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ELECTROMAGNETIC FIELD OF A ROTATING CLOSED SINGULAR MAGNETIC FLUX-LINE

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Abstract

The electromagnetic field due to the rotation of a circular singular magnetic flux-line is calculated. Averaging the resulting electric field over the period of rotation it is shown that by this procedure neither a static Coulomb charge nor an electric dipole moment can be generated.

In connection with work by H. Jehle[1] who tries to explain electric and other properties of particles by considering spinning, closed loops of quantized magnetic flux, the electromagnetic field due to one rotating loop is of interest. Especially the question of emerging static electric properties is of importance. This problem will be considered here. Our method to calculate the electromagnetic field originating from a singular magnetic flux-line is due to Dirac[2] where the singular flux-line becomes the famous Dirac-string[3]. This formalism will be used in this paper. The electromagnetic field-strength tensor is given by

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + {}^*G^{\mu\nu}(x) \qquad (1)$$

where ${}^*G^{\mu\nu}$ is the tensor dual to $G^{\mu\nu}$, that is

$${}^*G^{\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,G_{\alpha\beta} \qquad (2)$$

and $G_{\alpha\beta}(x)$ is a tensor concentrated on the singular line (Dirac-string) $y_\alpha(\tau,\sigma)$ (τ and σ parametrize this line in 4-dimensional space-time), given by

$$G_{\alpha\beta}(x) = g\int d\tau d\sigma\; \delta^4(x-y(\tau,\sigma))\,[\dot{y}_\alpha(\tau,\sigma)y'_\beta(\tau,\sigma) - \dot{y}_\beta(\tau,\sigma)y'_\alpha(\tau,\sigma)] \qquad (3)$$

with

$$\dot{y}_\alpha = \frac{\partial}{\partial\tau}\,y_\alpha\,, \qquad y'_\alpha = \frac{\partial}{\partial\sigma}\,y_\alpha\,, \qquad g = \text{magnetic charge.} \qquad (4)$$

Assuming that there exists only the singular Dirac-string the field equation has to be

$$\partial_\mu\, F^{\mu\nu}(x) = 0 \qquad (5)$$

or equivalently, substituting for $F^{\mu\nu}$ from (1),

$$\Box A^\nu(x) - \partial^\nu\partial_\mu A^\mu(x) + \partial_\mu\, {}^*G^{\mu\nu}(x) = 0\,, \qquad (6)$$

which becomes in the Lorentz-gauge, $\partial_\mu A^\mu = 0$,

$$\Box A^{\nu}(x) = - \partial_{\mu} {}^*G^{\mu\nu}(x) \ . \tag{7}$$

The standard solution of equ. (7) with no incoming free field is determined by use of the retarded Green's function to be

$$A^{\nu}(x) = \int d^4x' \, \frac{\delta(|\vec{x} - \vec{x}'| - (x^0 - x^{0\prime}))}{4\pi|\vec{x} - \vec{x}'|} \frac{\partial}{\partial x'^{\mu}} {}^*G^{\mu\nu}(x') \ . \tag{8}$$

Specializing to the case of interest, that is when the Dirac-string is a closed loop of unit radius rotating around the x^3-axis with circular frequency ω (see Fig. 1), the parametrization of the singular loop may be chosen to be

$$\begin{gathered} y_{\alpha}(\tau,\sigma) = (\tau,(1 + \cos\sigma)\cos\omega\tau,(1 + \cos\sigma)\sin\omega\tau,\sin\sigma) \\ -\infty < \tau < \infty \qquad 0 \leq \sigma < 2\pi \ . \end{gathered} \tag{9}$$

Inserting (9) into the defining equ. (3) for $G_{\alpha\beta}(x)$ one obtains for the dual tensor

$$\begin{aligned} {}^*G^{\mu\nu}(x) = {} & \frac{g}{|\sqrt{1-z^2}|} \, \Theta(1-|z|)\,\delta(x-(1+\sqrt{1-z^2})\cos\omega t)\,\delta(y-(1+\sqrt{1-z^2})\sin\omega t) \, \cdot \\ & \cdot \begin{pmatrix} 0 & \omega x\sqrt{1-z^2} & \omega y\sqrt{1-z^2} & \omega z(1+\sqrt{1-z^2}) \\ & 0 & \sqrt{1-z^2} & yz/(1+\sqrt{1-z^2}) \\ & & 0 & -xz/(1+\sqrt{1-z^2}) \\ & & & 0 \end{pmatrix} + \\ & + (\text{same expression with } \sqrt{1-z^2} \to -\sqrt{1-z^2}) \ . \end{aligned} \tag{10}$$

It is convenient to perform a partial integration in (8) in order to put the partial derivative in front of the integral. But since $G^{\mu\nu}$ is not restricted to a finite region in time (contrary to 3-space), one has to be careful about boundary terms originating from time derivatives. This difficulty can be circumvented by using instead of G a regularized version

$$*G^{reg}_{\mu\nu}(x) = e^{-\alpha t^2}\, *G_{\mu\nu}(x) \tag{11}$$

and taking the limit $\alpha \to 0$ after the calculation. The electromagnetic potential is given now by

$$A^{\nu}_{reg}(x) = \frac{\partial}{\partial x^{\mu}} \int d^3x' \frac{G^{\mu\nu}_{reg}(x^0 - |\vec{x}-\vec{x}'|, \vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \tag{12}$$

and one therefore obtains for the electric field

$$\begin{aligned} E^i_{reg}(x) = A^{o,i}_{reg}(x) - A^{i,o}_{reg}(x) = \frac{\partial^2}{\partial t^2} \int \frac{d^3x'}{4\pi} \{- \frac{1}{|\vec{x}-\vec{x}'|} *G^{oi}_{reg} + \frac{(\vec{x}-\vec{x}')^j}{|\vec{x}-\vec{x}'|^2} *G^{ji}_{reg} \\ - \frac{(\vec{x}-\vec{x}')^i(\vec{x}-\vec{x}')^j}{|\vec{x}-\vec{x}'|^3} *G^{jo}_{reg}\} + \frac{\partial}{\partial t} \int \frac{d^3x'}{4\pi} \{\frac{1}{|\vec{x}-\vec{x}'|^2} [\delta^{ij} - \\ - 3\frac{(\vec{x}-\vec{x}')^i(\vec{x}-\vec{x}')^j}{|\vec{x}-\vec{x}'|^2}] *G^{jo}_{reg} + \frac{(\vec{x}-\vec{x}')^j}{|\vec{x}-\vec{x}'|^3} *G^{ji}_{reg}\} + \\ + \int \frac{d^3x'}{4\pi} \frac{1}{|\vec{x}-\vec{x}'|^3} \{\delta^{ij} - \frac{3(\vec{x}-\vec{x}')^i(\vec{x}-\vec{x}')^j}{|\vec{x}-\vec{x}'|^2}\} *G^{jo}_{reg} \,. \end{aligned} \tag{13}$$

Since it is the static, time independent, and not the radiation field which is of interest one can average the electric field over one period of rotation, $\tau = 2\pi/\omega$, thus getting rid of all time-derivative terms in (13), since $G^{\mu\nu}$ is of course periodic in time with period τ and the limit $\alpha \to 0$ can be taken quite easily after averaging. Thus the result for the static part of the electric field is given by

$$\begin{aligned} \langle E^i(\vec{x})\rangle = \lim_{\alpha \to 0} \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dx^0 (A^{o,i}_{reg}(x) - A^{i,o}_{reg}(x)) = \\ = \frac{1}{2\pi} \int_0^{2\pi} d\phi\, d^3x' \frac{1}{4\pi|\vec{x}-\vec{x}'|^3} (\delta^{ij} - \frac{3(\vec{x}-\vec{x}')^i(\vec{x}-\vec{x}')^j}{|\vec{x}-\vec{x}'|^2}) \frac{\omega g}{|\sqrt{1-z'^2}|} \theta(1-|z'|) \cdot \\ \cdot \{\delta(x'-(1+\sqrt{1-z'^2})\cos\phi)\delta(y'-(1+\sqrt{1-z'^2})\sin\phi) \cdot \\ \cdot (x'\sqrt{1-z'^2}), y'\sqrt{1-z'^2}, z'(1+\sqrt{1-z'^2}))^j + (\text{same term } \sqrt{1-z'^2} \to -\sqrt{1-z'^2})\}, \end{aligned} \tag{14}$$

which vanishes of course in the static case ($\omega = 0$). From the above form one realizes immediately, that the large distance behaviour of the electrostatic field is of $O(1/r^3)$ at most, thus no electric charge is generated. Actually the field falls off even stronger at large distances, since due to the averaging procedure (integration over ϕ) and the symmetric integration over z' in (14) the electric dipole term ($\sim 1/r^3$) vanishes too. Therefore, in conclusion, the final result is that a rotating Dirac-loop does not generate any electric charge or electric dipole moment.

References

1. H. Jehle, Acta Phys. Austr. 52, 161 (1980) and references therein.
2. P.A.M. Dirac, Phys. Rev. 74, 817 (1948)
3. See for example J.D. Jackson, "Classical Electrodynamics", 2nd Ed., John Wiley and Sons, Inc., New York, 1975, p. 251.

Figure Caption

Fig. 1: Rotating loop.

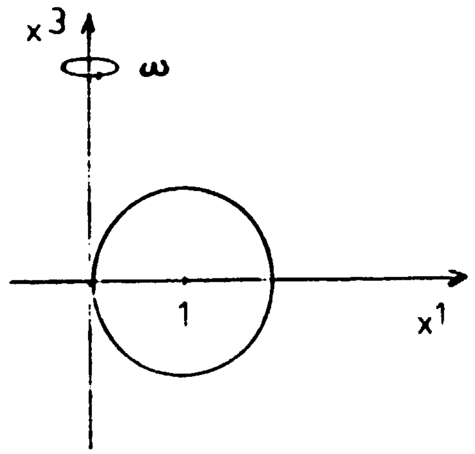

Fig.1