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KRONECKER PRODUCT OF Sp(2n) REPRESENTATIONS USING GENERALIZED YOUNG TABLEAUX

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A B S T R A C T

Using generalized Young tableaux, we obtain an explicit formula for the reduction of the Kronecker product of irreducible representation of the symplectic groups. This extends a previous work devoted to the case of orthogonal groups.

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1. INTRODUCTION

In the present paper, generalizing a method introduced by the authors (Girardi, Sciarrino and Sorba 1981-a,b) for the reduction of the Kronecker product of irreducible representations (IR's) of SO(n) groups we give general formulas for the direct product of two IR's of symplectic groups Sp(2n). This problem has been studied in the past by Littlewood, Wybourne and King using character theory and Schur functions. Another approach was taken by Fischler who used Young tableaux methods but his numerous rules in the general cases seem to give ambiguous results. The method we propose makes use of the weight vectors and reduce the problem to evaluate a sum of products of generalized Young tableaux(GYT's). GYT's are tableaux which can include negative boxes, their definitions and product rules were defined in earlier publications (Girardi, Sciarrino and Sorba 1981-a,b). The use of these GYT's seem to provide a consistent tool for the reduction of Kronecker product of IR's of classical group and whether it is relevant for exceptional groups is now under study. In section 2 we give a very short recall of Sp(2n) groups; in section 3 we present our result and finally in section 4 we give an illustrating example.

2. A REMINDER OF Sp(2n) GROUPS

The Lie algebra of Sp(2n) groups can be realized with the help of n(2n+1) infinitesimal generators Z^i_j (Gilmore)

$$\left[Z^i_j,\ Z^r_s\right] = \text{sign}\ (jr)\left\{Z^i_s\ \delta^{-j}_{-r} + Z^{-j}_{-r}\ \delta^i_s + Z^i_{-r}\ \delta^{-j}_s + Z^{-j}_s\ \delta^r_{-i}\right\}$$

$$i,j,r,s = \pm 1,\ \pm 2, \ldots,\ \pm n$$

$$Z^i_j = Z^{j\dagger}_i = -\ \text{sign}(ij)\ Z^{-j}_{-i}$$

If we relabel the generators $Z^{i'}_{j'}$ as follows

$$Z^{i'}_{j'} \to Z^i_j \qquad i = \begin{cases} 2i' & i' > 0 \\ 2(-i')-1 & i' < 0 \end{cases}$$

the n generators $H_i = Z^{2i}_{2i}$ $(i = 1,\ldots,n)$ commute with each other and span the maximal abelian (Cartan) subalgebra of Sp(2n). The IR's of Sp(2n) can be labelled by n integer positive numbers $\mu_i (\mu_i \geq \mu_{i+1})$ which are the highest eigenvalues(weight) of the generators H_i. Hereafter an IR will be labelled by the n-uple $(\mu_1,\mu_2,\ldots,\mu_n)$ or $[\mu]$. The dimensionality of the IR $[\mu]$ is

$$N[\mu] = \frac{P(\ell)}{P(\tau)} \qquad \text{where} \qquad \begin{aligned} \ell_i &= \mu_i + \tau_i \\ \tau_i &= n - i + 1 \end{aligned}$$

and P is a product of the form:

$$P(\ell) = \prod_{j=1}^{n} \ell_j \prod_{i\geq k}^{n} (\ell_k - \ell_i)(\ell_k + \ell_i)$$

Another way of labelling Sp(2n) IR's is by n non-negative integers q_i (Dynkin notation) which are related to ours as follows

$$q_i = \mu_i - \mu_{i+1} \quad (i = 1,\ldots,n-1)$$

$$q_n = \mu_n$$

Tables of dimension of IR's of Sp(2n) can be found in Wybourne and Patera and Sankoff.

3. RULES FOR THE PRODUCT OF TWO Sp(2n) IR's

The general formula for the product of two IR's $[\mu]$ and $[\nu]$ can be written as follows

$$[\mu] \times [\nu] = \Sigma_1 + \Sigma_2 + \Sigma_3 \tag{3.0}$$

$$\Sigma_1 = \sum_{k=o}^{a-1} (P_n^{2k} \times [\nu])_A \times [\mu] \tag{3.1}$$

$$\Sigma_2 = \sum_{k=1}^{a+b-1} \{(P_n^{2k} \times [\nu])_A \times [\mu] - (P_n^{2k} \times [\mu])_{NA} \times [\nu]\} \tag{3.2}$$

$$\Sigma_3 = \sum_{k=a+b}^{N} \{(P_n^{2k} \times [\nu])_A \times [\mu] - (P_n^{2a} \times [\mu])_{NA} \times (P_n^{2(k-a)} \times [\nu])_A\} \tag{3.3}$$

where P_n^{2k} is a n-row negative GYT, $P_n^{2k} = [0,\ldots,-\alpha_2,-\alpha_1]$ in which the α_i are positive even numbers such that $\alpha_i \geq \alpha_{i+1}$ and $\sum_{i=1}^{n} \alpha_i = 2k$. When P_n^{2k} acts on the IR $[\nu]$ the $\{\alpha_i\}$ must satisfy $\alpha_i \leq 2\nu_i$. For a given k there are several possible sets $\{\alpha_i\}$, except for the extreme case

$$k_{max} = N \equiv \sum_{i=1}^{n} \nu_i \tag{3.4}$$

where

$$P_n^{2N} = [-2\nu_n,\ldots,-2\nu_2,-2\nu_1] \tag{3.5}$$

is uniquely defined.

Finally a (resp.b) is the smallest integer such that $P_n^{2a} \times [\mu]$ (resp. $P_n^{2b} \times [\nu]$) gives a "not-allowed" (NA) GYT (see definition below).

For practical purposes it is more convenient to choose for $[\nu]$ the IR which has the least of boxes, this minimizes the number of operators P^{2k} to consider. The subscript "A" means that in the product one has to keep only the GYT's $[\lambda]$ which fulfill the conditions

i) $$\sum_{i=1}^{n} |\lambda_i| \leq N \tag{3.6}$$

ii) if λ_1 or $|\lambda_p|$ is equal to ν_1, $[\lambda]$ must not contain any label λ_j such that $|\lambda_j| > \nu_2$. If one of the λ_j satisfies $|\lambda_j| = \nu_2$, the $[\lambda]$ must not contain any label λ_k such that $|\lambda_k| > \nu_3$ and so on;

iii) a GYT which appears more than once in a product with P_n^{2k} $(\{\alpha_i\})$, for a fixed set $\{\alpha_i\}$ has to be considered only once. A GYT which appears in the product with P_n^{2k} $(\{\alpha_i\})$ for two different sets $\{\alpha_i\}$ has to be considered twice if

$$\sum_{i=1}^{n} |\lambda_i| \leq N - 2(k-1) \tag{3.7}$$

The subscript NA means that in the product one has to keep only the GYT's $[\lambda]$ which do not satisfy conditions i) and ii) or which should be neglected due to condition iii). In the r.h.s. of eq.(3.1) one has to keep only the GYT's such that $\lambda_i \geq 0$. A GYT which appears only with the minus sign has to be omitted.

If $a = 1$ in the r.h.s. of eq.(3.3) one has to add when it exists the following term

$$\sum_{k=1+b}^{N} \sum_{r=0}^{k-b-1} (-1)^r \left[\mu + \tilde{P}_n^{2(k-r)}\right] \times \left(P_n^{2r} \times [\nu]\right) \tag{3.8}$$

where $\left[\mu + \tilde{P}_n^{2(k-r)}\right]$ is a GYT whose rows are the rows of $[\mu]$ plus the rows of a negative GYT $\tilde{P}_n^{2(k-r)}$ $(\{\alpha_i\})$ which can be computed by the following recurrence formula $(k-r > 2)$

$$\begin{aligned} \tilde{P}_n^{2(k-r)} = & \sum_{\{\alpha_i\}} P_n^{2(k-r)} (\{\alpha_i\}) - \sum_{\{\beta_i\}} \left(P_n^2 \times P_n^{2(k-r-1)}(\{\beta_i\})\right) \\ & + \sum_{\ell} \sum_{\{\gamma_i\}} (-1)^{\ell-k-r} P_n^{2\ell} (\{\gamma_i\}) \times \tilde{P}_n^{2(k-r-\ell)} \end{aligned} \tag{3.9}$$

$$\tilde{P}_n^4 = |0,0,\ldots,-1,-3| \tag{3.10}$$

The new terms given by (3.10) are present only when the IR $[\mu]$ has, at least, as many null labels as the number of negative rows of $\tilde{P}_p^{2(k-r)}$. We give explicitly the $\tilde{P}_n^{2\ell}$ GYT's for $\ell = 3,4$

$$\tilde{P}_n^6 = [0,\ldots,0,-3,-3], \quad [0,\ldots,0,-1,-1,-4]$$
$$\tilde{P}_n^8 = [0,\ldots,0,-2,-3,-3], \quad [0,\ldots,0,-1,-1,-1,-5]; \quad [0,\ldots,0,-1,-3,-4] \qquad (3.11)$$

Finally we give compact formulae which can be deduced for completely symmetric or antisymmetric IR's

$$[\mu,0,\ldots,0] \times [\nu,\ldots,0] = \sum_{\ell=0}^{n} \sum_{k=0} [\mu + \nu - 2k - 2\ell, k, 0, \ldots, 0] \qquad (3.12)$$

where k satisfies $\mu \geq \nu \quad \mu + \nu - 2\ell \geq 2k \geq 0 \ ; \ \nu - \ell \geq k$

$$[\alpha,\ldots,\alpha] \times [\beta,\ldots,\beta] = \sum_{0 \leq k_1 \leq k_2 \leq \ldots \leq k_n} [\alpha + \beta - 2k_1, \alpha + \beta - 2k_2, \ldots, \alpha + \beta - 2k_n] \qquad (3.13)$$

4. AN ILLUSTRATIVE EXAMPLE

Let us calculate in Sp(10) the Kronecker product $[22000] \otimes [21000]$ of respective dimension 780 and 320. Hereafter the zero labels will be ignored when unnecessary. We shall operate on $[\nu] = [21]$, since it has only 3 boxes, we have to consider $k = 0,1,2,3$.

$$k = 0 \quad [22] \times [21] = [43] + [421] + [331] + [322] + [3211] + [2221] \qquad (4.1)$$

$k = 1$ there $P_5^2 = [0000-2]$ so

$$(P_5^2 \times [\nu])_A = ([0000-2] \times [21])_A = [2000-1] + [1100-1] + [10000] \qquad (4.2)$$

Then the products with $[\mu] = [22]$ give

$$\begin{aligned} [2000-1] \times [22] &= [41] + [32] + [311] + [221] \\ [1100-1] \times [22] &= [32] + [311] + [221] + [2111] \\ [1] \times [22] &= [32] + [221] \end{aligned} \qquad (4.3)$$

Now we consider if there are not allowed terms of the type $P_5^2 \times [\mu]$, and find one

$$(P_5^2 \times [\mu])_{NA} \times [\nu] = [2200-2]_{NA} \times [21] = [32] + [221] \qquad (4.4)$$

So from the terms obtained in (4.3) one substracts the contribution of (4.4).

k = 2 We have 2 different P_5^4, namely $[000-2-2]$ and $[0000-4]$

$$([000-2-2] \times [21])_A = [100-1-1] + [1000-2] + [0000-1] \qquad (4.5)$$

and

$$([0000-4] \times [21])_A = [1000-2] \qquad (4.6)$$

In view of rule iii) and eq.(3.7) we ignore (4.6) and only consider (4.5) for multiplying by $[\mu]$:

$$\begin{aligned} [100-1-1] &\times [22] = [21] + [111] \\ [1000-2] &\times [22] = [3] + [21] \\ [0000-1] &\times [22] = [21] \end{aligned} \qquad (4.7)$$

There is one not allowed contribution of the type occuring in (3.3); this gives

$$[2200-2]_{NA} \times [2000-1] = [21] \qquad (4.8)$$

to be substracted from (4.7).

k = 3 At this level we have only to consider $P_5^6 = (000-2-4)$ (from eq.3.5), this gives

$$\{[000-2-4] \times [21]\} \times \{[22]\} = [000-1-2] \times [22] = [1] \qquad (4.9)$$

Gathering all the terms obtained we have as final result:

$$\begin{aligned} &[22] \times [21] = [43] + [421] + [331] + [322] + [3211] + [2221] \\ &780 \times 320 = 42240 + 71500 + 35640 + 28028 + 32340 + 9152 \\ \\ &+ [41] + 2[32] + 2[311] + 2[221] + [2111] \\ &4928 + 2\times4620 + 2\times4212 + 2\times2860 + 1408 \\ \\ &+ [3] + 2[21] + [111] + [1] \\ &220 + 2\times320 + 110 + 10 \end{aligned}$$

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