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GENERAL TREATMENT OF THE NON-LINEAR R_ξ GAUGE CONDITION

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A B S T R A C T

It is shown that the non-linear R_ξ gauge condition already introduced for the standard $SU(2) \times U(1)$ model can be generalized for any gauge model with the same type of simplification, namely the suppression of any coupling of the form: (massless gauge boson).(massive gauge boson).(unphysical Higgs).

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A variant of the R_ξ gauge condition[1] has first been proposed by Fujigawa[2] in the O(3) model. An explicit study[3] of this non linear R_ξ gauge in the standard $SU(2) \times U(1)$ model of weak and electromagnetic interactions has allowed to show out virtues of such a gauge condition: simpler Feynman rules and in particular the vanishing of couplings $A^\gamma_\mu W^\pm \phi^\mp$ where $\phi^\mp$ are the components of the unphysical Higgs swallowed up by the $W^\mp$ bosons. The computational interest of such a gauge condition has been illustrated in the transition Higgs $\to \gamma\gamma$ in Ref.(4) where seven of the nine dominant diagrams containing W's in the loop and present in the usual linear gauge disappear in this new gauge, while in the Faddeev-Popov ghost sector, only one new diagram appears. This new gauge condition was simply obtained by replacing in the usual R_ξ gauge term $\mathcal{L}_G$ the derivative ∂_μ by the covariant derivative with respect to the unbroken $U(1)_{e.m.}$ group i.e. : $\partial_\mu \to \partial_\mu - ieA^\gamma_\mu . Q$: one notices immediately that the presence of this new term in the gauge functions will make the gauge condition non linear

Computational simplifications are of course even more important when large groups and representations are considered, which is the case in grand unified theories. Therefore, it seems that this class of non linear gauge conditions can be convenient for calculating in any gauge model. Indeed let G be the gauge group under consideration which is spontaneously broken down to its subgroup S via the Higgs representation $\mathcal{H}$, then one can show that using the S-covariant derivative instead of the usual ∂_μ in the gauge condition for the massive gauge bosons will insure the vanishing of all tri-linear couplings of the form : (massless gauge boson).(massive gauge bosons).(unphysical Higgs). The Faddeev-Popov ghost sector due to this new gauge condition is worked out in detail which allows in the case of a particular gauge model and for a definite physical process, to choose, between the usual linear R_ξ gauge condition and the non linear one, the more adequate one.

The proof of such a property is very simple and is based on linear algebra. To simplify the notations, we will assume the gauge group G to be simple, and therefore the presence of only one coupling constant (the generalisation to a semi-simple group is straightforward and will be rapidly discussed at the end of this letter). In the following G is supposed to be compact. Let $\mathcal{G}$ be the Lie algebra of G, $\mathcal{S}$ the Lie algebra of the unbroken subgroup S, it is always possible to choose a basis for $\mathcal{G}$ made by a set of generators t_s ($s = 1,2,\ldots, \sigma = \dim.\mathcal{S}$) of $\mathcal{S}$ completed by elements t_B ($B = \sigma + 1,\ldots, \dim\mathcal{G} - \dim\mathcal{S} = \gamma - \sigma$) associated with the "broken part" of $\mathcal{G}$. The elements of this basis will be suitably normalized with the help of the Killing form on $\mathcal{G}$. The covariant derivative with respect to G acting on the representation space of $\mathcal{D}(G)$ is defined as:

$$D_\mu = \partial_\mu - g\, A_\mu^G\, T_G = \partial_\mu - g(A_\mu^s . T_s + A_\mu^B T_B) \tag{1}$$

where the T_G's are the representatives of the generators t for the considered representation $\mathcal{D}(G)$ and the gauge fields $A_\mu^G(x)$ are functions of $x \in M^4$. If $\mathcal{H}$ is the representation space of $\mathcal{D}(G)$ there exists in $\mathcal{H}$ a scalar product

$$<H,H'> \quad \forall \quad H,H' \in \mathcal{H} \tag{2}$$

making this representation $\mathcal{D}(G)$ unitary, since G is compact. Then the matrices T will satisfy the condition:

$$<T_G H,H'> + <H,T_G H'> = 0 \; . \tag{3}$$

Now, let us consider the Higgs kinetic term:

$$\mathcal{L}_{Higgs} = <D_\mu H, D_\mu H> \tag{4}$$

H being the most general element of the chosen (reducible or irreducible) Higgs representation $\mathcal{H}$. The quantity $\mathcal{L}_{Higgs}$ is real, as is any part in the Lagrangian density. Therefore even in the case when the Higgs representation $\mathcal{H}$ is complex, we will use a real scalar product on $\mathcal{H}$ defined as follows:

$$(H,H') = \mathrm{Re} <H,H'> = (H',H) \tag{5}$$

for any couple of vectors H and H' in $\mathcal{H}$.

Let us call H_o the direction along which H gets a non zero vacuum expectation value v :

$$<H> = v\, H_o \quad \text{with } (H_o,H_o) = ||H_o||^2 = 1 \tag{6}$$

Then:

$$T_s(H_o) = 0 \qquad s = 1,\ldots,\sigma \tag{7}$$

while the $T_B(H_o)$ $B = \sigma + 1, \ldots \gamma$ span a linear subspace in $\mathcal{H}$ of dimension $\gamma - \sigma$.

Now, without loss of generality, we can choose a basis for:

$$\mathcal{G} = \mathcal{S} \odot \mathcal{B} \tag{8}$$

such that the following two conditions are satisfied :

(a) its Killing form is a multiple of the identity

(b) the generators T_B verify:

$$(T_B(H_o), T_{B'}(H_o)) = \mu_B^2 \delta_{BB'} = \rho_{BB'} \; . \tag{9}$$

The gauge boson masses are given by $g^2 v^2 \mu_B^2$.

Let us rapidly show that this is always possible. We can in a first step choose a basis of $\mathcal{G}$ such that the Killing form appears as a multiple of the identity. Then considering the matrix $\mathcal{M}^2$ defined by : $\mathcal{M}^2_{BB'} = (T_B(H_o), T_{B'}(H_o))$ which is real and symmetric, we can diagonalize it with the help of an orthogonal matrix O : $\mathcal{M}^2_{BB'} \to (O\, T_B(H_o), O\, T_{B'}(H_o)) = D_{BB'}$. Such an orthogonal transformation on the broken generators T_B will not affect the diagonal Killing form.

Moreover, since the subgroup S leaves invariant the scalar product:

$$(ST_BS^{-1}(H_o), ST_{B'}S^{-1}(H_o)) = (T_B(H_o), T_{B'}(H_o)) \tag{10}$$

the matrix D will appear as a multiple of the identity on each subspace $\mathcal{B}_i$ of $\mathcal{B}$ irreducible under S, i.e.:

$$\mathcal{B} = \bigoplus_{i=1}^{k} \mathcal{B}_i \ , \quad [\mathcal{S}, \mathcal{B}_i] \subseteq \mathcal{B}_i \tag{11}$$

and can be written as a direct sum : $D = \oplus \mu^2_i \, \mathbb{1}_{\dim \mathcal{B}_i}$.

So we make a shift on the vector H

$$H = H' + v\, H_o \tag{12}$$

and we also separate H' into two pieces, orthogonal with respect to the real scalar product above defined

$$H' = h^B . T_B(H_o) + \Phi \tag{13}$$

the h^B being real functions, and

$$(\Phi, T_B(H_o)) = 0 \quad B = \sigma + 1, \dots \gamma \ . \tag{14}$$

The h^B are therefore the "unphysical" Higgs field components which are "eaten up" by the corresponding ghost gauge bosons $\partial_\mu A^B_\mu$. One remark important for our problem is that the action of the unbroken subgroup S on the unphysical Higgs subspace is identical to its action on the (soon) massive gauge bosons: indeed, infinitesimally, one has

$$T_S(A^B_\mu . T_B) \overset{\text{def}}{=} A^B_\mu \, [T_S, T_B] \tag{15}$$

and also:

$$T_S(h^B\, T_B(H_o)) \overset{\text{def}}{=} h^B\, [T_S, T_B]\, (H_o) \tag{16}$$

since $T_S(H_o) = 0$ or using the property $[\mathcal{S}, \mathcal{B}] \subseteq \mathcal{B}$ where $\mathcal{B}$ is such that $\mathcal{G} = \mathcal{S} \oplus \mathcal{B}$ as vector space and $c^{B'}_{SB}$ are the structure constants :

$$T_S(A_\mu^B\ T_B) = c_{SB}^{B'}\ A_\mu^B\ T_{B'} \tag{17}$$

$$T_S(h^B\ T_B(H_o)) = c_{SB}^{B'}\ h^B\ T_{B'}(H_o)\ . \tag{18}$$

So, let us develop the expression of $\mathcal{L}_{Higgs}$:

$$\begin{aligned}
(D_\mu H,\ D_\mu H) &= (\partial_\mu\phi,\partial_\mu\phi) + \partial_\mu h^B\ \partial_\mu h^{B'}(T_{B'}(H_o),T_B(H_o)) \\
&\quad + g^2v^2\ A_\mu^B A_\mu^{B'}(T_B(H_o),T_{B'}(H_o)) \\
&\quad + 2g\left[A_\mu^B\partial_\mu h^{B'}(\phi,T_BT_{B'}(H_o)) - A_\mu^G\partial_\mu h^B\ h^{B'}(T_B(H_o),T_G\ T_{B'}(H_o))\right. \\
&\quad\quad \left. - A_\mu^G(\partial_\mu\phi,T_G\phi) - A_\mu^B h^{B'}(\partial_\mu\phi,T_BT_{B'}H_o)\right] \\
&\quad + g^2\left[A_\mu^G A_\mu^{G'}(T_G(\phi),T_{G'}(\phi)) + 2\ A_\mu^G A_\mu^{G'}h^B(T_G(\phi),T_{G'}\ T_B(H_o))\right. \\
&\quad\quad \left. + A_\mu^G A_\mu^{G'}h^B h^{B'}(T_GT_B(H_o),T_{G'}T_{B'}(H_o))\right] \\
&\quad + 2g^2v\ A_\mu^B A_\mu^{B'}\left[(T_B(\phi),T_{B'}(\phi)) + h^{B''}(T_BT_{B''}(H_o),T_{B'}(H_o))\right] \\
&\quad + 2g^2v\ A_\mu^S A_\mu^B h^{B'}(T_ST_{B'}(H_o),T_B(H_o)) \\
&\quad - 2g\ v\ A_\mu^B\partial_\mu h^{B'}(T_B(H_o),T_{B'}(H_o))\ .
\end{aligned} \tag{19}$$

The last term ("unwanted" term) can be cancelled if we work in a 't Hooft gauge:

$$\mathcal{L}_{LGauge} = -\frac{1}{\xi}\left|\left|(\partial_\mu A_\mu^B + \xi\ g\ v\ h^B)T_B(H_o)\right|\right|^2 - \frac{1}{2\eta}\ (\partial_\mu A_\mu^S T_S)^2 \tag{20}$$

where $-\frac{1}{2\eta}\ (\partial_\mu A_\mu^S T_S)^2$ is, up to the factor $-\frac{1}{2\eta}$, the square of the Killing form on the unbroken part $\partial_\mu A_\mu^S . T_S$, and can be rewritten as: $-\frac{1}{2\eta}\sum_S\ (\partial_\mu A_\mu^S)^2$. We will not be interested by this part in the following.

The first term in the r.h.s. of eq.(20) can be rewritten as:

$$-\frac{1}{\xi}\left|\left|(\partial_\mu A_\mu^B + \xi\ g\ v\ h^B)T_B(H_o)\right|\right|^2 = -\frac{1}{\xi}\ \mathcal{G}_L^B\ \mathcal{G}_L^{B'}\ (T_B(H_o),T_{B'}(H_o)) \tag{21}$$

using eq.(9), with the $\mathcal{G}_L^B$ being the Gauge functions defined as:

$$\mathcal{G}_L^B = \partial_\mu A_\mu^B + \xi\ g\ v\ h^B\ . \tag{22}$$

A natural extension of this gauge to a non-linear one is

$$\mathcal{L}_{NLGauge} = -\frac{1}{\xi}\left|\left|(\partial_\mu - g\, A_\mu^S\, T_S)A_\mu^B\, T_B(H_o) + \xi\, g\, v\, h^B\, T_B(H_o)\right|\right|^2 - \frac{1}{2\eta}\sum_S (\partial_\mu\, A_\mu^S)^2 \tag{23}$$

which will then allow to make disappear not only the last term, but also the last but one in eq.(19). Rewritting $\mathcal{L}_{NLGauge}$ as :

$$\mathcal{L}_{NLGauge} = \mathcal{L}^B_{NLGauge} + \mathcal{L}^S_{Gauge} \tag{24}$$

with

$$\mathcal{L}^B_{NLGauge} = -\frac{1}{\xi}\; \mathcal{G}^B_{NL}\, \mathcal{G}^{B'}_{NL}\, (T_B(H_o), T_{B'}(H_o)) \tag{25}$$

and

$$\mathcal{G}^B_{NL}\; T_B(H_o) = (\partial_\mu\, A_\mu^B + \xi\, g\, v\, h^B)T_B(H_o) - g\, A_\mu^S\, A_\mu^B\, T_S\, T_B(H_o)$$

which can be rewritten using (16)

$$\mathcal{G}^B_{NL}\; T_B(H_o) = \left[\partial_\mu\, A_\mu^B + \xi\, g\, v\, h^B - g\, c^B_{SB''}\, A_\mu^S\, A_\mu^{B''}\right] T_B(H_o) \tag{26}$$

one obtains the relation

$$\mathcal{G}^B_{NL} \equiv \mathcal{G}^B_L - g\, c^B_{SB''}\, A_\mu^S\, A_\mu^{B''} \tag{27}$$

and therefore

$$\begin{aligned} \mathcal{L}_{NLGauge} = \mathcal{L}_{LGauge} - \frac{1}{\xi}\; \Big[& g^2\, c^B_{SB''}\, c^{B'}_{S'B'''}\, A_\mu^S\, A_\mu^{S'}\, A_\mu^{B''}\, A_\mu^{B'''} - \\ & - 2\, g\, c^B_{SB''}\, A_\mu^S\, A_\mu^{B''}\, (\partial_\mu A_\mu^{B'})\Big]\, (T_B(H_o),\, T_{B'}(H_o)) \\ & + 2g^2 v c^B_{SB''}\, A_\mu^S\, A_\mu^{B''}\, h^{B'}(T_B(H_o), T_{B'}(H_o)). \end{aligned} \tag{28}$$

Looking at the last term in eq.(28), we recognize that it is exactly the opposite of the last but one term in eq.(19): indeed, because of the antihermiticity of the T operators:

$$\begin{aligned} & 2g^2 v\, A_\mu^S\, A_\mu^B\, h^{B'}(T_S\, T_{B'}(H_o),\, T_B(H_o)) = \\ & = -2g^2 v\, A_\mu^S\, A_\mu^B\, h^{B'}(T_{B'}(H_o),\, T_S\, T_B(H_o)) \\ & = -2g^2 v\, c^{B''}_{SB}\, A_\mu^S\, A_\mu^B\, h^{B'}(T_{B'}(H_o),\, T_{B''}(H_o)) \\ & = -2g^2 v\, c^B_{SB''}\, A_\mu^S\, A_\mu^{B''}\, h^{B'}(T_{B'}(H_o),\, T_B(H_o)) \end{aligned} \tag{29}$$

and therefore the terms (massless gauge boson).(massive gauge boson).(unphysical Higgs) are not present in $\mathcal{L}_{Higgs} + \mathcal{L}_{NLGauge}$.

We have now to consider the Faddeev-Popov ghost pieces, and to study how this part is affected by this non-linear gauge as compared with the linear one. Let us recall that the Faddeev-Popov part can be written

$$\mathcal{L}_{F.P.} = - \xi^{\alpha} \rho_{\alpha\beta} M^{\beta}_{\gamma} \eta^{\gamma} \tag{30}$$

where M is the matrix defined by :

$$M^{\beta}_{\gamma} = \frac{\delta \mathcal{G}^{\beta}}{\delta \omega^{\gamma}} \tag{31}$$

ω^{γ} being the infinitesimal parameters of the gauge group G ($U = \exp g \vec{\omega}.\vec{T}$), and ξ^{α} and η^{γ} the (anticommuting, scalar) Faddeev-Popov fields. Finally $\rho_{\alpha\beta}$ is the matrix tensor, diagonal according to conditions (a) and (b) which diagonalize the Higgs kinetic term and the gauge boson A^{B}_{μ} physical mass matrix.

From eqs (27, 30, 31) it is easy to analyse the new terms appearing in $\mathcal{L}_{F.P.}$. Actually :

$$\mathcal{L}_{N.L.F.P.} = \mathcal{L}_{L.F.P.} - \xi_{B} \rho_{BB'} \frac{\delta \mathcal{N}^{B'}}{\delta \omega^{G}} \eta^{G} \tag{32}$$

with

$$\mathcal{N}^{B} = - g \, c^{B}_{SB'} A^{S}_{\mu} A^{B'}_{\mu} \; . \tag{33}$$

Under the action of $U = \exp g \vec{\omega}.\vec{T}$ the gauge bosons transform as :

$$\vec{A}_{\mu}.\vec{T} \to \vec{A}_{\mu}'.\vec{T} = \frac{1}{g} (\partial_{\mu} U) U^{-1} + U.\vec{A}.\vec{T}\, U^{-1} \tag{34}$$

or infinitesimally:

$$\delta A^{G}_{\mu} = \partial_{\mu} \omega^{G} + g \, c^{G}_{G'G''} \omega^{G'} A^{G''}_{\mu} \; . \tag{35}$$

It follows that the new term appearing in $\mathcal{L}_{F.P.}$ is simply :

$$- \xi_{B''} \rho_{B''B} \left(\frac{\delta \mathcal{N}^{B}}{\delta \omega^{S}} \eta^{S} + \frac{\delta \mathcal{N}^{B}}{\delta \omega^{B'}} \eta^{B'} \right) \tag{36}$$

with

$$\left\{ \begin{aligned} \frac{\delta \mathcal{N}^{B}}{\delta \omega^{S}} &= - g \, c^{B}_{SB'} A^{B'}_{\mu} \partial_{\mu} - g^{2} c^{S'}_{SS''} c^{B}_{S'B'} A^{S''}_{\mu} A^{B'}_{\mu} \\ &\qquad - g^{2} c^{B'}_{SB''} c^{B}_{S'B'} A^{S'}_{\mu} A^{B''}_{\mu} \\ \frac{\delta \mathcal{N}^{B}}{\delta \omega^{B'}} &= - g \, c^{B}_{SB'} A^{S}_{\mu} \partial_{\mu} - g^{2} c^{S}_{B'B''} c^{B}_{SB'''} A^{B''}_{\mu} A^{B'''}_{\mu} \\ &\qquad - g^{2} c^{B''}_{B'G} c^{B}_{SB''} A^{G}_{\mu} A^{S}_{\mu} \end{aligned} \right. \tag{37}$$

The first relation can be rewritten as follows after use of the Jacobi identity :

$$\frac{\delta \mathcal{A}^B}{\delta \omega^S} = - i\, c^B_{SB'}\, A^{B'}_\mu \partial_\mu - g^2 K\left(T_B, \left[T_S, \left[\dot{A}^{S'} T_{S'}, A^{B'} T_{B'}\right]\right]\right) / K(T_B, T_B) \quad (38)$$

where the use of the Killing scalar product $K(T_B,X)$ expresses that one has to pick up only the coefficient of T_B in X. The second relation is more complicated to write in an analogous way.

Let us summarize the situation. Following the kind of physical process one has to calculate, the non-linear or the linear R_ξ gauge may prove more convenient. One can choose either to suppress all diagrams of the type (massless gauge boson).(massive gauge bosons).(non physical Higgs) and then enlarge the Faddeev-Popov ghost sector (eq.37) with a gauge condition of the type (eq.23), or to keep the usual linear gauge condition (eq.21) without perturbing the F.P. sector but keeping the above mentioned tri-linear couplings. Owing to the simple forms of eq.(37) which necessitate only to calculate double commutators in the Lie algebra of $\mathcal{G}$ one can easily decide the most economical way.

Another question is whether a non-linear gauge condition can be defined by replacing in $\mathcal{L}_{L.G.}$ ∂_μ by the covariant derivative D_μ with respect to a subgroup bigger than the unbroken one S. One sees immediately that by choosing D_μ in the whole algebra $\mathcal{G}$ does not help more than limiting ourselves to $\mathcal{S}$. Indeed, in $\mathcal{L}_{N.L.G.}$, one will have :

$$\left\{\partial_\mu - g\,(A^S T_S + A^B T_B)\right\}(A^B T_B) \equiv \left\{\partial_\mu - g\, A^S T_S\right\}(A^B . T_B) \quad (39)$$

since $$\left\{A^S T_S + A^B T_B\right\}(A^B T_B) = \left[A^S T_S + A^B T_B,\, A^B . T_B\right]$$

However, it could be sometimes interesting to define in $\mathcal{L}_{LNG}$ the covariant derivative with respect to a subgroup S' bigger than S but smaller than G itself, i.e. $G \supsetneq S' \supsetneq S$, and in particular with respect to subgroups S' of the form: $S' = S \times U(1)$. In the case of the electro-weak $SU(2) \times U(1)$ gauge group, the choices in $\mathcal{L}_{NLG}$ of D_μ with respect to $U(1)_{e.m.}$ or $U(1)_{T_3} \times U(1)_Y$ induced more or less simple contributions for other diagrams [3].

Finally, let us mention that the non-linear gauge conditions can be used in the case of semi-simple gauge group $G = \prod_{i=1}^{n} G_i$. One has then simply to take care of the different coupling constants g_i associated with the simple components G_i. If the unbroken subgroup S is a direct product of subgroups $S = \prod_{i=1}^{n} S_i$, $S_i \subset G_i$ then the non-linear gauge condition will be defined by introducing $\hat{D}_\mu = \partial_\mu - \sum_i g_i A^{S_i} . T_{S_i}$. If S is not of this form, but contains for example a "diagonal" subgroup built from several isomorphic subgroups of different G_i, then one has to redefine the physically relevant coupling constants one wants to keep : this is again the case of the electro-weak $SU(2) \times U(1)$ group in which the unbroken part $U(1)_{em}$ is generated by the combination $T_3 + Y$ with T_3 and Y being respectively generators of the $SU(2)$ and $U(1)$ part.

A detailed study of this non-linear gauge condition in the grand unified model SU(5) will be found in Ref. 5.

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