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SIXTH ORDER COUPLING RESONANCES FROM THE BEAM-BEAM INTERACTION IN e^+e^- STORAGE RINGS

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Abstract : The non-linear beam-beam interaction can excite coupling resonances that enlarge vertical beam dimensions in flat-beam operated machines and are hence harmful to luminosity achievements. Two such resonances are examined here using a first order averaging procedure : $2\,Q_y - 4\,Q_x$ = integer and $4\,Q_y - 2\,Q_x$ = integer. The effect on vertical amplitudes, when frequency split falls within predicted resonance band limits, is investigated. Effective resonance bands, taking into account amplitude distributions, are indicated and the limitations of the method are discussed.

1. INTRODUCTION

Charged particles circulating in a storage ring, experience, at each interaction point, sharp non-linear impulses caused by the oppositely circulating charged bunches. These kicks may, under certain conditions, add up to increase amplitudes of particle oscillations. Consequently, transverse beam dimensions are enlarged and the luminosity drops. Even more, particles can hit the vacuum chamber walls with a corresponding short lifetime. The performances of any storage ring and in particular the achievements in integrated luminosity are thus ultimately limited by these strongly non-linear beam-beam interactions.

No clear explanation of the observed beam-beam limit and beam blow up is so far established. Isolated non-linear resonances are, however, believed to play an important role, according to experimental data and to computer simulations. In particular, the beam-beam force can excite coupling resonances when operating point is such that :

$$p\, Q_y - q\, Q_x = \text{integer} \;;\; p,\, q \text{ positive integer}$$

In the case of a flat beam, such a difference resonance produces an exchange of oscillation energy between vertical and radial directions, which significantly increases the vertical beam dimension.

The lowest, i.e. 4^{th}-order non-linear difference resonance excited by the beam-beam force :

$$2\, Q_y - 2\, Q_x = \text{integer}$$

has been studied by B. Montague[3], using a first order averaging method, as developed by A. Schoch[1]. Computer simulations[6,7] show its excitation and are in qualitative agreement with the predictions[3].

Here, we study the 6^{th}-order resonances :

$$2\, Q_y - 4\, Q_x = \text{integer}$$

and

$$4\, Q_y - 2\, Q_x = \text{integer}$$

in the same way. The use of this first order procedure is believed to be valid provided the following assumptions are made :

a) The charge distribution of the oppositely circulating bunches is assumed to remain unaffected by the studied beam-beam interactions ; this is quite true if the incoming beam is much weaker than the reverse

b) The resonance studied is supposed to be isolated enough so that no other resonance is sufficiently excited to be taken into account in a first order scheme. Furthermore, pushing the averaging method to higher orders yields terms proportional to $\Delta Q_e^i / \Delta q^i$, where

ΔQ_e^i, $i = 1,..$ are the excitation widths of nearby low order resonances and Δq^i, $i = 1,..$ their respective distances to operating point. We here assume that the ratios : $\Delta Q_e^i / \Delta q^i$, $i = 1,..$ are small enough. This is true as long as the resonances studied are well isolated, and as long as the strong beam current is not too high. Resonance widths are effectively proportional to the strength of the beam-beam interaction :

$$\xi_{x,y} = \frac{r_o N}{2\pi\gamma} \frac{\beta_{x,y}}{\sigma_{x,y}(\sigma_x + \sigma_y)}$$

where N is the number of particles per bunch in the strong beam
r_o the classical electron radius
γ the Lorentz energy factor
$\beta_{x,y}$ the betatron amplitudes at interaction point
$\sigma_{x,y}$ the r.m.s. values of the strong beam particle distribution

Assumption a) allows one to account for the succession of kicks by only adding the potential created by the opposite beam to the weak beam particle hamiltonian :

$$\mathcal{H}(x,x',y,y',\theta) = \mathcal{H}_o + \mathcal{H}'(x,y,\theta)$$

where $\mathcal{H}' \equiv$ beam potential expansion

The method of variation of constants with the Floquet solution for the unperturbed problem is then used to derive an action-phase hamiltonian. As made possible assuming b), the latter is treated in perturbation theory to the first order with the beam-beam strength as an expansion parameter. From this approximated hamiltonian, two invariants are derived, one of which allows a reduction of the number of degrees of freedom from two to one. Further on, a condition for resonance to occur is derived, which yields resonance band limits for the two resonances studied. The second invariant is then used to investigate beating ranges. Finally, the limitations of the method and its possible further improvements are discussed.

2. PERTURBATION THEORY TO THE FIRST ORDER

The perturbation hamiltonian H' is the potential deriving from the charge distribution in the opposite bunch. This distribution is assumed to be gaussian as expected from natural behaviour and is not distorted by the interaction with the weaker beam, according to a). An expansion of it is given in Appendix. Further multiplying it by a series of δ-functions to account for the kicks being very localised and equally spaced by the superperiod $2\pi/S$ enables one to write

$$H'(x,y,\theta) = \sum_{k_1 k_2} V_{k_1 k_2}\, x^{k_1}\, y^{k_2} \sum_{n=-\infty}^{+\infty} e^{-inS\theta} \qquad (1)$$

Following Schoch[1] , we get an action-phase hamiltonian using the Floquet functions as zero-order solutions

$$H_1(a_x\; a_y\; \phi_x \theta) = \sum_{\substack{nk_1k_2 \\ \ell_1+m_1=k_1 \\ \ell_2+m_2=k_2}} V_{\ell_1 m_1 \ell_2 m_2}\; a_x^{\frac{k1}{2}}\; a_y^{\frac{k2}{2}} \times e^{i[(\ell_1-m_1)(Q_x\theta+\phi_x)+(\ell_2-m_2)(Q_y\theta+\phi_y)-nS\theta]} \qquad (2)$$

where
$$\begin{cases} V_{\ell_1 m_1 \ell_2 m_2} = \binom{\ell_1+m_1}{m_1}\binom{\ell_2+m_2}{m_2} w_1^{\ell_1}\, \bar{w}_1^{m_1}\, u_1^{\ell_2}\, \bar{u}_1^{m_2}\, V_{k_1 k_2} \\ V_{k_1 k_2} \text{ are given in Appendix} \end{cases} \qquad (3)$$

and where the Floquet factors $(W_1,\ u_1)$, defined by the Floquet functions

$$x = a_x^{1/2}\{W_1(\theta)\; e^{i(Q_x\theta+\phi_x)} + \bar{W}_1(\theta)\; e^{-i(Q_x\theta+\phi_x)}\}$$
$$y = a_y^{1/2}\{u_1(\theta)\; e^{i(Q_y\theta+\phi_y)} + \bar{u}_1(\theta)\; e^{-i(Q_y\theta+\phi_y)}\} \qquad (4)$$

are written
$$\begin{cases} W_1 = \bar{W}_1 = \sqrt{\dfrac{\beta_x}{2R}} \\ u_1 = \bar{u}_1 = \sqrt{\dfrac{\beta_y}{2R}} \end{cases} \qquad (5)$$

Von Zeipel's procedure[4)] provides a perturbative averaging procedure which is developed to the first order, according to b). It is then equivalent to the usual neglecting of fast oscillating terms. Thus, Keeping only zero and low frequency terms in (2), i.e terms satisfying :

$$(\ell_1 - m_1)Q_x + (\ell_2 - m_2)Q_y - nS \simeq 0 \tag{6}$$

one gets, in the cases of the two 6th-order resonances studied :

$$\begin{cases} 2Q_y - 4Q_x = 2q_{01} + 2\Delta q_1 & (7) \\ 4Q_y - 2Q_x = 2q_{02} + 2\Delta q_2 & (8) \end{cases}$$

$$H_1 = h_1(a_x, a_y) + \tilde{H}_1(a_x, a_y, \phi_x, \phi_y, \theta) \tag{9}$$

with

$$h_1(a_x, a_y) = V_{1100}a_x + V_{0011}a_y + V_{2200}a_x^2 + V_{0022}a_y^2 + V_{1111}\, a_x + V_{3300}\, a_x^3 + V_{0033}\, a_y^3 + V_{2211}\, a_x^2\, a_y + V_{1122}\, a_x\, a_y^2$$

$$\tilde{H}_1(a_x, a_y, \phi_x, \phi_y, \theta) = 2\, V_{4002}\, a_x^2\, a_y \cos[-2\Delta q_1\theta + 4\phi_x - 2\phi_y] \tag{10}$$

in the case of (7)

$$\tilde{H}_1(a_x, a_y, \phi_x, \phi_y, \theta) = 2\, V_{0240}\, a_x\, a_y^2 \cos[-2\Delta q_2\theta + 2\phi_x - 4\phi_y] \tag{11}$$

in the case of (8)

The two resonances are excited by the beam-beam force and are isolated, as discussed in 1). This means, in terms of (7) and (8), that q_{oi} and Δq_i should satisfy $q_{oi} = \frac{nS}{2}$ and Δq_1 or Δq_2 small.

The coefficient, calculated from the V_{k_1,k_2} potential coefficients given in Appendix and from (3,5), are

$$V_{1100} = - V_0\, \frac{\beta_x}{2R}\, \frac{2}{\sigma_x(\sigma_x + \sigma_y)}$$

$$V_{0011} = - V_0\, \frac{\beta_y}{2R}\, \frac{2}{\sigma_y(\sigma_x + \sigma_y)}$$

$$V_{2200} = V_0 \left(\frac{\beta_x}{2R}\right)^2 \frac{2\sigma_x + \sigma_y}{2\sigma_x^3 (\sigma_x + \sigma_y)^2}$$

$$V_{0022} = V_0 \left(\frac{\beta_y}{2R}\right)^2 \frac{2\sigma_y + \sigma_x}{2\sigma_y^3 (\sigma_x + \sigma_y)^2} \qquad (12)$$

$$V_{1111} = V_0 \left(\frac{\beta_x}{2R}\right)\left(\frac{\beta_y}{2R}\right) \frac{2}{\sigma_x \sigma_y (\sigma_x + \sigma_y)^2}$$

$$V_{3300} = - V_0 \left(\frac{\beta_x}{2R}\right)^3 \frac{1}{3} \quad \frac{3\sigma_y^4 + 8\sigma_x^4 - 7\sigma_y \sigma_x^3 - 7\sigma_x^2 \sigma_y^2 + 3\sigma_x \sigma_y^3}{\sigma_x^5 (\sigma_y^2 - \sigma_x^2)^2 (\sigma_x + \sigma_y)}$$

$$V_{0033} = - V_0 \left(\frac{\beta_y}{2R}\right)^3 \frac{1}{3} \quad \frac{3\sigma_x^4 + 8\sigma_y^4 - 7\sigma_x \sigma_y^3 - 7\sigma_x^2 \sigma_y^2 + 3\sigma_y \sigma_x^3}{\sigma_y^5 (\sigma_x^2 - \sigma_y^2)^2 (\sigma_x + \sigma_y)}$$

$$V_{2211} = - V_0 \left(\frac{\beta_x}{2R}\right)^2 \left(\frac{\beta_y}{2R}\right) \frac{\sigma_y^3 + 3\sigma_x^3 + \sigma_x \sigma_y^2 - 5\sigma_x^2 \sigma_y}{\sigma_y \sigma_x^3 (\sigma_y^2 - \sigma_x^2)^2 (\sigma_x + \sigma_y)}$$

$$V_{1122} = - V_0 \left(\frac{\beta_x}{2R}\right)\left(\frac{\beta_y}{2R}\right)^2 \frac{\sigma_x^3 + 3\sigma_y^3 + \sigma_y \sigma_x^2 - 5\sigma_y^2 \sigma_x}{\sigma_x \sigma_y^3 (\sigma_x^2 - \sigma_y^2)^2 (\sigma_x + \sigma_y)}$$

$$V_{02110} = \frac{1}{12} V_{1122}$$

$$V_{4002} = \frac{1}{12} V_{2211}$$

It can be seen that constant terms from the potential expansion given in [A] have been included up to the order of the resonance [here 6]. This is sufficient since that expansion converges fast enough for not to large amplitudes (see Appendix).

3. TRANSFORMATION OF THE HAMILTONIAN. REDUCTION TO A ONE-DIMENSIONAL PROBLEM

The perturbation Hamiltonian $\mathcal{H}_1$ is transformed, first by a scaling operation, second by a canonical transformation which will reduce it to a one-dimensional Hamiltonian, and third by another scaling operation.

Writing $\mathcal{H}_1$ in terms of $A_x = \frac{2}{R} a_x$, $A_y = \frac{2}{R} a_y$ and then multiplying it by an appropriate $\frac{2}{R}$ factor to preserve hamiltonian scaling gives :

$$\mathcal{H}_2 = h_2(A_x, A_y) + \tilde{\mathcal{H}}_2(A_x, A_y, \phi_x, \phi_y, \theta) \tag{13}$$

with

$$h_2(A_x, A_y) = K_{20} A_x + K_{02} A_y + K_{40} A_x^2 + K_{04} A_y^2 + K_{22} A_x A_y + K_{60} A_x^3 + K_{06} A_y^3 + K_{42} A_x A_y^2 + K_{24} A_x^2 A_y$$

$$\tilde{\mathcal{H}}_2(A_x, A_y, \phi_x, \phi_y, \theta) = \frac{1}{6} K_{42} A_x^2 A_y \cos[-2\Delta q_1 \theta + 4\phi_x - 2\phi_y] \tag{14}$$

in the case of (7)

$$\tilde{\mathcal{H}}_2(A_x, A_y, \phi_x, \phi_y, \theta) = \frac{1}{6} K_{24} A_x A_y^2 \cos[-2\Delta q_2 \theta + 2\phi_x - 4\phi_y] \tag{15}$$

in the case of (8)

and with
$$K_{k_1, k_2} = \left(\frac{R}{2}\right)^{\frac{k_1 + k_2 - 2}{2}} V_{\ell_1 m_1 \ell_2 m_2} \tag{16}$$

The scaled hamiltonian we have obtained is two-dimensional. The use of an invariant typical of the coupled motion we are investigating will permit us to reduce it to a one dimensional hamiltonian.

$\mathcal{H}_2$ has two invariants. In the case of (7), they are :

$$C_{11} = A_x + 2 A_y \tag{17}$$

$$C_2 = \mathcal{H}_2 + (Q_x - \frac{ns}{6}) A_x + (Q_y - \frac{ns}{6}) A_y \tag{18}$$

C_{11} is simply derived equating $\frac{\partial H_2}{\partial \phi_x}$, $\frac{\partial H_2}{\partial \phi_y}$ and using Hamilton's equations

$\frac{\partial H_2}{\partial \phi_x} = \frac{dA_x}{d\theta}$, $\frac{\partial H_2}{\partial \phi_y} = -\frac{dA_y}{d\theta}$; one gets :

$$\frac{\partial H_2}{\partial \phi_x} + 2\,\frac{\partial H_2}{\partial \phi_y} = 0$$

consequently, $$-\frac{dA_x}{d\theta} - 2\,\frac{dA_y}{d\theta} = 0$$

which yields (17)

C_2 is obtain similarly calculating :

$$\frac{dH_2}{d\theta} = \frac{\partial \tilde{H}_2}{\partial \theta}$$

$$\frac{dH_2}{d\theta} = \frac{1}{6}\,K_{42}A_x^2\,A_y\,\frac{\partial}{\partial \theta}\cos[4(Q_x\theta + \phi_x) - 2(Q_y\theta + \phi_y) - nS\theta]$$

$$\frac{dH_2}{d\theta} = (Q_x - \frac{ns}{6})\,\frac{\partial H_2}{\partial \phi_x} + (Q_y - \frac{ns}{6})\,\frac{\partial H_2}{\partial \phi_y}$$

$$\frac{dH_2}{d\theta} = -(Q_x - \frac{ns}{6})\,\frac{dA_x}{d\theta} - (Q_y - \frac{ns}{6})\,\frac{dA_y}{d\theta}$$

which yields (18)

Invariant C_{11} can be used to carry through a canonical transformation with the generating function :

$$G_1(\phi_x\phi_y\alpha_1C_{11}\theta) = \phi_x\,\frac{C_{11}}{2}\,(1-2\alpha_1) + \phi_y\,\frac{C_{11}}{4}\,(1+2\alpha_1)$$

$$+\,\alpha_1\,\frac{C_{11}}{2}\,\Delta q_1\theta \qquad (19)$$

The two canonical momenta (A_x , A_y) are then replaced by (C_{11}, α_1) where C_{11} is the invariant and where

$$\alpha_1 = \frac{2A_y - A_x}{2C_{11}} \tag{20}$$

The physical range of α_1 is [-.5, +.5]. From (19), we get

$$A_x \equiv \frac{\partial G_1}{\partial \phi_x} = \frac{C_{11}}{2}(1-2\alpha_1)$$

$$A_y \equiv \frac{\partial G_1}{\partial \phi_y} = \frac{C_{11}}{4}(1+2\alpha_1) \tag{21}$$

$$\frac{\partial G_1}{\partial \alpha_1} = \frac{C_{11}}{2}(\phi_y - 2\phi_x + \Delta q_1 \theta) \equiv \frac{C_{11}}{4}\psi_1$$

$$\frac{\partial G_1}{\partial C_{11}} = \frac{\phi_x A_x + \phi_y A_y}{C_{11}} + \frac{\alpha_1}{2}\Delta q_1 \theta \equiv \phi \tag{22}$$

where we put $-\psi_1 \equiv -2\Delta q_1 \theta + 4\phi_x - 2\phi_y$. The hamiltonian is transformed into

$$H_3(\alpha_1, C_{11}, \phi, \frac{C_{11}}{4}\psi_1) = H_2 + \alpha_1 \frac{C_{11}}{2}\Delta q_1 \tag{23}$$

Here, one clearly sees how the number of degrees of freedom has been reduced since

$$\frac{\partial H_3}{\partial \phi} = -\frac{dC_{11}}{d\theta} = 0$$

Consequently, H_3 depends only on $(\alpha_1, \frac{C_{11}}{4}\psi_1)$.

At last, scaling $\frac{C_{11}}{4}\psi_1$ into ψ_1, we get

$$H_4(\alpha_1, \psi_1) = \frac{4}{C_{11}} H_3(\alpha_1, \frac{C_1}{4}\psi_1) = \frac{4}{C_{11}}(H_2 + \alpha_1 \frac{C_{11}}{2}\Delta q_1) \tag{24}$$

In the case of resonance (8), invariant C_2 still holds, but C_{11} has to be rewritten :

$$C_{12} = 2\,A_x + A_y \tag{25}$$

Carrying through similar transformations, $\mathcal{H}_2$ is similarly reduced to

$$\mathcal{H}_4(\alpha_2,\ \psi_2) = \frac{4}{C_{12}}\left(\mathcal{H}_2 + \alpha_2 \frac{C_{12}}{2}\ \Delta q_2\right) \tag{26}$$

with $$\alpha_2 = \frac{A_y - 2A_x}{2\ C_{12}}\ ;\ |\alpha_2| \leq 1/2 \tag{27}$$

$$-\psi_2 \equiv -\,2\ \Delta q_2 \theta + 2\ \phi_x - 4\ \phi_y \tag{28}$$

The explicit expressions are the following :

$$\mathcal{H}_4(\alpha_1,\ \psi_1) = \Delta Q_{e1}\ \Big[(\nu_1 + \frac{1}{3}\cos\psi_1)\ \alpha_1^3 + (\eta_1 - \frac{1}{6}\cos\psi_1)\alpha_1^2 + (\chi_1 - \frac{1}{12}\cos\psi_1)\alpha_1 + (B_1 + \frac{1}{24}\cos\psi_1)\Big] \tag{29}$$

with
$$\left\{\begin{aligned}
\Delta Q_{e1} &= K_{42}\ C_{11}^2 \\
\nu_1 &= \frac{4}{K_{42}}\left[\frac{K_{06}}{8} - K_{60} - \frac{K_{24}}{4} + \frac{K_{42}}{2}\right] \\
\eta_1 &= \frac{2}{K_{42}}\left[3\left(K_{60} + \frac{K_{06}}{8}\right) - \left(\frac{K_{24}}{4} + \frac{K_{42}}{2}\right)\right] + \frac{4}{C_{11}K_{42}}\left[K_{40} + \frac{K_{04}}{4} - \frac{K_{22}}{2}\right] \\
\chi_1 &= \frac{1}{K_{42}}\left[-3\left(K_{60} - \frac{K_{06}}{8}\right) + \left(\frac{K_{24}}{4} - \frac{K_{42}}{2}\right)\right] + \frac{4}{C_{11}K_{42}}\left[\frac{K_{04}}{4} - K_{40}\right] + \frac{2}{C_{11}^2K_{42}}\left[2\left(\frac{K_{02}}{2} - K_{20}\right) + \Delta q_1\right] \\
B_1 &= \frac{1}{2K_{42}}\left[\left(K_{60} + \frac{K_{06}}{8}\right) + \left(\frac{K_{24}}{4} + \frac{K_{42}}{2}\right)\right] + \frac{1}{C_{11}K_{42}}\left[K_{40} + \frac{K_{04}}{4} - \frac{K_{22}}{2}\right] + \frac{1}{C_{11}^2K_{42}}\left[2\ K_{20} + K_{02}\right]
\end{aligned}\right. \tag{30}$$

in the case of resonance (7) and,

$$H_4(\alpha_2,\psi_2) = \Delta Q_{e2}\ [(\nu_2 - \frac{1}{3}\cos\psi_2)\alpha_2^3 + (\eta_2 - \frac{1}{6}\cos\psi_2)\alpha_2^2 + (\chi_2 + \frac{1}{12}\cos\psi_2)\alpha_2 + (B_2 + \frac{1}{24}\cos\psi_2)] \tag{31}$$

$$\text{with}\begin{cases} \Delta Q_{e2} = C_{12}^2 K_{24} \\ \nu_2 = \frac{4}{K_{24}}\left[K_{06} - \frac{K_{60}}{8} - \frac{K_{24}}{2} + \frac{K_{42}}{4}\right] \\ \eta_2 = \frac{2}{K_{24}}\left[3\ \frac{K_{60}}{8} + K_{06} - \frac{K_{24}}{2} + \frac{K_{42}}{4}\right] \\ \quad + \frac{4}{C_{12}K_{24}}\left[\frac{K_{40}}{4} + K_{04} - \frac{K_{22}}{2}\right] \\ \chi_2 = \frac{1}{K_{24}}\ -\left[3\ \frac{K_{60}}{8} - K_{06}\right] + \left[\frac{K_{24}}{2} - \frac{K_{42}}{4}\right] \\ \quad + \frac{4}{C_{11}K_{24}}\left[K_{04} - \frac{K_{40}}{4}\right] + \frac{2}{C_{11}^2 K_{14}}\left[2(K_{02} - \frac{K_{10}}{2}) + \Delta q_2\right] \\ B_2 = \frac{1}{2K_{211}}\left[\left(\frac{K_{60}}{8} + K_{06}\right) + \left(\frac{K_{24}}{2} + \frac{K_{42}}{4}\right)\right] \\ \quad + \frac{1}{C_{12}K_{24}}\left[\frac{K_{40}}{4} + K_{04} + \frac{K_{22}}{2}\right] + \frac{1}{C_{12}^2 K_{24}}\left[K_{20} + 2K_{02}\right] \end{cases} \tag{32}$$

in the case of resonance (8).

Finally, the $K_{k_1 k_2}$ are re-expressed in terms of the beam shape parameters, $\sqrt{k} = \frac{\sigma_y}{\sigma_x}$, $\varepsilon_{x,y} = \frac{\sigma^2_{x,y}}{\beta_{x,y}}$ and in terms of the strengths of the beam-beam interaction in the horizontal and vertical planes, already defined in 1) as :

$$\xi_{x,y} = -\frac{r_o N}{2\pi\gamma}\ \frac{\beta_{x,y}}{\sigma_{x,y}(\sigma_x + \sigma_y)} \tag{33}$$

We get :

$$\begin{cases} K_{20} = S\,\xi_x \\ K_{02} = S\,\xi_y \end{cases}$$

$$\begin{cases} K_{40} = -\dfrac{S\,\xi_x}{16\varepsilon_x}\,\dfrac{2+\sqrt{k}}{1+\sqrt{k}} \\[2ex] K_{04} = -\dfrac{S\,\xi_y}{16\varepsilon_y}\,\dfrac{2\sqrt{k}+1}{1+\sqrt{k}} \\[2ex] K_{22} = -\dfrac{S}{4}\sqrt{\dfrac{\xi_x\,\xi_y}{\varepsilon_x\,\varepsilon_y}}\;\dfrac{k^{1/4}}{1+\sqrt{k}} \end{cases} \qquad (34)$$

$$\begin{cases} K_{60} = \dfrac{S}{96}\,\dfrac{\xi_x}{\varepsilon_x^2}\,\dfrac{3k^2+8-7\sqrt{k}-7k+3k^{3/2}}{(k-1)^2} \\[2ex] K_{06} = \dfrac{S}{96}\,\dfrac{\xi_y}{\varepsilon_y^2}\,\dfrac{3+8k^2-7k^{3/2}-7k+3k^{1/2}}{(k-1)^2} \\[2ex] K_{42} = \dfrac{S}{4}\,\dfrac{\xi_x}{\varepsilon_x^2}\sqrt{\dfrac{\xi_y}{\varepsilon_y^2}}\;k^{2/3}\,\dfrac{k^{3/2}+3+k-5k^{1/2}}{(k-1)^2} \\[2ex] K_{24} = \dfrac{S}{4}\sqrt{\dfrac{\xi_x}{\varepsilon_y^2}}\;\dfrac{\xi_y}{\varepsilon_y^2}\;k^{1/3}\,\dfrac{1+3k^{3/2}+k^{1/2}-5k}{(k-1)^2} \end{cases}$$

4. RESONANT CONDITION AND RESONANCE BAND LIMITS FOR EQUAL BEAM-BEAM STRENGTHS

A particle is on resonance when

$$\frac{d\psi_i}{d\theta} = \frac{\partial H_4}{\partial \alpha_i}(\alpha_i, \psi_i) = 0 \qquad i = 1,2 \qquad (35)$$

i.e. when

$$3\left(\nu_{\substack{1\\2}} \pm \frac{1}{3}\cos\psi_{\substack{1\\2}}\right)\alpha^2_{\substack{1\\2}} + 2\left(\eta_{\substack{1\\2}} - \frac{1}{6}\cos\psi_{\substack{1\\2}}\right)\alpha_{\substack{1\\2}} + \left(\chi_{\substack{1\\2}} \mp \frac{1}{12}\cos\psi_{\substack{1\\2}}\right) = 0 \qquad (36)$$

A resonance can be crossed if at least one of the solutions of (36) falls inside the physical interval [-1/2,1/2] for $\alpha_{\substack{1\\2}}$. A sufficient condition for no resonance to be crossed is thus :

$$|\alpha_1 \pm| > \frac{1}{2} \qquad \text{in the case of resonance (7)} \qquad (37)$$

$$|\alpha_2 \pm| > \frac{1}{2} \qquad \text{in the case of resonance (8)} \qquad (38)$$

where $\alpha_{\substack{1\\2}} \pm$ are the two solutions of (36) given by :

$$\alpha_1 \pm = \frac{-(\eta_1 - \frac{1}{6}\cos\psi_1) \pm \sqrt{(\eta_1 - \frac{1}{6}\cos\psi_1)^2 - 3(\nu_1 + \frac{1}{3}\cos\psi_1)(\chi_1 - \frac{1}{12}\cos\psi_1)}}{3(\nu_1 + \frac{1}{3}\cos\psi_1)}$$

$$\alpha_2 \pm = \frac{-(\eta_2 - \frac{1}{6}\cos\psi_2) \pm \sqrt{(\eta_2 - \frac{1}{6}\cos\psi_2)^2 - 3(\nu_2 - \frac{1}{3}\cos\psi_2)(\chi_2 + \frac{1}{12}\cos\psi_2)}}{3(\nu_2 - \frac{1}{3}\cos\psi_2)}$$

and where $\alpha_1 \pm$, $\alpha_2 \pm$ are supposed real, which is true only if

$$\left(\eta_{\substack{1\\2}} - \frac{\cos\psi_{\substack{1\\2}}}{6}\right)^2 - 3\left(\nu_{\substack{1\\2}} \pm \frac{\cos\psi_{\substack{1\\2}}}{3}\right)\left(\chi_{\substack{1\\2}} \mp \frac{\cos\psi_{\substack{1\\2}}}{12}\right) \geq 0$$

In the case of flat beams operated with equal beam-beam strengths in the two planes ($\xi_x = \xi_y$), we have $h = \sqrt{k}$, and we put $h = \frac{1}{16}$ as a typical value for a storage ring operated with flat beams, as LEP or PETRA.

From (34), (32) and (30), we thus get :

$$\begin{cases} \nu_1 = 42,14 \\ \eta_1 = 76,33 - \frac{1}{\sigma_1} \; 12,66 \\ \chi_1 = 35,8 - \frac{1}{\sigma_1} 6,76 - 23,59 \; \frac{1 - R_1}{\sigma_1^2} \end{cases} \tag{39}$$

$$\begin{cases} \nu_2 = 62,698 \\ \eta_2 = 95,95 - \frac{1}{\sigma_2} 7,39 \\ \chi_2 = 48,86 - \frac{1}{\sigma_2} 7,83 + 3,81 \; \frac{1 + R_2}{\sigma_2^2} \end{cases} \tag{40}$$

where, for $i = 1,2$ $\begin{cases} \sigma_i = \frac{C_{1i}}{\varepsilon_x} & \text{are the reduced invariants} \\ R_i = \frac{\Delta q_i}{S\xi_y} & \text{are the reduced frequency splits} \end{cases}$

The above numerical values allow us to drop $\cos\psi_i$ terms, which are all small, from the expressions of $\alpha_1 \pm$ and $\alpha_2 \pm$. Inequalities (37 and (38) yield resonance band limits for the two resonances studied. The largest bands are produced for $\sigma = 1$

$$-4,27 \leq R_1 \leq 1,13 \tag{41}$$

in the case of (7)

$$-47,36 \leq R_2 \leq -0,87 \tag{42}$$

in the case of (8)

Their physical meaning follows from eq.(35) : if $R_i = \frac{\Delta q_i}{S\xi_y}$ falls within the limits of corresponding resonance band, particles can start to beat, thus increasing their vertical dimensions. Whether a large number of beam particles will or not beat appreciably due to these resonances depends however on two other factors : the size of the beating ranges and their average over the particle amplitude distribution.

This point will be further explained in the next paragraph. It will turn out reducing notably effective resonance widths as well as their harmfulness.

5. BEATING RANGES

As done in[2,3], beating of particles can be studied using invariant C_2. It can be easily shown that $\mathcal{H}_3$, the hamiltonian obtained through the canonical transformation described above, is equal to just C_2. Hence, individual particles can be described by

$$\mathcal{H}_4(\alpha_i,\ \psi_i) \ = \ \text{constant} \qquad\qquad i = 1,2$$

which can be written, for resonance (7), putting $\cos\psi_1 = \pm 1$:

$$(\nu_1 \pm \tfrac{1}{3})\alpha_1^3 + (\eta_1 \mp \tfrac{1}{6})\alpha_1^2 + (\chi_1 \mp \tfrac{1}{12})\alpha_1 \pm \frac{1}{24} = \frac{B_4(1)}{\Delta Q_{e1}} - B_1 = \lambda_1 \qquad (43)$$

and for resonance (8), putting $\cos\psi_2 = \pm 1$

$$(\nu_2 \mp \tfrac{1}{3})\alpha_2^3 + (\eta_2 \mp \tfrac{1}{6})\alpha_2^2 + (\chi_2 \pm \tfrac{1}{12})\alpha_2 \pm \frac{1}{24} = \frac{B_4(2)}{\Delta Q_{e2}} - B_2 = \lambda_2 \qquad (44)$$

For fixed values of R_i and σ_i, the limit values of α_i are given by two curves, obtained solving eq.(43) and (44) as a function of $B_4(i)/\Delta Q_{ei} - B_i \equiv \lambda_i$, respectively for $i = 1, 2$.

Fig. 1 to 4 show the set of limiting curves in the $(\lambda_i,\ \alpha_i)$-plane corresponding to a set of R_i values, for different σ_i values. It can be seen in fig. 1 and 3 that particles do beat over approximatively all the calculated resonance bands for each of the two resonances, however with quite small amplitudes, since the limiting curves are very close to each other. Effectively for each value of R_i, particles move along the vertical between the two limiting curves. Significant beating occurs when initial values of $(\alpha_i,\ C_{1i})$ correspond to a point near the vertical tangent to the curves. In fig. 2 is shown a "magnification" of the center of fig. 1. Beating doesn't exceed 10 % in α_i.

Fig. 4 shows the behaviour for another value of σ_1. The range of R_1 values for which beating is appreaciable within the physical limits of α_1 is considerably reduced : from $-4 < R_1 < 1$ to $0 < R_1 < 1$. The <u>effective sizes of the bands</u> therefore depend strongly on the number of particles populating each (α_1, σ_1). Since, for a flat beam, typical particles have

$$\begin{cases} A_x \simeq \varepsilon_{x/2} \\ A_y \simeq \varepsilon_{y/2} = \frac{k}{h} \frac{\varepsilon_x}{2} \end{cases}$$

$\begin{cases} \alpha_1 \sim -1/2 \\ \sigma_1 \sim 1/2 \end{cases}$ has the largest population, in the case of resonance (7)

and $\begin{cases} \alpha_2 \sim -1/2 \\ \sigma_2 \sim 1 \end{cases}$ in the case of resonance (8)

Hence, in the case of resonance (7), for example, the effective resonance band is a rather narrow strip close to $R \sim .9$. In other parts of the full resonance band, particles also beat, but as they are very few, the effect is hardly significant. Similarly, in the case of resonance (8), the effective resonance band is a strip located between $R \sim -1$ and $R \sim -3$.

6. CONCLUSIONS AND PROSPECTS

The results of this first order calculation seem reasonably in accordance with what one expects for higher order resonances such as the 6th-order coupling resonances examined here. They appear to be wider than the 4th-order coupling resonance : $2\,Q_y - 2\,Q_x$ = integer, investigated in[3]. Associated beating ranges are also smaller by a factor 4 in amplitude: which is consistent with large widths.

Furthermore, as in the 4th-order case, the two 6th-order resonances exhibit asymmetries typical of non-linear resonances. These asymmetries here appear through the difference in behaviour on both sides of the resonances. According to our calculations, beating appears sooner when approaching the resonances from below than from above. However, as only the effective resonance bands are relevant to the behaviour of the beam, the effective asymmetry is reversed in the case of resonance (7) : a large fraction of the beam will beat only when approaching that resonance from above.

As the beating range is rather limited and as its average over the particle distribution in (σ_i, α_i) has the effect of reducing the large widths to narrower effective ones, the two resonances seem less dangerous than one could have feared.

How these results will prove relevant and accurate for a high energy storage ring such as PETRA or LEP is a measure of the completeness of the method used. Two limitations and improvements to make up for them are believed to be of importance.

Firstly, as previously mentioned, pushing the averaging method to higher orders yields resonance mixing effects whose importance grow with the beam-beam strengths $\xi_{x,y}$. This is quite natural since $\xi_{x,y}$ is the expansion parameter of the perturbation method. A first order procedure is hence sufficient if currents are not too high. What "not too high" means depends on the resonance one is working on and on its surrounding (vicinity of other resonances of lower order than the studied one). In the case of the 4th-order resonance, the only resonances of lower order excited by the beam-beam interaction are the one-dimensional $2\,Q_{x,y}$ = integer. The first order approximation in the perturbation method is hence good for currents below the stochasticity limit provided one is far from those two one-dimensional resonances. In the case of 6th-order resonances, many lower order resonances can contribute in, for example, 2nd-order terms of the perturbative expansion. Taking into account such corrections could change predictions a great deal, even though general features would remain.

Secondly, the evolution of the particle distribution due to quantum fluctuations and radiation damping, has not been taken into account, even though it could transport particles all over phase space (and thus over (α_i, σ_i)-space). A more global treatment taking account of the distribution would be appreaciable.

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APPENDIX : Potential expansion from the gaussian charge distribution

Following Montague[2], the potential deriving from a gaussian charge distribution

$$\rho(x,y) = \rho_o \, e^{-\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)} \qquad \text{(A1)}$$

is written in an integral form derived by Houssais :

$$V(x,y) = - \frac{\rho_o \, a \, b}{4\varepsilon_o} \int_o^\infty \frac{1-e^{-\left(\frac{x^2}{a^2+t} + \frac{y^2}{b^2+t}\right)}}{(a^2+t)(b^2+t)} \, dt \qquad \text{(A2)}$$

Expanding it yields :

$$V(x,y) = \frac{\rho_o ab}{4\varepsilon_o} \sum_{n=1} \frac{(-1)^n}{n!} \sum_{j_1+j_2=n} \binom{j_1+j_2}{j_2} x^{2j_1} y^{2j_2} \times \int_o^\infty \frac{dt}{(a^2+t)^{j_1+1/2} (b^2+t)^{j_2+1/2}} \qquad \text{(A3)}$$

Keeping terms up to the 6th order and using a reduction formula[5] to calculate the integrals gives :

$$V(x,y) = - x^2 \, B_{10} - y^2 \, B_{01} + \frac{1}{2} x^4 \, B_{20} + \frac{1}{2} y^4 \, B_{02} + x^2 \, y^2 \, B_{11} - \frac{1}{6} x^6 \, B_{30} - \frac{1}{2} x^4 \, x^2 \, B_{21} - \frac{1}{2} x^2 \, y^4 \, B_{12} - \frac{1}{6} y^6 \, B_{03} \qquad \text{(A4)}$$

with :

$$B_{10} = V_0 \frac{2}{a(a+b)}$$

$$B_{01} = V_0 \frac{2}{b(a+b)}$$

$$B_{20} = \frac{2}{3} V_0 \frac{2a+b}{a^3(a+b)^2}$$

$$B_{02} = \frac{2}{3} V_0 \frac{2b+a}{b^3(a+b)^2}$$

$$B_{11} = V_0 \frac{2}{a\,b(a+b)^2} \qquad \text{(A5)}$$

$$B_{03} = \frac{2}{15} V_0 \frac{3a^4 + 8b^4 - 7b^3a - 7a^2b^2 + 3ba^3}{b^5(b^2-a^2)^2\,(a+b)}$$

$$B_{30} = \frac{2}{15} V_0 \frac{3b^4 + 8a^4 - 7ba^3 - 7a^2b^2 + 3b^3a}{a^5(b^2-a^2)^2\,(a+b)}$$

$$B_{21} = \frac{2}{3} V_0 \frac{b^3 + 3a^3 + ab^2 - 5a^2b}{ba^3\,(b^2-a^2)^2\,(a+b)}$$

$$B_{12} = \frac{2}{3} V_0 \frac{a^3 + 3b^3 + a^2b - 5ab^2}{ab^3\,(b^2-a^2)^2\,(a+b)}$$

and where $V_0 = \frac{\rho_o\, a\, b}{4\,\varepsilon_o}$ has to be scaled to normalise to the dynamical variables, as shown further down.

Putting $a = \sqrt{2}\,\sigma_x$ and $b = \sqrt{2}\,\sigma_y$ in (A5) and multipling by the corresponding powers of w_1 and u_1 given in (5) gives the coefficients of (12).

Using Hamilton's equation on eq.(13) gives, for vanishing small amplitudes :

$$\frac{\partial H_2}{\partial A_x} = \frac{d\phi_x}{d\theta} \simeq K_{20} = V_{1100}$$

V_{1100} is hence the linear beam-beam time shift. Writing, for S interaction points,

$$\frac{\Delta\phi_x}{2\pi} = V_{1100} = S\,\xi_x = -\,S\,\frac{r_o N}{2\pi\gamma}\,\frac{\beta_x}{\sigma_x(\sigma_x + \sigma_y)} \qquad \text{(A6)}$$

enables us, through a comparison with coefficient V_{1100} given in (12), to identify

$$V_0 = \frac{r_o N\,R\,S}{2\pi\gamma} \qquad \text{(A7)}$$

we thus simply replace V_0 in (12) by its expression given in (A7) to account for the scaling operations.

Neglecting terms of order higher than 6 in the potential of (A4) is a good approximation if those terms are small enough. Since high order terms grow faster than low order terms when x or y is increased, the expansion will not converge as fast for particles far out in the tails of the distributions and, in particular when the beam blow up causes the vertical dimension to be increased. The accuracies of the 4th-and 6th-order approximations were checked numerically : we found the 6th-order approximation to be accurate to less than 1 % for particles at 1 σ_y, to 10 % for particles at 2 σ_y and to only 50 % at 3 σ_y. Hence, potential (A4) is believed to be a good approximation for particles up to around 2 σ_y, which is sufficient since beating ranges are small. In the case of the 4th order coupling resonance :

$$2\,Q_y - 2\,Q_x = \text{integer}$$

studied in[3], beating ranges are bigger and the 4th-order approximation is less good. However, since particles in fact only spend a fraction of their time at the maximum values of the y-displacement corresponding to to their amplitudes, the approximation is not believed to be bad.

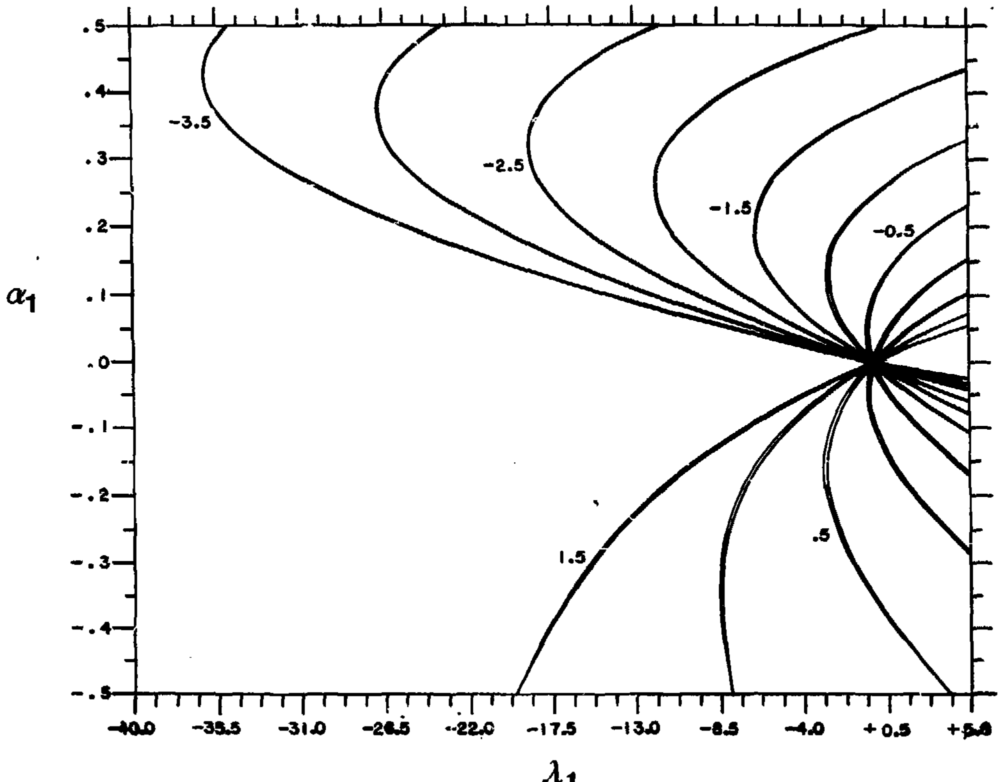

FIG 1 : Beating range boundaries for resonance $2Q_y - 4Q_x$ = integer, corresponding to $\sigma_1 = C_{11}/\varepsilon_x = 1$. The boundaries of α_1 for constant λ_1 limited by $\cos\psi_1 = \pm 1$ are calculated over the full resonance band : from $R_1 \equiv \Delta q_1/S\xi_y = -3.5$ to $R_1 = 1.5$

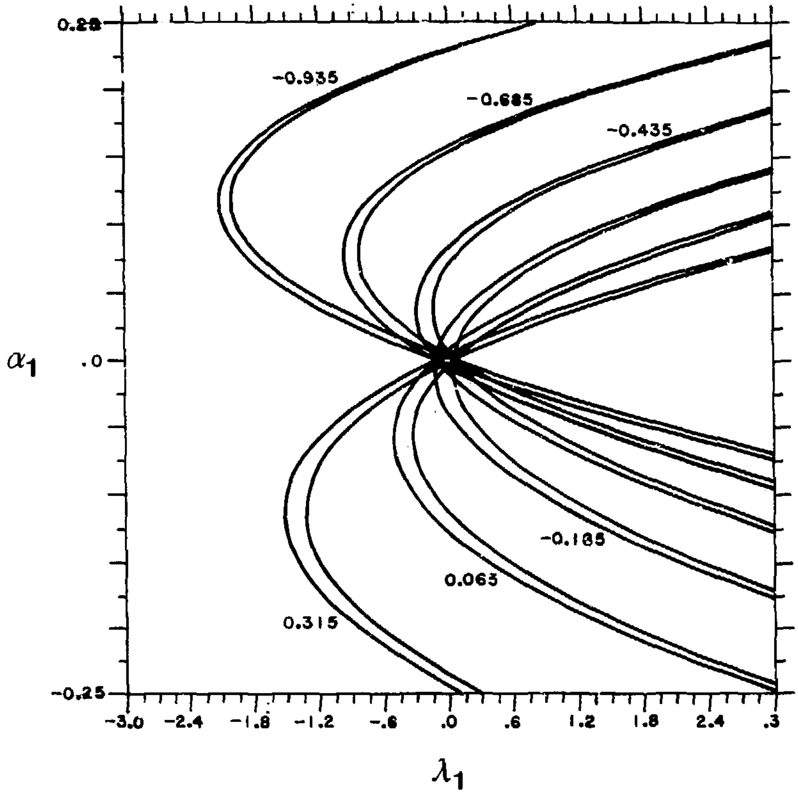

FIG 2 : "Zooming" of the central region of fig. 1 : the boundaries are calculated as in fig. 1 but only for part of the resonance band : from $R_1 \equiv \Delta q_1/S\xi_y = -0.935$ to $R_1 = 0.315$. Beating doesn't exceed 10 %

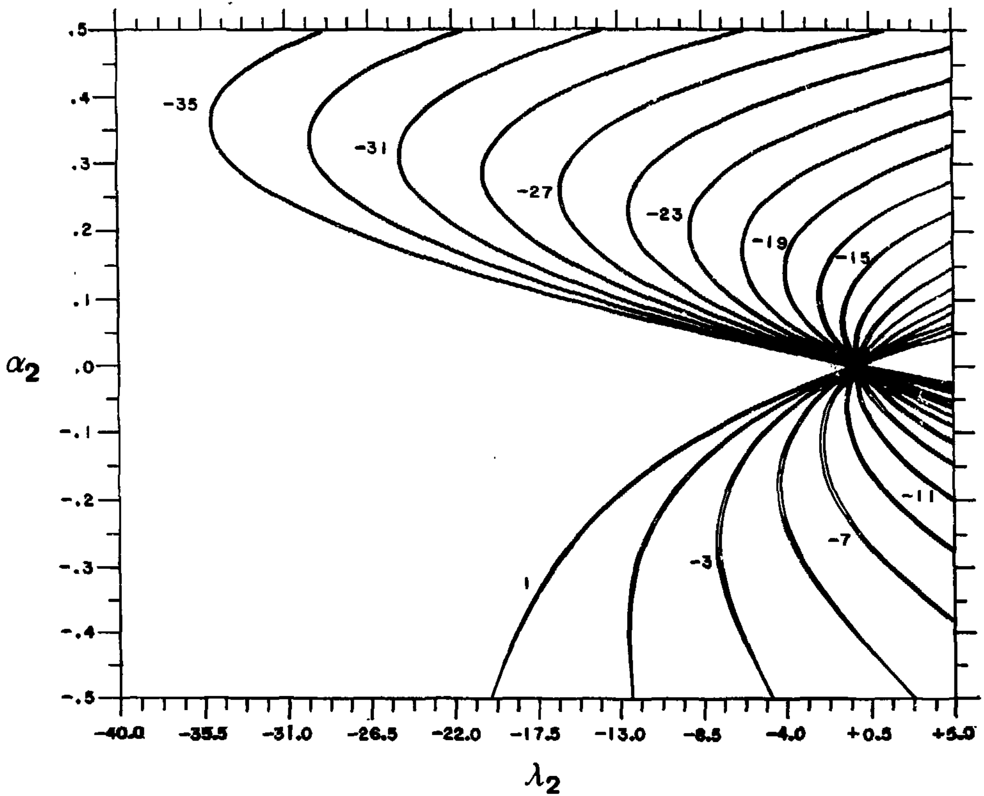

FIG 3 : Beating range boundaries for resonance $4Q_y - 2Q_x$ = integer, corresponding to $\sigma_1 = C_{12}/\epsilon_x = 1$. The boundaries of α_2 for constant λ_2 limited by $\cos\psi_2 = \pm 1$ are calculated over the full resonance band : from $R_2 \equiv \Delta q_2/S\xi_y = -35$ to $R_2 = 1$

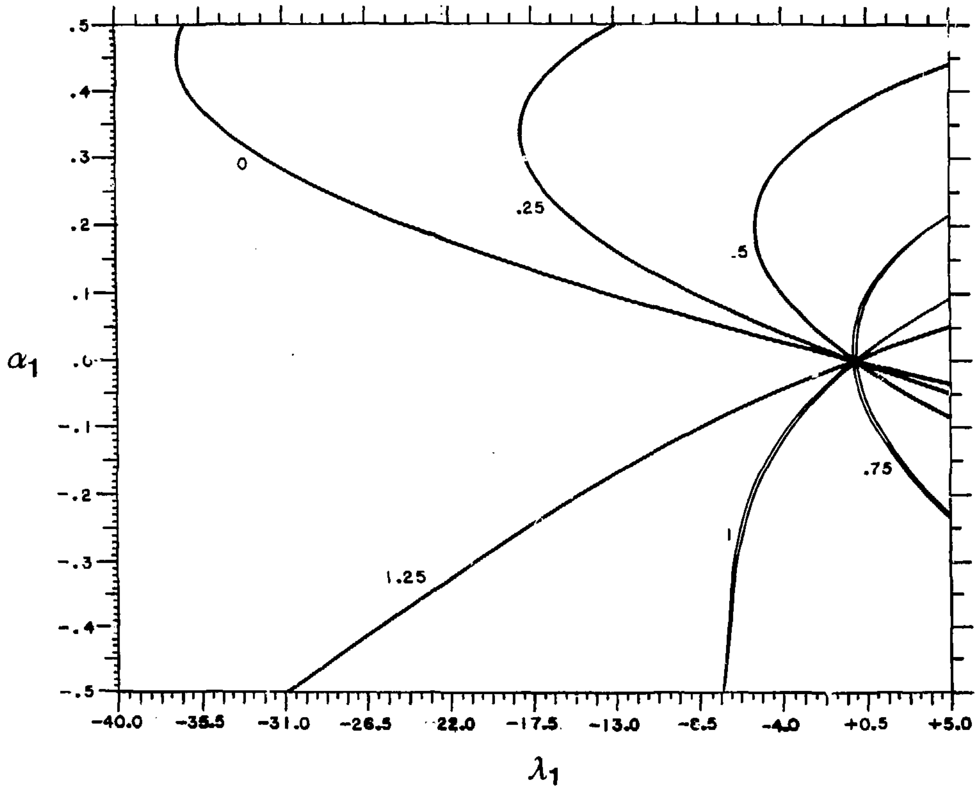

FIG 4 : Beating range boundaries for resonance $2Q_y - 4Q_x$ = integer, corresponding to $\sigma_1 \equiv C_{11}/\epsilon_x = .5$. The boundaries are calculated as in fig. 1 but for $R_1 \equiv \Delta q_1/S\xi_y = 0$ to $R_1 = 1.25$