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IMPROVED ANALYTICAL METHOD FOR STUDYING BEAM-BEAM DRIVEN NON-LINEAR RESONANCES

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SUMMARY

A compact averaging procedure is worked out to study the beam-beam effect in storage rings, in the weak beam-strong beam case. In order to avoid difficulties with polynomial approximations beam-beam potential deriving from the assumed Gaussian charge distribution is used.

INTRODUCTION

In a storage ring, particles experience, each time they intersect with the on-coming bunches, sharp non-linear impulses. These may add up to increase particle amplitudes thus hurting luminosity goals and lifetimes. As for non-linear problems in general, the motions of the "weak beam" particles are described by a non-integrable Hamiltonian. Two approaches have been developed. The first one is to express the problem as an iterative mapping and to use a computer to calculate the evolution. Results from such simulations[1] show the behaviour of particles during a great number of turns as well as fundamental characteristics of non-integrable problems : beyond a threshold, a large number of trajectories become "chaotic" and unpredictable.

The second one is to approximate the problem with an integrable one. Analytical calculations can then, in principle, be achieved. Although results will be qualitatively false in the sense that the fundamental "chaotic" property of the system is, a priori, suppressed, they can, up to a threshold in beam-current or in amplitude, be very good quantitatively (see 1)). The most popular approximation is the use of a first order averaging procedure : among the many resonant terms appearing in the non-approximated Hamiltonian, only one is kept, assuming that the system is close enough to the corresponding resonance. Simulations[2,3] as well as the operation of existing storage rings[4] show the resonances do play an important role, before the beam-beam limit is reached.

In this paper, we compare polynomial approximations of the beam-beam potential with the exact one, and we present operational expressions using the latter. Illustration is given in the case of the coupling resonance $2Q_x - 2Q_y$ = integer.

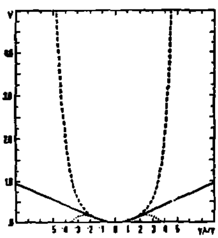

Figure 1 - Exact potential (solid line) and truncated potentials to 4th order (dotted line) and to 6th order (dashed line) for $x = 0$

Truncated and exact potentials

The electromagnetic potential deriving from a Gaussian charge distribution can be written in an integral form as done in Ref. 5 :

$$V(x,y) = V_o \int_0^\infty \frac{1-e^{-\left(\frac{x^2}{2\sigma_x^2+t}+\frac{y^2}{2\sigma_y^2+t}\right)}}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)}}\,dt \qquad (1)$$

For convenience, most authors expand the potential in polynomial series[5,6,7], truncate them at a certain order, and then proceed to transforming and averaging.
A truncation at the 4th order gives

$$V_4(x,y) = -x^2B_{10} - y^2B_{01} + \frac{1}{2}x^4B_{20} + \frac{1}{2}y^4B_{02} + B_{11}x^2y^2 \qquad (2)$$

In[6], the truncation is made at the 6th order, yielding

$$V_6(x,y) = V_4(x,y) - \frac{1}{6}x^6B_{30} - \frac{1}{2}x^4y^2B_{21} - \frac{1}{2}x^2y^4B_{12} - \frac{1}{6}y^6B_{03} \qquad (3)$$

Coefficients B_{nm} can be found in appendix of[5].

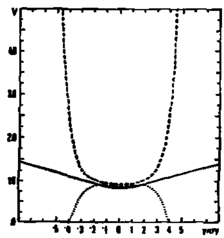

Figure 2 - Exact potential (solid line) and truncated potentials to 4th order (dotted line) and to 6th order (dashed line) for $x = 0.5\ \sigma_x$

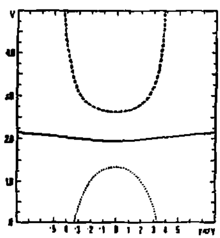

Figure 3 - Exact potential (solid line) and truncated potentials to 4 th order (dotted line) and to 6th order (dashed line) for $x = \sigma_x$

Such truncated expansions are believed to be valid as long as one only considers particles with small amplitudes. Since the series are alternating, it seems obvious that the convergence is going to be extremely slow for large amplitudes. What a "large" or a "small" amplitude means, has never, to my knowlegde, been qualified. An answer is given here through a comparison of $V(x,y)$ with $V_4(x,y)$ and $V_6(x,y)$, in the case of a flat beam $\sigma_x/\sigma_y = 16$. They are plotted as function of y, for three fixed values of x, in Fig. 1-3.

Roughly, it can be said that the truncated expressions and their first derivatives (relevant to resonance widths) are very close to the exact ones for $x < \frac{1}{2}\sigma_x$, $y < 2\sigma_y$.

Dimensionless non-truncated potential

The potential given in (1) is first reexpressed in terms of dimensionless quantities :

$$X = x/\sqrt{2\sigma_x\sigma_y} \quad Y = y/\sqrt{2\sigma_x\sigma_y} \quad f = \sigma_y/\sigma_x \quad T = t/2\sigma_x\sigma_y$$

$$V(x,y) = V_0 \int_0^\infty \frac{1-e^{-\left(\frac{X^2}{1/f+T} + \frac{Y^2}{f+T}\right)}}{\sqrt{(1/f+T)(f+T)}}\, dt \tag{4}$$

The Hamiltonian and its transformation

As done in[5,6], the motion of a weak-beam particle can be described by the following Hamiltonian

$$H(X\ \dot{X}\ Y\ \dot{Y}\ \theta) = H_0 + H'(X,\ Y,\ \theta) \tag{5}$$

where $H'(X,Y,\theta) = V(X,Y)\sum_{k=-\infty}^{\infty} e^{-ikS\theta}$ is the perturbing term due to the succession of localised kicks. H_0 represents unperturbed motion, S = superperiodicity

The method of variation of constants with the Floquet solutions for the unperturbed problem is then used to obtain an action-phase Hamiltonian. Setting :

$$X = \left(\frac{\beta_x A_x}{2\sigma_x\sigma_y}\right)^{1/2} \cos(Q_x\theta + \phi_x)$$
$$Y = \left(\frac{\beta_y A_y}{2\sigma_x\sigma_y}\right)^{1/2} \cos(Q_y\theta + \phi_y) \tag{6}$$

in (4), we obtain

$$H_1(A_x,A_y,\phi_x,\phi_y,\theta) = V_0 \sum_{k=-\infty}^{\infty} e^{-ikS\theta}$$

$$\times \int_0^\infty dT\, \frac{1-e^{-[u_x(1+\cos 2\psi_x) + u_y(1+\cos 2\psi_y)]}}{\sqrt{(1/f+T)(f+T)}} \tag{7}$$

where $u_x = \frac{\beta_x A_x}{4\sigma_x\sigma_y}\,\frac{1}{1/f+T}$ $\quad u_y = \frac{\beta_y A_y}{4\sigma_x\sigma_y}\,\frac{1}{f+T}$

$$\psi_x = Q_x\theta + \phi_x \qquad \psi_y = Q_y\theta + \phi_y$$

Using the following expansion in terms of modified Bessel functions

$$e^{-x\cos y} = I_0(x) + 2\sum_{n=1}^{\infty}(-1)^n \cos(ny)\, I_n(x) \tag{8}$$

we obtain :

$$H_1 = \sum_k e^{-ikS\theta}\Big[H_0(A_xA_y) + \sum_{nm} H_{nm}(A_xA_y)\cos(2n\psi_x)\cos(2m\psi_y)\Big] \tag{9}$$

where $H_0(A_x,A_y) = V_0 \int_0^\infty \frac{1-e^{-(u_x+u_y)}I_o(u_x)I_o(u_y)}{\sqrt{(1/f+T)(f+T)}}\, dT$

$$H_{nm}(A_x,A_y) = (-1)^{n+m-1}2V_0 \int_0^\infty \frac{e^{-(u_x+u_y)}I_n(u_x)I_m(u_y)}{\sqrt{(1/f+T)(f+T)}}\, dT \tag{10}$$

Expression (9) is a compact non-approximated way of writing the Hamiltonian, and the problem of it being non-integrable remains. A similar approach including all orders of the potential can be found in[9]. The integrals given in (10), are, of course, not easy to calculate analytically. However, the smooth behaviour of $y = e^{-x}I_n(x)$ allows a quite accurate computation.

First order averaging procedure

Von Zeipel's procedure[10] provides a perturbative averaging scheme which is developed to the first order as in[5,6,7]. It is then equivalent to the usual neglecting of fast oscillating terms. Assuming the system is close to an isolated resonance, i.e. verifies

$$2\,n\,Q_x + 2\,m\,Q_y = k\,S + 2\,\Delta q \qquad \Delta q \text{ small} \tag{11}$$

the averaged Hamiltonian is :

$$H_1 = H_0(A_x,A_y) + H_{|n||m|}(A_x,A_y)\cos(2n\psi_x + 2m\psi_y - kS\theta) \tag{12}$$

We now have a completely integrable problem since two invariants of the motion can be derived

$$C_1 = -m\,A_x + n\,A_y$$

$$C_2 = H_1 + \left(Q_x - \frac{n}{2|n|}\cdot\frac{kS}{|n|+|m|}\right)A_x + \left(Q_y - \frac{m}{2|m|}\cdot\frac{kS}{|n|+|m|}\right)A_y \tag{13}$$

Let us also notice that the Hamiltonian given in (9) makes it quite easy to develop the averaging method to higher orders.

Illustration in the case of the coupling resonance $2\,Q_x - 2\,Q_y$ = integer

Putting n = - m = 1 in (9 - 13) and reducing the number of degrees of freedom through a canonical transformation with generating function

$$G_2(\phi_x,\phi_y,\alpha,C_1,\theta) = \phi_x\, C_1[\phi_x(\tfrac{1}{2}-\alpha) + \phi_y(\tfrac{1}{2}+\alpha)] + \alpha\Delta q\theta \tag{14}$$

we obtain the following expression for the invariant C_2 (with $S\xi_x = S\xi_y$)

$$C_2 = 2\left(\frac{\sigma_x^2}{\beta_x} + \frac{\sigma_y^2}{\beta_y}\right)S\xi_y\Big[H_0\big(C_1(\tfrac{1}{2}-\alpha),\ C_1(\tfrac{1}{2}+\alpha)\big)$$
$$+ H_{11}\big(C_1(\tfrac{1}{2}-\alpha), C_1(\tfrac{1}{2}+\alpha)\big)\times\cos(2\psi_x - 2\psi_y - kS\theta) + \alpha\Delta q\, C_1\Big] \tag{15}$$

On fig. 4 is shown a plot of the limiting values of C_2 corresponding to $\cos(2\psi_x - 2\,Q_y - kS) = \pm 1$ and for particles initially at $x = 0.25\,\sigma_x$, $y = 0.25\,\sigma_y$.

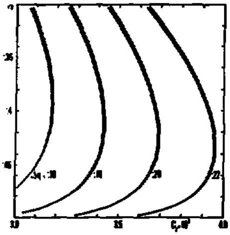

Figure 4 – Limiting invariant curves for the coupling resonance $2\,Q_x - 2\,Q_y$ = integer, corresponding to (x,y)-initial values of $\left(\frac{1}{4}\,\sigma_x, \frac{1}{4}\,\sigma_y\right)$, $\xi_y = 0.12$, and for various values of Δq

Particle trajectories are represented by vertical lines limited by the two curves. The amplitude of the beating is given by the length of these lines, which may become important near the resonance. The closeness of the 2 curves is explained by the respective order of magnitude of the computed functions H_0 and H_{11}. In general $|H_{11}/H_0| \sim 10^{-3}$.

Conclusions and prospects

Motivations to present a practical and realistic method can be listed by order of importance

1°) allow more or less reliable predictions to be made concerning beam blow-up versus (Q_x, Q_y)-working points, below the stochastic limit.

2°) answer the question : "up to which threshold is a non-integrable system quantitatively well approximated by an integrable one in the case of beam-beam interaction ?"

3°) verify the conjecture :"the usual operating region in which a resonance behaviour prevails corresponds to the zone below the above mentionned threshold".

ACKNOWLEDGMENTS

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