B. In ref.9, for example, the terminology adjoint means both what I have called adjoint and dual - be careful!!).

Inner Product (Clebsch-Gordan Expansions)

Suppose we have two (or more) spaces in which the particles 1, 2, ... n have states defined. For example, we could declare the orbital wave functions of n particles, the spin wave function, the flavor wave function, the color wave function, etc. In each space we can declare the symmetry with respect to transformations with respect to S_n.
The question now is to find the symmetries of the combined spaces. For two spaces in which we have defined states $\Phi(f'Y')$ and $\Gamma(f''Y'')$, in general

$$\Phi(f'Y')\Gamma(f''Y'') = \sum_{f(Y)} (f'Y'f''Y''|fY)\ \Psi(fY) \qquad (2.4a)$$

where Ψ is the combined $\Phi\Gamma$ space. The coefficients $(f'Y'f''Y''|fY)$ are the Clebsch-Gordan coefficients for S_n.
The eq. 2.4a has an inverse

$$\Psi(fY) = \sum_{Y'Y''} (f'Y'f''Y''|fY)\ \Phi(f'Y')\Gamma(f''Y'') \qquad (2.4b)$$

Let us examine this expansion in some special cases:

S_2. $\psi([2](11)) = \sqrt{\tfrac{1}{2}}\,(\psi_\alpha\psi_\beta+\psi_\beta\psi_\alpha)$:$\phi([2](11)) = \sqrt{\tfrac{1}{2}}\,(\phi_\alpha\phi_\beta+\phi_\beta\phi_\alpha)$

$\psi([11](21)) = \sqrt{\tfrac{1}{2}}\,(\psi_\alpha\psi_\beta-\psi_\beta\psi_\alpha)$:$\phi([11](21)) = \sqrt{\tfrac{1}{2}}\,(\phi_\alpha\phi_\beta-\phi_\beta\phi_\alpha)$

Clearly

$$\Psi([2](11))_1 = \psi([2](11))\phi([2](11)) \qquad (2.5a)$$

$$\Psi([2](11))_2 = \psi([11](21))\phi([11](21)) \qquad (2.5b)$$

$$\Psi([11](21))_1 = \psi([2](11))\phi([11](21)) \qquad (2.5c)$$

$$\Psi([11](21))_2 = \psi([11](21))\ \phi([2](11)). \qquad (2.5d)$$

Thus

$$([2](11)[2](11)|[2](11)) = 1$$

$$([11](21)[11](21)|[2](11)) = 1$$

$$([2](11)[11](21)|[11](21)) = 1$$

$$([11](21)[2](11)|[11](21)) = 1$$

We have had to distinguish two different states in eqs. 2.5a and b with subscripts because the overall symmetry is not enough to completely specify the states. In this situation it is usual to declare the symmetries of the subspaces. Thus

$$\Psi([2](11))_1 \equiv \Psi([[2][2]]\ [2](11))$$

$$\Psi([11](21))_1 \equiv \Psi([[2][11]]\ [11](21)) \quad \text{etc.}$$

with the symmetries in the ψ and ϕ spaces being given in that order.

In general one has to work hard to get the CG-coefficients of S_n (see Appendix B), but they are simple (as above) in a few cases.

Example 1:

Consider the state $\Psi = \sqrt{\frac{1}{d_f}} \sum_Y \psi(fY)\phi(fY)$

$$\equiv \sqrt{\frac{1}{d_f}} \sum_{pqy} \psi(f(pqy))\phi(f(pqy))$$

where d_f is the dimension of the representation f. Now

$$P_{n,n-1}\,\Psi = \sqrt{\frac{1}{d_f}} \sum_{pqy} (\alpha^f_{pq}\, \psi(f\,pqy) + \beta^f_{pq}\, \psi(f(qpy)))$$

$$(\alpha^f_{pq}\phi(f\,pqy) + \beta^f_{pq}\, \phi(f(qpy)))$$

$$= \sqrt{\frac{1}{d_f}} \sum_{pqy} \{[(\alpha^f_{pq})^2 + (\beta^f_{pq})^2]\}\psi(f(pqy))\phi(f(pqy))$$

$$+ [\alpha^f_{pq}\, \beta^f_{pq} + \beta^f_{pq}\, \alpha^f_{qp}]\, \psi(f\,pqy)\phi(f(qpy))$$

$$\equiv \sqrt{\frac{1}{d_f}} \sum_{pqy} \psi(f(pqy))\phi(f(pqy))$$

$$\equiv \psi$$

since $(\alpha^f_{pq})^2 + (\beta^f_{pq})^2 = 1$ and $\alpha^f_{pq} = -\alpha^f_{qp}$.

Since the states $\psi(fy)$ and $\phi(fy)$ are defined in the standard Young representation then the above proof holds for all subgroups S_2, S_3, $\cdots S_{n-1}$ of S_n. Thus Ψ is symmetric w.r.t. all local transpositions and hence is a completely Symmetric function, i.e. transforms like the Young Tableau

$[n] =$  (n boxes)

Hence $(fYfY|[n]) \equiv 1$.

Example 2:

Consider the state:

$$\Psi = \sqrt{\frac{1}{d_f}} \sum_{Y(\tilde{Y})} \psi(fY)\phi(\tilde{\tilde{f}}\tilde{\tilde{Y}})$$

$$= \sqrt{\frac{1}{d_f}} \sum_{pqy} \psi(f(pqy))\phi(\tilde{\tilde{f}}\,\widetilde{\widetilde{pqy}}) \qquad (2.7)$$

Again

$$P_{n\ n-1}\Psi = \sqrt{\frac{1}{d_f}} \sum_{pqy} [\alpha^f_{pq}\,\psi(f(pqy)) + \beta^f_{pq}\,\psi(f\ qpy)]*$$

$$[-\alpha^f_{pq}\,\phi(\tilde{\tilde{f}}(\widetilde{\widetilde{pqy}})) - \beta^f_{pq}\,\phi(\tilde{\tilde{f}}(\widetilde{\widetilde{qpy}}))]$$

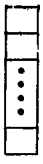
$$= \sqrt{\frac{1}{d_f}} \sum_{pqy} \{[-(\alpha^f_{pq})^2 - (\beta^f_{pq})^2]\,\psi(f(pqy))\phi(\tilde{\tilde{f}}(pqy))$$

$$[-\alpha^f_{pq}\beta^f_{pq} - \beta^f_{pq}\alpha^f_{qp}]\,\psi(f(pqy))\phi(\tilde{\tilde{f}}(qpy))$$

$$\equiv -\sqrt{\frac{1}{d_f}} \sum_{pqy} \psi(f(pqy))\phi(\tilde{\tilde{f}}\ pqy)$$

$$= -\psi$$

Again since $\Psi(f(pqy))$ and $\phi(\hat{\tilde{f}}\ \widehat{\widetilde{pqy}})$ also belong to irreducible representations of $S_2, S_3, \cdots S_{n-1}$, the function Ψ is antisymmetric w.r.t. all local transposition operators P_{12}, P_{23}, $\cdots$ $P_{n-2\ n-1}$. Hence Ψ transforms like the antisymmetric representation

$[1^n]$  (n boxes)

Example 3:

Prove that if $\Psi = \phi([n](1^n))\psi([f]Y)$ then Ψ transforms like the Y Yamanouchi label for the symmetry [f]. Note that we could consider ϕ to be a symmetric operator.

Example 4:

Prove that if $\Psi = \psi([f]Y)\phi([1^n](n,n-1\cdots 1))$ then Ψ transforms like the $\tilde{Y}$ Yamanouchi label for the symmetry $[\tilde{f}]$.

THE RELEVANCE OF S_n TO FERMION PHYSICS

- The Use of Antisymmetric Functions

I find even among experienced theorists a misunderstanding as to why we use antisymmetric functions to describe collections of fermions. It is true that the Pauli principle prevents any two "like" particles being at the same space-time point or, equivalently, in the same state. The Pauli principle doesn't say anything about unlike particles however. Thus an electron with spin-up and another electron with spin-down can both be in the state ϕ_α. The point is that there is only one such physical state which we could write $\phi_\alpha^\uparrow \phi_\alpha^\downarrow$. We could also write the state $\phi_\alpha^\downarrow \phi_\alpha^\uparrow$ but we must not consider both ways of writing the state otherwise we would be double counting. Thus we could declare a system of states by declaring an order (spin-up states are written first and spin-down second, for example). Such an approach is not useful for algebraic manipulations. For this we declare the physical state to be described by the antisymmetric combination $\sqrt{\frac{1}{2}}(\phi_\alpha^\uparrow \phi_\alpha^\downarrow - \phi_\alpha^\downarrow \phi_\alpha^\uparrow)$ and consider the symmetric combination $\sqrt{\frac{1}{2}}(\phi_\alpha^\uparrow \phi_\alpha^\downarrow + \phi_\alpha^\downarrow \phi_\alpha^\uparrow)$ to be redundant. The advantage of the scheme of representing collections of fermions by antisymmetric states only is that we never double count and, if we use algebra for a state, e.g. $\sqrt{\frac{1}{2}}(\phi_\alpha^s \phi_\alpha^{s'} - \phi_\alpha^{s'} \phi_\alpha^s)$ for undeclared spins s and s', we find that when $s = s'$, the state vanishes, i.e. the Pauli principle is obeyed. The antisymmetric representation has

particular advantage when we consider symmetries with respect to transformations in selected subspaces, e.g. spin, or isospin or color. If the transformations are such as to give two particles exactly the same quantum numbers, the antisymmetry of the total state will guarantee that the function will disappear, i.e. ensure the Pauli principle is obeyed. Even when there are no like particles the use only of antisymmetric states ensures no double counting.

We showed in the previous section how to construct antisymmetric states from two spaces (say spin and orbital) by taking the adjoint/dual symmetry in one space with the standard symmetry in the other. For more than two spaces we have to use in principle the Clebsch-Gordan series to construct from one set of spaces a function of adjoint/dual symmetry to the (Clebsch-Gordon summed) symmetry of the remaining functions (see ref. 13 and 14 for an example in the case of six quark states).

CALCULATION OF ONE-BODY MATRIX ELEMENTS WITH ANTISYMMETRIC FUNCTIONS

Let Ψ_n and $\overline{\Psi}_n$ be two antisymmetric functions of n particles. And let $T = \sum_i T(i)$ be a symmetric one-body operator. We can write

$$\langle \Psi_n | T | \overline{\Psi}_n \rangle = n \langle \Psi_n | T(n) | \overline{\Psi}_n \rangle \tag{2.8}$$

Any antisymmetric function of n particles can be written as a sum of products of an antisymmetric function of the first (n-1) particles with the state of the last particle, i.e.

$$\Psi_n = \sum_{\alpha\beta} C_{\alpha\beta} \Psi_{n-1}(\alpha)\psi(\beta) \tag{2.9}$$

where α denotes the structure of the function of (n-1) particles and β that of the remaining particle (as selected from the n-particle states of Ψ_n) and $C_{\alpha\beta}$ are the fractional parentage coefficients.

Example:

$$\text{if } \Psi_3 = \sqrt{\frac{1}{6}} (\alpha\beta\gamma - \alpha\gamma\beta + \gamma\alpha\beta - \gamma\beta\alpha + \beta\gamma\alpha - \beta\alpha\gamma)$$

$$\equiv \sqrt{\frac{1}{3}} (\widetilde{\alpha\beta}\ \gamma + \widetilde{\beta\gamma}\ \alpha + \widetilde{\gamma\alpha}\ \beta)$$

where

$$\widetilde{\alpha\beta} = \sqrt{\frac{1}{2}} (\alpha\beta - \beta\alpha) \text{ etc.}$$

In this case

$$C_{\alpha\beta,\gamma} \equiv C_{\beta\gamma,\alpha} \equiv C_{\gamma\alpha,\beta} \equiv \sqrt{\frac{1}{3}}$$

Combining eqs. 2.8 and 2.9 we can write

$$\langle \Psi_n | T | \bar{\Psi}_n \rangle = n \sum_{\substack{\alpha\beta \\ \bar{\alpha}\,\bar{\beta}}} C_{\alpha\beta} C_{\bar{\alpha}\,\bar{\beta}} \langle \Psi_{n-1}(\alpha) | \bar{\Psi}_{n-1}(\bar{\alpha}) \rangle \langle \psi(\beta) | T | \bar{\psi}(\bar{\beta}) \rangle$$

Thus the matrix element of a single-body operator between n-particle antisymmetric states can be written in terms of sums of products of fractional parentage coefficients, overlaps of (n-1)-particle functions and single-particle matrix elements. The extension to m-body operators is now obvious

$$\langle \Psi_n | T(m) | \bar{\Psi}_n \rangle = \binom{n}{m} \sum_{\substack{\alpha\beta \\ \bar{\alpha}\,\bar{\beta}}} C_{\alpha\beta} C_{\bar{\alpha}\,\bar{\beta}} \langle \Psi_{n-m}(\alpha) | \Psi_{n-m}(\bar{\alpha}) \rangle \langle \psi_m(\beta) | T(m) | \bar{\Psi}_m(\beta) \rangle$$

where now $\binom{n}{m}$ is a binomial coefficient; the $C_{\alpha\beta}$ now stand for the fractional parentage coefficients reducing an n-particle antisymmetric state to sums of products of (n-m)-particle antisymmetric states with m-particle antisymmetric states; the overlap function is now for (n-m)-particle states and we have to know m-body matrix elements for T_m.

FRACTIONAL PARENTAGE COEFFICIENTS IN FACTORED FORM

If antisymmetric states are constructed in factored form similar to eq. 2.7 (page 36), e.g.

$$\Psi_n([f]) = \sqrt{\frac{1}{d_f}} \sum_{py} \Phi_n([f](py))\Omega_n([\tilde{f}](\widetilde{py})) \qquad (2.11)$$

then the fractional parentage coefficients can also be written in factored form. Let us consider an example and then we can write down the general formula.

Example

Earlier we showed that three particle functions of [21] symmetry could be written

$$\Phi_3([21](211)) = \sqrt{\frac{2}{3}}\,\alpha\alpha\,\beta - \sqrt{\frac{1}{3}}\sqrt{\frac{1}{2}}\,(\alpha\beta+\beta\alpha)\alpha$$

$$\equiv \sqrt{\frac{2}{3}}\,\alpha^2\beta - \sqrt{\frac{1}{3}}\,\overline{\alpha\beta}\,\alpha \quad \text{with} \quad \overline{\alpha\beta} = \sqrt{\frac{1}{2}}(\alpha\beta+\beta\alpha)$$

and

$$\Phi_3([21](121)) = \widetilde{\alpha\beta}\,\alpha \quad : \quad \widetilde{\alpha\beta} = \sqrt{\frac{1}{2}}\,(\alpha\beta-\beta\alpha)$$

Let us consider similar types of states in the Ω-space, i.e.

$$\Omega_3([\tilde{21}](\overline{211})) = \sqrt{\frac{2}{3}}\,a^2b - \sqrt{\frac{1}{3}}\,\overline{ab}\,a$$

and

$$\Omega_3([\tilde{21}](\overline{121})) = -\,\widetilde{ab}\,a$$

The antisymmetric state of three particles has the form

$$\Psi_3 = \sqrt{\tfrac{1}{2}}\,[\phi_3([21](211))\Omega_3([21]\overline{121}) + \phi_3([21](121)\Omega_3[21]\overline{211})]$$

$$\equiv \sqrt{\tfrac{1}{2}}\,[(\sqrt{\tfrac{2}{3}}\,\alpha^2\beta - \sqrt{\tfrac{1}{3}}\,\overline{\alpha\beta}\;\alpha)(-\widetilde{a\bar{b}}\;a) + (\widetilde{\alpha\beta}\;\alpha)(\sqrt{\tfrac{2}{3}}\,a^2b - \sqrt{\tfrac{1}{3}}\,\bar{a}b\;a)]$$

$$\equiv \sqrt{\tfrac{1}{2}}\,[\;+\sqrt{\tfrac{2}{3}}\,(-1)(\alpha^2\widetilde{a\bar{b}})(\beta a) - \sqrt{\tfrac{1}{3}}\,(-1)(\overline{\alpha\beta}\;\widetilde{a\bar{b}})(\alpha a)$$

$$(+1)\sqrt{\tfrac{2}{3}}\,(\widetilde{\alpha\beta}\;a^2)\alpha b + 1(-\sqrt{\tfrac{1}{3}})(\widetilde{\alpha\beta}\;\overline{ab})(\alpha a)]$$

Clearly Ψ_3 has been written in terms of antisymmetric states of the first two particles with the last particle having the Greek/Roman structure (βa) or (αa) or (αb) or (αa). The cfp's have factored forms, e.g. $(\sqrt{\frac{2}{3}})(-1)$, with each factor arising from the one-body reduction of each symmetry. Matrix elements of a one-body operator now have the form

$$(\Psi_3|T_1|\Psi_3') = 3\;\tfrac{1}{2}\Big[\tfrac{2}{3}\,\langle\alpha^2ab|\alpha'^2a'b'\rangle\langle\beta a|T|\beta'a'\rangle$$

$$+\tfrac{1}{3}\,\langle\overline{\alpha\beta}\;\widetilde{ab}|\overline{\alpha'\beta'}\;\widetilde{a'b'}\rangle\langle\alpha a|T|\alpha'a'\rangle$$

$$-\tfrac{1}{3}\sqrt{2}\,\langle\alpha^2\widetilde{ab}|\overline{\alpha'\beta'}\;\widetilde{a'b'}\rangle\langle\beta a|T|\alpha'a'\rangle$$

$$-\tfrac{1}{3}\sqrt{2}\,\langle\overline{\alpha\beta}\;\widetilde{ab}|\alpha'^2\;\widetilde{a'b'}\rangle\langle\alpha a|T|\beta'a'\rangle$$

$$+\tfrac{2}{3}\,\langle\widetilde{\alpha\beta}\;a^2|\widetilde{\alpha'\beta'}\;a'^2\rangle\langle\alpha b|T|\alpha'b'\rangle$$

$$+\tfrac{1}{3}\,\langle\widetilde{\alpha\beta}\;\overline{ab}|\widetilde{\alpha'\beta'}\;\overline{a'b'}\rangle\langle\alpha a|T|\alpha'a'\rangle$$

continued ...

$$- \frac{1}{3}\sqrt{2}\,\langle\alpha\beta\; a^2\; \alpha'\beta'\; \overline{a'b'}\rangle\langle\alpha b|T|\alpha' a'\rangle$$

$$- \frac{1}{3}\sqrt{2}\,\langle\alpha\beta\; \overline{ab}|\alpha'\beta'\; a'^2\rangle\langle\alpha a|T|\alpha' b'\rangle\Big] \qquad (2.12)$$

Note in this last expression that even if the primed states are not orthogonal to the unprimed states, overlaps like $\langle\overline{\alpha\beta}|\widetilde{\alpha'\beta'}\rangle \equiv 0$ because of Theorem #5 (- states belonging to different representations are orthogonal - page 20).

Note if T_1 is replaced by $\sum_i 1 \equiv 3$ then for the diagonal matrix element $\langle\Psi_3|T|\Psi_3\rangle = 3$. [Check that the R.H.S. of eq. 2.12 does indeed give this number.] Such a check can always be made on a final result to ensure that mistakes have not been made. For a two-body operator note $\sum_{i<j} 1 \equiv \frac{n(n-1)}{2}$. For an m-body operator $\sum_{i<j<k\cdots} 1 \equiv \binom{n}{m}$.

For the general reduction of eq. 2.11 for a one-body operator we would write each factor

$$\Phi_n([f](py)) = \sum_{\alpha\beta} (\alpha[f_p]:\beta|\}[f])\Phi_{n-1}(\alpha[f_p]y)\phi_1(\beta) \qquad (2.13a)$$

where (: |}) denotes a fractional parentage coefficient for the symmetry [f] derived by removing one square from the p^1 -row such that the (n-1)-particle state has symmetry $[f_p]$ (cf.

Appendix A for notation) and Yamanouchi label y with general structure α. In the example we had

$$(\alpha^2[2]:\beta|\}[21]) = \sqrt{\frac{2}{3}}$$

$$(\overline{\alpha\beta}[2]:\alpha|\}[21]) = -\sqrt{\frac{1}{3}}$$

$$(\tilde{\alpha\beta}[11]:\alpha|\}[21]) = +1 \quad \text{etc.}$$

Similarly

$$\Omega_n([f]py) = \sum_{ab} (a[f_p]:b|\}[f])\Omega_{n-1}(a[f_p]\tilde{y})\omega_1(b) \tag{2.13b}$$

Substituting eqs. 2.13a and b into eq. 2.11 we get

$$\Psi_n([f]) = \sum_p \sqrt{\frac{d_{f_p}}{d_f}} \sum_{\alpha a} (\alpha[f_p]:\beta|\}[f])(a[f_p]:b|\}[f])$$

$$\Psi_{n-1}(\alpha a[f_p])\phi_1(\beta)\omega_1(b) \tag{2.14}$$

where the summation over the (n-1)-particle Yamanouchi label y in eq. 2.11 yields (with a normali ation coefficient d_{f_p}) the (n-1)-particle antisymmetric state

$$\Psi_{n-1}(\alpha a[f_p]) = \sqrt{\frac{1}{d_{f_p}}} \sum_y \Phi_{n-1}(\alpha[f_p]y)\,\Omega_{n-1}(a[\tilde{f_p}]\tilde{y}) \tag{2.15}$$

We have then in eq. 2.14 reduced the n-particle antisymmetric state to sums of products of cfp's (now in factored form), of an (n-1) antisymmetric state and the last particle.

PART III

THE UNITARY GROUPS U_m, SU_m

The unitary groups U_m is the set of unitary transformations among m objects. Any unitary matrix can be diagonalized by means of another unitary matrix S to yield eigenvalues with unit modulus. Thus

$$U = S^{-1}\begin{bmatrix} e^{i\phi_1} & & & \\ & e^{i\phi_2} & & \\ & & \ddots & \\ & & & e^{i\phi_m} \end{bmatrix} S$$

$$= \sum_n \frac{1}{n!} S^{-1}\begin{bmatrix} i\phi_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & i\phi_m \end{bmatrix}^n S$$

$$= \exp\left\{ iS^{-1}\begin{bmatrix} \phi_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \phi_m \end{bmatrix} S \right\} \qquad (3.1)$$

Since S is unitary and $\begin{bmatrix} \phi_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \phi_m \end{bmatrix}$ is a real diagonal matrix then $S^{-1}\begin{bmatrix} \phi & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \phi \end{bmatrix}S = H$ is Hermitian. If ϕ_i are restricted s.t. $0 < \phi_i \leq 2\pi$ then there is a 1:1 correspondence between the unitary transformation and the Hermitian matrix. When $H \equiv 0$ this corresponds to $U = I$.

The Hermitian matrix can always be written in terms of a set of m^2 linearly independent matrices

$$G_{ij\pm} = [(E_{ij} + E_{ji}) \pm i(E_{ij} - E_{ji})] \frac{1}{(1+\delta_{ij})}$$

where E_{ij} is the mxm matrix with zeros everywhere except at the ij element, where there is a 1. Thus we can always write a unitary matrix U as

$$U = \exp(i \sum_{\alpha} C_{\alpha} G_{\alpha}) \tag{3.2}$$

with the arbitrary coefficients C_{α}. When the C_{α} are <u>very</u> small we can write

$$U \approx 1 + i \sum_{\alpha} C_{\alpha} G_{\alpha} \tag{3.3}$$

The G_{α} are known as the infinitesimal operators of U_m or the <u>generating operators</u>.

The property that products of unitary matrices form a unitary matrix implies that the commutators of G_{α} are expandable in terms of the Generating matrices

$$[G_{\alpha}, G_{\beta}] = \sum_{\gamma} \varepsilon_{\alpha\beta}^{\gamma} G_{\gamma} \quad \{\text{N.B. } [E_{ij}, E_{k\ell}] = E_{i\ell}\delta_{jk} - E_{kj}\delta_{i\ell}\}$$

If a unitary transformation is supposed to act on a system of n particles then

$$U = U(1)\, U(2) \cdots U(n)$$

$$\equiv \exp i \sum_{\alpha} C_{\alpha}(G_{\alpha}(1) + G_{\alpha}(2) + \cdots G_{\alpha}(n)) \tag{3.4}$$

This follows since transformation on the space of different particles commute. The generating operators for an n-particle space are symmetric functions of particle numbers.

The group SU_m of unimodular unitary matrices is the set of transformations formed from all the generators of U_m except the identity. Thus there are m^2-1 generators for SU_m which are trace-less. [Prove that the unitary matrices so formed form a group.]

Because the unitary transformations are made up of generating operators which are symmetric in particle number (i.e. transform like the Young Symmetry [n]) it follows that the action of the unitary transformation on a state $\psi([f]Y)$ with symmetry [f] and Yamanouchi label Y, yields a state also with symmetry [f] and Yamanouchi label Y (see example 3 on page 37). Thus unitary transformations only transform among states with the same Young Symmetry. The states of the given Young Symmetry thus form a basis for a representation of U_m. Thus the U_2 transformations among two states α and β transform among the two particle states with symmetry [2]

$$\alpha\alpha, \sqrt{\frac{1}{2}}\,(\alpha\beta + \beta\alpha), \beta\beta \tag{3.5}$$

but do not couple these states to the state with symmetry [11]

$$\sqrt{\frac{1}{2}}\,(\alpha\beta - \beta\alpha) \tag{3.6}$$

We can thus use the Young Symmetry label [f] also to describe the representations of U_m. Of course the dimension of the representation [f] for transformations of U_m is not the same as the dimension of [f] for S_n. In the case of U_2 given above the [2] symmetry is one dimension w.r.t. S_2 and can be represented by any one (or combination of) the states in eq. 3.5. The transformations of U_2 transform among the three states of eq. 3.5 (but neither more nor less). Thus the three states in eq. 3.5 form a three-dimensional basis for U_2. The one state in eq. 3.6 forms a one-dimensional basis for U_2.

The tableau [f] for SU_m denotes a representation with dimension

$$d([f]\ U_m) = \prod_{1\le i<j\le m} \frac{f_i - f_j + j - i}{j-i} \tag{3.7}$$

where f_i is the number of squares in the i^{th} row of the tableau f. [For the proof of this see ref. 1, eq. 10.25 and Sec. 10.4]. Irreducible bases for U_m are also irreducible for the subgroups SU_m, however, representations differing only by columns of m boxes are equivalent. The representations of SU_m are described by the set of m-1 numbers: $(f_1-f_2,\ f_2-f_3\ \cdots, f_{m-1}-f_m)$.

For SU_2 the representations are thus described by one number which, for this special case, is usually written $S = (f_1-f_2)/2$ - i.e. a spin quantum number.

Note that from eq. 3.7 the dimension of a representation of U_2 or SU_2 is

$$d([f]\ SU_2) = f_1 - f_2 + 1 \equiv 2S+1$$

- i.e. the familiar dimensions of a state with spin S.

THE CLASSIFICATION OF STATES

Single fermion states are endowed with a number of properties associated with symmetries in the Universe. Thus we now recognize that states have an orbital/spin character, flavor, color - and perhaps other quantum numbers as yet to be recognized. We wish to construct many-fermion states in which the character of the state in the separate space (orbital/spin, flavor, etc.) is manifest. How do we do this?

First note that collections of fermions can be described by antisymmetric functions only. Thus we can consider these functions to be made up of an orbital/spin state with some Young Symmetry $[f]$ and a combined flavor, color (and whatever else) - space with dual-adjoint symmetry $\tilde{f}$. We divide the problem in this way because there are only a finite number of flavors and colors (at least to our knowledge at the moment) but an infinite number of orbital states. So whatever restrictions there are in the problem will be determined by the flavor/color space. Actually it is tempting to include the spin also with the flavor/color space and I have done this in refs. 13 and 14, but in the discussion of quarks within baryons it is perhaps wise to always treat quarks as relativistic particles, i.e. keep orbital and spin spaces closely associated. This is the approach I shall follow in these lectures.

In the classification of states, we shall therefore only consider the possible flavor and color symmetries in a symmetry f of the combined flavor/color space. For the sake of argument we shall consider only a two-dimensional flavor space and three-dimensional color space. The combined flavor/color space is thus six dimensional. Our classification will be such as to give the reduction

$$U_6(\text{flavor/color}) \to U_2(\text{flavor}) \times U_3(\text{color})$$

For one particle we have

$$\square_6 \to (\square_2 \times \square_3)_6 \tag{3.8}$$

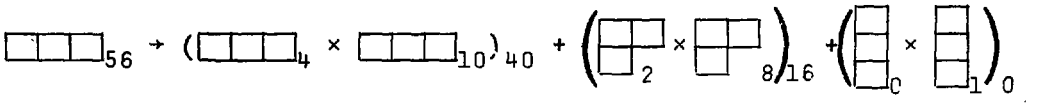
where the subscript numbers give the dimension of the Young Tableau w.r.t. the appropriate unitary group using eq. 3.7. For two particles in $\square\square$ symmetry in U_6 we know from our earlier discussions (cf. eq. 2.6) that the symmetries in both the U_2 and U_3 spaces have to be the same. Thus

$$\square\square_{21} \to (\square\square_3 \times \square\square_6)_{18} + (\begin{smallmatrix}\square\\\square\end{smallmatrix}_1 \times \begin{smallmatrix}\square\\\square\end{smallmatrix}_3)_3 \tag{3.9a}$$

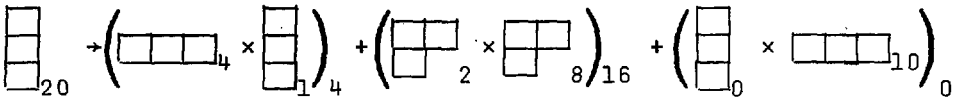
Also following the discussion around eq. 2.7 we can immediately write

$$\begin{smallmatrix}\square\\\square\end{smallmatrix}_{15} = (\square\square_3 \times \begin{smallmatrix}\square\\\square\end{smallmatrix}_3)_9 + (\begin{smallmatrix}\square\\\square\end{smallmatrix}_1 \times \square\square_6)_6 \tag{3.9b}$$

Note that the number of states formed by putting two particles in a six-dimensional space is $6^2 = 36$ of which 21 are symmetric ($\square\square$) and 15 are antisymmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$). Consider now three particles. Again the completely symmetric ($\square\square\square$) and completely antisymmetric states are easy to write down



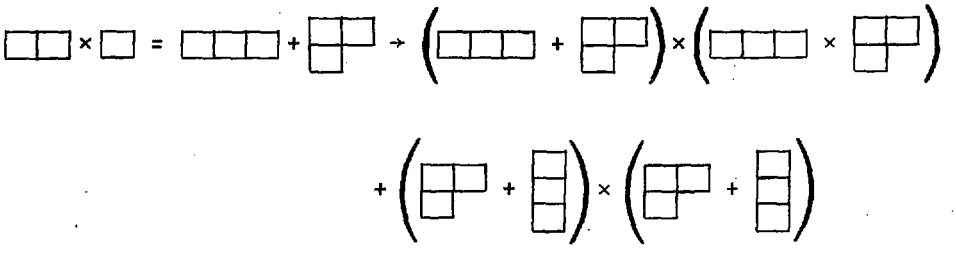
(3.10a)



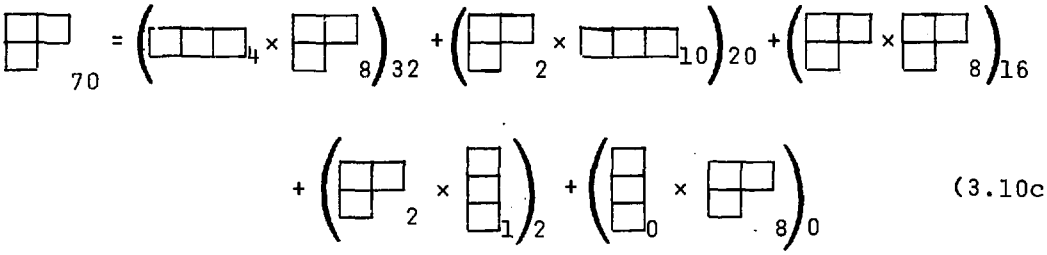
(3.10b)

Note that for a two-dimensional flavour space we cannot construct a state with symmetry $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ - hence the zero dimension.

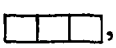
We have now to discover the breakdown of the $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ symmetry in the six-dimensional space. Returning to eq. 3.9a we see that by adding a single square to both the LHS and RHS we find



Since we know the breakdown of  for U_6 in eq. 3.10a we can deduce



(3.10c)

We note from the dimensions of  ,  ,  of U_6 that

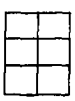
$$6^3 = 56 + 20 + (2 \times 70)$$

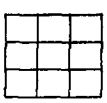
The factor of 2 is needed for the 70-dimensional representation because  is 2-dimensional w.r.t. S_3.

PHYSICS OF BARYON RESONANCES

Let us now see what these rather abstract statements tell us about the physical world of collections of quarks.

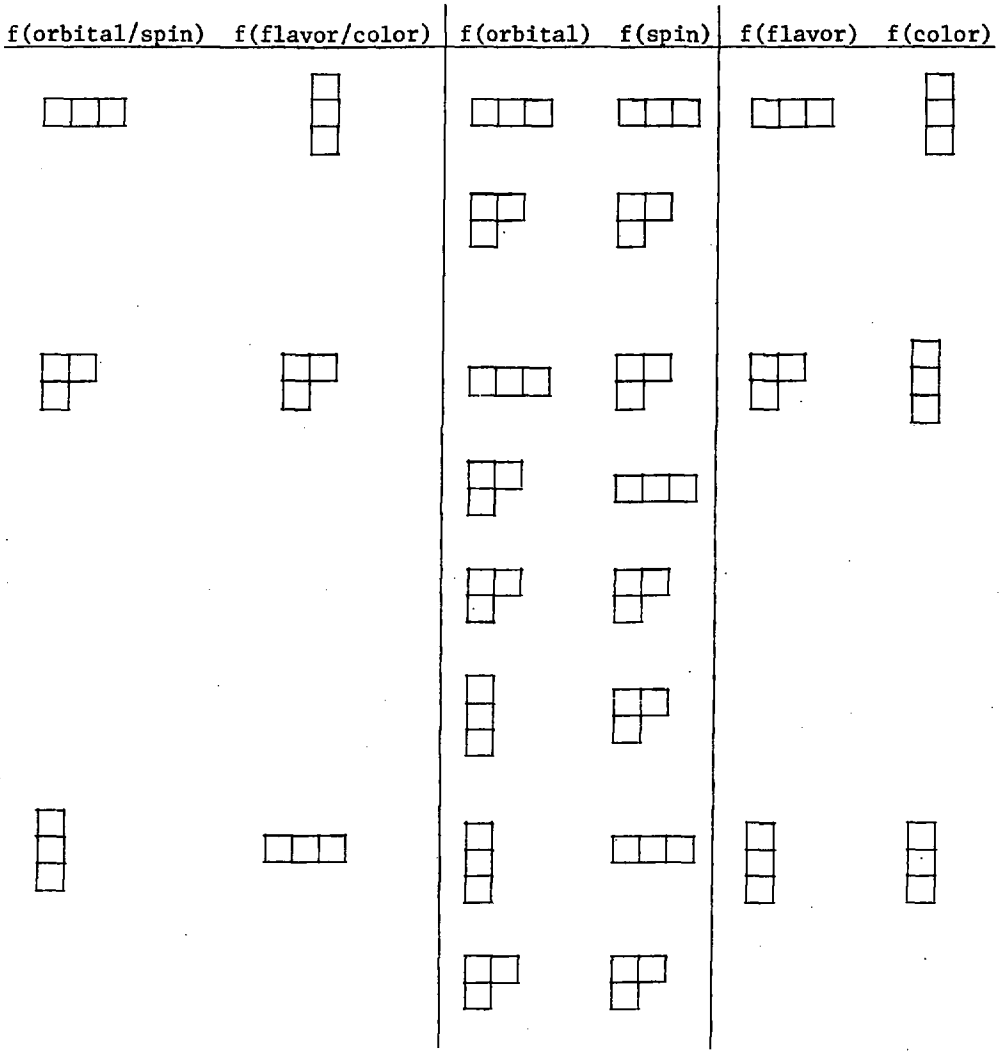
First we note that the only matter that has been revealed to us in experiments is colorless matter - more correctly color-singlet matter. This is matter for which the Young Tableau come *in columns of three* squares, e.g.

3-quarks →  [111] 6-quarks →  [222]

9-quarks →  [333] etc.

All these reps are equivalent w.r.t. SU_3(color). Picking out just these representations of U_3 *from* eq. 3.10 we can write the classification of Table I. In this table we have also given the breakdown of the combined orbital/spin space into separate orbital and spin symmetries. In this latter case we consider only those symmetries which exist for a two-dimensional spin space. For the flavor symmetry we have written the [111] symmetry even though this will not exist for a two-dimensional flavor space.

TABLE I



Let us now rewrite the table using the familiar spin S and isospin T labels (-assuming now a 2-dimensional flavor space)

f(orbital/spin)	f(flavor/color)	f(orbital)	Spin	Isospin	Color
[3]	$[1^3]$	[3]	3/2	3/2	$[1^3]$
		[21]	1/2	3/2	$[1^3]$
[21]	[21]	[3]	1/2	1/2	$[1^3]$
		[21]	3/2	1/2	$[1^3]$
		[21]	1/2	1/2	$[1^3]$
		$[1^3]$	1/2	1/2	$[1^3]$

Now we can begin to understand the classification of baryon resonances. If three quarks are in s orbits the orbital symmetry is [3] then the only spin and isospin labels are $(\frac{3}{2}\ \frac{3}{2})$ and $(\frac{1}{2}\ \frac{1}{2})$ which we identify with the Δ and N.

If one quark is in a p state (i.e. s^2p configuration) we could have

$$\text{orbital symmetry} = \begin{cases} [3] \quad (ST) = (\frac{3}{2}\ \frac{3}{2}),\ (\frac{1}{2}\ \frac{1}{2}) \\ [21] \quad (ST) = (\frac{1}{2}\ \frac{3}{2}),\ (\frac{3}{2}\ \frac{1}{2}),\ (\frac{1}{2}\ \frac{1}{2}) \end{cases}$$

The first set with orbital symmetry [3] are spurious states of excited centre of mass motion.

[The centre of mass co-ordinate is $R = \sum_i r_i$ which is symmetric in all particles. Thus multiplying an s^3 state by R we get a state with 1-quantum excitation but still with [3] symmetry (using example 3 on page 37). The one-quantum excitation is clearly on the centre of mass. Hence the $s^2p[3]$ state is spurious.]

Thus the lowest negative parity resonance is expected to have [21] orbital symmetry with classification

	L	S	J	T	
$(s^2p)_{[21]}$	1	1/2	1/2	3/2	(Δ)
			3/2		
	1	1/2	1/2	1/2	(N)
			3/2		
	1	3/2	1/2	1/2	(N)
			3/2		
			5/2		

Just this number of states have been observed in the low energy N/Δ spectrum and Isgur and Karl (ref. 12) have shown this space to carry the properties of the observed states.

[Exercise - classify the two-quantum states. Beware of spurious states of both [3] and [21] orbital symmetries!]

The calculation of nucleon/delta resonances can be done directly using relative coordinates

$$\rho_1 = \sqrt{1/2}\ (r_1 - r_2)$$

$$\rho_2 = \sqrt{1/6}\ (r_1 + r_2 - 2r_3)$$

which was in fact the approach taken by Isgur and Karl (ref. 18). For example the state having os excitation on both the ρ_1 and ρ_2 coordinate in an oscillator has the structure

$\exp -\{(\rho_1^2+\rho_2^2)/2b^2\}$ with oscillator length parameter b. Clearly this state is symmetric with respect to P_{12} and P_{23} and hence belongs to the [3] symmetry of S_2. A p-orbit on the ρ_i coordinate has the structure $\psi_i \sim \rho_i \exp-(+\rho_1^2+\rho_2^2)/2b^2$. Clearly ψ_1 belongs to the [11] representation of S_2, ψ_2 the [2] representation and ψ_1, ψ_2 carry the [21] representation with Yamanouchi labels (121) and (211), respectively. The states are therefore identical to the internal structures of the single-particle state s^2p of [21] symmetry.

The use of internal coordinates is straightforward and one might think should be preferred. When it comes to using relativistic mechanics the independent particle picture has to be used, however, despite the difficulties associated with the definition of the centre of mass. My own approach here is to select states, using the Symmetry classification, such that they are non-spurious in the non-relativistic limit.

APPENDIX A

DEFINITION AND NOTATION

Notation

f - a Young Tableau for a symmetry of S_n, the Symmetry group of n particles.

Y - a Yamanouchi label[8,9] for the Young Symmetry f.

Y=(pqy) - a Yamanouchi label in which the n^{th} particle is in row p of f, the $(n-1)^{th}$ particle in row q and the remaining particles have a Yamanouchi distribution y. The labelling refers to the standard-Young-Yamanouchi representation (cf. below).

Y=([pq]y) - a Yamanouchi label for the diagonalised Young-Yamanouchi-Rutherford representation below in which the n^{th} and $(n-1)^{th}$ particle has definite (but unspecified) symmetry. Where the symmetry is to be specified we write $\overline{pq}$ or $\tilde{pq}$ to denote the symmetric or antisymmetric pairing.

$f_p(f_{pq})$ - a Young Tableau of S_{n-1} (S_{n-2}) derived from the tableau f of S_n by the removal of a square in the p^{th} (and q^{th}) row(s).

$\tilde{f}$ - the Tableau of the dual symmetry to f obtained by interchanging rows and columns.

f,f',f" or Y,Y',Y" etc. - when used in the same expression refer to orbital, color and isospin-spin spaces, respectively.

Definitions

1. Standard Young-Yamanouchi representation[9] for the transposition $P_{n,n-1}$ in the symmetry f is defined as follows for the three possible types of Yamanouchi labelling:

case i for p=q $\quad P_{n,n-1}|f(pqy)\rangle = +1|f(pqy)\rangle$

case ii for p>q with Y=(pqy) existing but not Y=(qpy)

$$P_{n,n-1}|f(pqy) = -1|f(pqy)\rangle$$

case iii for p>q with Y=(pqy) and (qpy) existing

$$P_{n,n-1}|f(pqy)\rangle = -\frac{1}{\mu}|f(pqy)\rangle + \frac{\sqrt{\mu^2-1}}{\mu}|f(qpy)\rangle$$

$$P_{n,n-1}|f(qpy)\rangle = \frac{\sqrt{\mu^2-1}}{\mu}|f(pqy)\rangle + \frac{1}{\mu}f(qpy)\rangle$$

Here μ is the axial distance (number of lines crossed counting along rows and columns) between the n^{th} particle and $(n-1)^{th}$ particle in the Young Tableau f.

In general we write

$$P_{n,n-1}|f(pqy)\rangle = \alpha^f_{pq}|f(pqy)\rangle + \beta^f_{pq}|f(qpy)\rangle \qquad (A-1)$$

with α^f_{pq} and β^f_{pq} taking the above values depending on the situation (e.g. $\beta^f_{pq} \equiv 0$ in cases i and ii).

2. Adjoint Young-Yamanouchi representation for the transposition $P_{n,n-1}$ in the symmetry f is the same as that for the standard representation except that all the β^f_{pq} coefficients are negative.

3. Diagonalised Young-Yamanouchi-Rutherford representation[9] is such that the last pair of particles have definite symmetry. In terms of the standard representation we may write for the three cases:

Case i $\quad |f([\bar{pq}]y)> \equiv |f(pqy)>$

Case ii $\quad |f([\tilde{pq}]y)> = |f(pqy)>$

Case iii $\quad (p>q)\ |f([\bar{pq}]y)> = (\sqrt{\mu+1}\ |f(qpy)> + \sqrt{\mu-1}|f(pqy)>)/\sqrt{2\mu}$

$$|f([\tilde{pq}]y)> = (\sqrt{\mu-1}|f(qpy)> - \sqrt{\mu+1}|f(pqy)>)/\sqrt{2\mu}$$

In general we write

$$|f([pq]y)> = \gamma^{f}_{pq}|f(pqy)> + \delta^{f}_{pq}|f(qpy)> \qquad (A\text{-}2)$$

with the above definitions for γ and δ depending on the situation.

APPENDIX B

CLEBSCH-GORDAN COEFFICIENTS FOR THE SYMMETRIC GROUP S_n

With the notation of Appendix A, a state $|f''Y''\rangle$ can be constructed from two sets of states $|fY\rangle$ and $|f'Y'\rangle$ by the Clebsch-Gordan expansion

$$|f''Y''\rangle = \sum_{YY'} S(fY\ f'Y'|f''Y'')\ |fY\rangle|f'Y'\rangle \qquad \text{(B-1)}$$

where $S(fY\ f'Y'|f''Y'')$ is the Clebsch-Gordan (CG) coefficient. Hammermesh (ref. 1 - Section 7.14) has shown how the CG coefficients for S_n can be related to these for S_{n-1} by a matrix K. With $Y = (pqy)$ (cf. Appendix A) we may write

$$S(f\ (pqy)\ f'(p'q'y')|f''(p''q''y''))$$

$$= K(f\ p\ f'p'|f''p'')\ S(f_p(qy)f'_{p'}(q'y')|f''_{p''}(q''y''))$$

$$\equiv K_2(f(pq)f'(p'q')|f''(p''q''))\ \ S(f_{pq}\ y\ f'_{p'q'}\ y'|f''_{p''q''}\ y'') \qquad \text{(B-2)}$$

t equality is as given by Hammermesh. The second follows by a second application of the theorem relating the CG for S_{n-1} to those for S_{n-2} and the definition

$$K_2(f(pq)\ f'(p'q')|f''(p''q'') = K(f\ p\ f'p'|f''p'')K(f_p\ q\ f'_{p'}q'|f''_{p''}q'') \qquad \text{(B-3)}$$

Hammermesh also shows, with the notation of Appendix A, that the K matrices may be calculated using the following relationships:

$$K(f\ p\ f'p'|f''p'')\ K(f_p\ q\ f'_p,q'|f''_{p''}q'')(\alpha^{f}_{pq}\ \alpha^{f'}_{p'q'} - \alpha^{f''}_{p''q''})$$

$$+ K(f\ p\ f'q'|f''p'')\ K(f_p\ q\ f'_q,p'|f''_{p''}q'')\ \alpha^{f}_{pq}\ \beta^{f'}_{p'q'}$$

$$+ K(f\ q\ f'p'|f''p'')\ K(f_q\ p\ f'_p,q'|f''_{p''}q'')\ \beta^{f}_{pq}\ \alpha^{f'}_{p'q'}$$

$$+ K(f\ q\ f'q'|f''q'')\ K(f_q\ p\ f'_q,p'|f''_{p''}q'')\ \beta^{f}_{pq}\ \beta^{f'}_{p'q'}$$

$$= K(f\ p\ f'p'|f''q'')\ K(f_p\ q\ f'_p,q'|f''_{q''}p'')\ \beta^{f''}_{p''q''} \qquad \text{(B-4)}$$

(with α and β defined in A-1) and the ortho-normal condition

$$\sum_{pp'} K(f\ p\ f'p'|f''p'')\ K(f\ p\ f'p'|\bar{f}''\bar{p}'')\delta(f''_{p''}\ \bar{f}''_{\bar{p}''}) \qquad \text{(B-5)}$$

$$= \delta(f''\bar{f}'')\ \delta(p''\bar{p}'')$$

The above expressions allow the determination of K relating the CG of S_n to S_{n-1} once the K relating the CG of S_{n-1} to S_{n-2} are known. Since (trivially) $K(11\ 11|11) \equiv 1$, all K matrices can be determined by iteration. The K_2 matrices being products of K matrices (cf. eq. B-3) are thus also determined.

The determination of the K matrices above is for the standard Young-Yamanouchi representation. We define a $\overline{K}$ matrix for the diagonalised Young-Yamanouchi-Rutherford representation by

$$\overline{K}(f[pq]\ f'[p'q']|f''[p''q''])$$

$$= (\gamma^{f}_{pq} + \delta^{f}_{pq}\ \hat{P}_{pq})(\gamma^{f'}_{p'q'} + \delta^{f'}_{p'q'}\ \hat{P}_{p'q'})(\gamma^{f''}_{p''q''} + \delta^{f''}_{p''q''}\ \hat{P}_{p''q''})$$

$$\times K_2(f(pq)\ f'(p'q')|f''(p''q''))$$

with $\hat{P}_{pq}\ K_2(f(pq)\ f'(p'q')|f''(p''q')$

$$= K_2(f(qp)\ f'(p'q')|f''(p''q'')) \text{ etc.}$$

and the γ and δ defined in eq. (A-2).

The elements of the $\overline{K}$ matrices required in the quark-cluster problem are listed in Table 5 of ref. 13. Note that the $\overline{K}$ matrices have the property that all elements are zero unless the product of the symmetries of [pq] and [p'q'] is equal to the symmetry for [p"q"].

The $\overline{K}$ matrices relate the CG coefficients of S_n to those for S_{n-2} in the d-YYR representation.

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