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RANDOMIZED RANDOM WALK ON A RANDOM WALK *

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ABSTRACT

This paper discusses generalizations of the model introduced by Kehr and Kunter of the random walk of a particle on a one-dimensional chain which in turn has been constructed by a random walk procedure. The superimposed random walk is randomised in time according to the occurrences of a stochastic point process. The probability of finding the particle in a particular position at a certain time instant is obtained explicitly in the transform domain. It is found that the asymptotic behaviour for large time of the mean-square displacement of the particle depends critically on the assumed structure of the basic random walk, giving a diffusion-like term for an asymmetric walk or a square root law if the walk is symmetric. Many results are obtained in closed form for the Poisson process case, and these agree with those given previously by Kehr and Kunter.

I. INTRODUCTION

The basic problem of random walks has been well discussed in the classic text on probability theory of Feller [1] and in many modern texts on Markov chains and stochastic processes [2)-4)]. Applications of the theory of random walks to a wide range of physical problems have also been extensively documented in Barber and Ninham [5].

Recently Kehr and Kunter [6] introduced an interesting and novel model of a random walk on a random walk. In it, they consider the random walk of a particle on a linear chain which does not extend uniformly in one direction, but which in itself is constructed by another random walk formulation (see Fig.1 of their paper). In a way, we may picture this as a second random walk superimposed on the path or folded chain of the first or basic random walk.

There are many physical situations which can be appropriately modelled by random walk of a particle along random paths. As has been mentioned by Kehr and Kunter [6], a realistic example is afforded by imagining a particle in an amorphous substance which has a tendency to hop along an irregular path more easily. The "reptation" of a polymer chain discussed by De Gennes [7),8)] where the diffusive motion of a polymer in a tube which is subject to random deformation is another example. Richards [9] considered the hopping motion of particles which interact through Coulomb repulsion. He advances a model by assuming that the repulsion is sufficiently strong to forbid two or more particles from occupying the same lattice site, nearest-neighbour and longer-range repulsions being neglected. Thus, essentially, the particles are assumed non-interacting except for the exclusion requirement which only allows hops to unoccupied sites. For any particular tagged particle in the linear chain, as other neighbouring particles are allowed to move, it will experience a fluctuating range of unoccupied sites to fill. This of course is akin to the case of a random walk on a random walk. Of particular interest is the prediction that the mean-square displacement satisfies the relation $\langle x^2 \rangle(t) \propto t^{1/2}$ instead of the diffusion-like relation $\langle x^2 \rangle(t) \propto t$ for non-interacting particles.

In Kehr and Kunter's discussions, both the basic and superimposed random walks are assumed to be symmetric (equal forward and backward transition probabilities), homogeneous (constant transition probability from site to site) and infinite (over the integers ...-2, -1, 0, 1, 2,...). They initially formulate the solution in discrete time where many basic properties of the solution including moments, approximate formulas for the probability density

$W_n(x)$ of finding a value x after n steps transitions via the standard Gaussian approximation and the saddle point approximation of Daniels. Extensive numerical simulations have been performed by them to confirm conclusions from their derivations. They later discuss rather briefly the generalization of the random walk on a random walk to continuous time for the case when the steps of transitions take place at time instants according to a stationary Poisson process. The solution for the basic formula for $W(x,t)$, the probability of finding the particle in state x, is obtained in a double Fourier-Laplace form, from which they deduce asymptotic behaviour of the moments $\langle x^2 \rangle(t)$ and other physical quantities of interest such as the frequency-dependent coefficient and the incoherent dynamic structure factor. They have shown, among other things, that the "anomalous" square-root law for mean-square displacement is a natural consequence of the random walk on a random walk model, in both discrete and continuous time.

In this paper we shall look at the continuous time generalization mentioned above (calling it randomised random walk following the terminology of Feller [10]) in some detail and in a more general setting. We find that generally most results are expressible in closed form. The asymptotic laws for moments derived by Kehr and Kunter via Tauberian arguments are easily deduced from asymptotic behaviours of the solutions in the time domain. Lastly, the superposed randomised random walk is generalized allowing transitions to occur according to a stationary renewal process. It is shown that the asymptotic square-root law for mean-square displacement still holds true in this case provided the basic random walk is symmetric, else a diffusion-like law ($\propto t$) will be in force. It is rather interesting that the model gives rise to two types of asymptotic laws dependent on assumptions on the parameter (the transition probability rule) of the basic random walk.

II. RANDOMISED RANDOM WALK: POISSON PROCESS TRANSITIONS

We assume that both the basic and superimposed random walks are homogeneous and infinite but may be asymmetric. Let $p_0(x)$ ($p_1(x)$) be the transition probability of the basic (superimposed) random walk, where $p_i(+1) = p_i$, $p_i(-1) = q_i$ and $0 \leq p_i = 1 - q_i \leq 1$, $i = 0,1$. Define the corresponding structure factor, $\sigma_i(u)$, of the random walk as the Fourier transform of $p_i(x)$, $i = 0,1$, i.e.

$$\sigma_i(u) = \sum_x \exp(iux)\, p_i(x) = \cos u + i(p_i - q_i)\sin u. \tag{1}$$

For the basic random walk let $P_\nu(x)$ be the probability of starting with $x = 0$, and finding a value x after ν steps. It is not hard to find an explicit formula for $P_\nu(x)$, as is given in Ref.6 for the symmetric case of $p_0 = q_0 = 1/2$. For the superimposed random walk (in discrete time) the $P_n(\nu)$ is the probability of finding a particular value of ν after n steps. On account of the independence in transition, it is not hard to see that the Fourier transform of $P_\nu(x)$ is

$$\tilde{P}_\nu(u) = [\sigma_0(u)]^{|\nu|} \tag{2}$$

with a similar formula for $P_n(\nu)$.

For the randomised random walk on a random walk let $P(\nu,t)$ be the probability of the superimposed walk to be in "site" ν at time t, and $W(x,t)$ the probability of eventually finding the particle which performs this continuous time random walk to be at a value x at time t. By enumerating all paths leading to the value x, we immediately have the equation given by Kehr and Kunter

$$W(x,t) = \sum_\nu P(\nu,t)\, P_\nu(x). \tag{3}$$

When the transitions occur according to a Poisson process, $P(\nu,t)$ can be very simply derived using the following combinatorial arguments of Feller [10]. For the superimposed walk, the ν^{th} step leads to the position $x \geq 0$ iff among the first ν transitions $(\nu + x)/2$ are positive and $(\nu - x)/2$ negative. This is impossible unless $\nu - x = 2k$ is even. In this case the probability of the position x just after the ν^{th} jump is $\binom{k+2x}{k+x} p_1^{k+x} q_1^x$. In the Poisson process with parameter λ, the probability that up to epoch t exactly $\nu = 2k + x$ jumps occur is $(\lambda t)^\nu \exp(-\lambda t)/\nu!$. Hence for $x \geq 0$

$$P(\nu,t) = \exp(-\lambda t) \sum_{k=0}^{\infty} \frac{(\lambda t)^{k+2x}}{(k+2x)!} \binom{k+2x}{k+x} p_1^{k+x} q_1^{x} = \exp(-\lambda t)\, \beta^x I_x(\alpha t) \tag{4}$$

where $\beta = p_1/q_1$, $\alpha = 2\lambda\,(p_1 q_1)^{1/2}$.

$I_n(z)$ is the modified Bessel function of order n with imaginary argument [11]. Note that, since $I_n(z) = I_{-n}(x)$, the probability distribution $P(\nu,t)$ is defined for all $\nu = 0, \pm 1, \pm 2, \ldots$

The fact that the Bessel functions satisfy the Neumann identity [10]

$$I_\nu(t+\tau) = \sum_{k=-\infty}^{\infty} I_k(t)\, I_{\nu-k}(\tau)$$

implies that

$$P(\nu, t+\tau) = \sum_{k=-\infty}^{\infty} P(k,t)\, P(\nu-k,\tau)$$

which is the Chapman-Kolmogorov equation for a Markov process. That $P(\nu,t)$ is Markovian enables us to obtain (4) as a solution to the following master equation [12]:

$$(\partial/\partial t)P(\nu,t) = q_1\lambda P(\nu+1,t) + p_1\lambda P(\nu-1,t) - \lambda P(\nu,t) \quad , -\infty < \nu < \infty$$

subject to the initial condition $P(\nu,0) = \delta_{\nu,0}$. Solutions similar to (4) have also been considered in connection with particle kinetics and Ising model [13),14)].

From (2), (3) and (4), the Fourier transform of $W(x,t)$ is given by

$$\tilde{W}(u,t) = \sum_{\nu=-\infty}^{\infty} P(\nu,t)\left[\sigma_0(u)\right]^{|\nu|}$$

$$= \exp(-\lambda t)\left\{\sum_{\nu=0}^{\infty}(\beta^\nu + \beta^{-\nu})\, I_\nu(\alpha t)\left[\sigma_0(u)\right]^\nu - I_0(\alpha t)\right\}. \quad (5)$$

Further reduction of the general expression in (5) does not seem possible. We shall, however, be able to discuss general properties of the solution on the basis of it. It is obvious from (5) that since $\beta = (p_1/q_1)^{1/2}$, $\beta' = (q_1/p_1)^{1/2}$ would yield the same expression. This means that in the superimposed randomised random walk, p_1 and q_1 can be interchanged without affecting the probability $W(x,t)$.

In the sequel, we shall be mainly interested in the case of a symmetric basic random walk ($p_0 = q_0 = 1/2$), and we rewrite (5) as

$$\tilde{W}(u,t) = \exp(-\lambda t)\left\{\sum_{\nu=0}^{\infty}(\beta^\nu + \beta^{-\nu})(\cos u)^\nu\, I_\nu(\alpha t) - I_0(\alpha t)\right\}. \quad (6)$$

If we make use of the expansions [15]

$$(\cos u)^{2n} = 2^{-2n}\left\{\sum_{k=0}^{n-1} 2\binom{2n}{k}\cos 2(n-k)u + \binom{2n}{n}\right\}$$

$$(\cos u)^{2n-1} = 2^{2n-2}\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos(2n-2k-1)u$$

and the fact that

$$\langle x^r\rangle(t) = (-i)^r\,(\partial/\partial u)^r\,\tilde{W}(u,t)\Big|_{u=0}$$

it is easily shown after some algebra that for $n = 1,2,\ldots$

$$\langle x^{2n-1}\rangle(t) = 0$$

$$\langle x^{2n}\rangle(t) = 2\exp(-\lambda t)\sum_{r=1}^{\infty}\left[(\beta/2)^r + (\beta/2)^{-r}\right] I_r(\alpha t)\sum_{k=0}^{[r/2]}\binom{r}{k}(r-2k)^{2n}. \quad (7)$$

In order to bring the solution in (7) to bear on Kehr and Kunter's results, we shall first of all consider the special case for which $p_0 = q_0 = 1/2$, corresponding to a symmetric superimposed walk. In this case, $\beta = 1$ and $\alpha = \lambda$, and we have

$$\langle x^{2n}\rangle(t) = 4\exp(-\lambda t)\sum_{r=1}^{\infty} 2^{-r}\, I_r(\lambda t)\sum_{k=0}^{[r/2]}\binom{r}{k}(r-2k)^{2n}.$$

For $n = 1$ and $n = 2$, we further note the identities

$$\sum_{k=0}^{[r/2]}\binom{r}{k}(r-2k)^2 = r\,2^{r-1}$$

and

$$\sum_{k=0}^{[r/2]}\binom{r}{k}(r-2k)^4 = (3r^2-2r)\,2^{r-1}$$

and therefore

$$\langle x^2\rangle(t) = 2e^{-\lambda t}\sum_{r=1}^{\infty} r\, I_r(\lambda t) = \lambda t\, e^{-\lambda t}\left[I_0(\lambda t)+I_1(\lambda t)\right] \tag{8}$$

$$\langle x^4\rangle(t) = 2e^{-\lambda t}\sum_{r=1}^{\infty}(3r^2-2r) I_r(\lambda t) = 3\lambda t - 2\lambda t\, e^{-\lambda t}\left[I_0(\lambda t)+I_1(\lambda t)\right]. \tag{9}$$

The final results in (8) and (9) are arrived at with the help of the following recurrence relation [15]:

$$I_{\nu-1}(z) - I_{\nu+1}(z) = (2\nu/z)\, I_\nu(z). \tag{10}$$

From the following well-known asymptotic formula as $t \to \infty$

$$I_\nu(t) = e^{t}\,(2\pi t)^{-1/2}\left[\sum_{\kappa=0}^{\nu}(-1)^\kappa(\nu,\kappa)(2t)^{-\kappa} + O(t^{-n-1})\right] \tag{11}$$

where

$$(\nu,\kappa) = \Gamma(1/2+\nu+\kappa)/(\kappa!\,\Gamma(1/2+\nu-\kappa)) \quad , \quad (\nu,0)=1$$

it is immediate from (8) and (9) that

$$\langle x^2\rangle(t) \sim (2\lambda t/\pi)^{1/2} \tag{12}$$

$$\langle x^4\rangle(t) \sim 3\lambda t. \tag{13}$$

The formula in (12) is given by Kehr and Kunter using Tauberian arguments. The slight disagreement in the constant between (10) and their formula (34) arises from the fact that essentially their procedure corresponds to taking $W(x,t) = \exp(-2\lambda t)\, I_x(2\lambda t)$.

We note in passing that since [16]

$$\sum_{\nu=1}^{\infty}\nu\, I_{\nu+\alpha}(t) = (e^{t}/2)\int_0^t e^{-z}\, I_\alpha(z)\,dz$$

The expression in (8) can alternately be expressed as

$$\langle x^2\rangle(t) = \int_0^{\lambda t} e^{-z}\, I_0(z)\,dz.$$

The asymptotic behaviour of the moments is slightly more complicated if $\beta \neq 1$. Consider, for example, the second moment

$$\langle x^2\rangle(t) = 2e^{-\lambda t}\sum_{r=1}^{\infty} r(\beta^r+\beta^{-r})\, I_r(\alpha t). \tag{14}$$

On using (10), the first term can be written as

$$\begin{aligned}&2e^{-\lambda t}\sum_{r=1}^{\infty} r\beta^r I_r(\alpha t)\\ &= 2e^{-\lambda t}\left\{(\beta-\beta^{-1})\sum_{r=0}^{\infty}\beta^r I_r(\alpha t) + I_1(\alpha t)+\beta^{-1}I_0(\alpha t)\right\}\\ &= \alpha t\, e^{-\lambda t}\left\{(\beta-\beta^{-1})\exp\left[\alpha t(\beta+\beta^{-1})/2\right] J\left(\frac{\alpha t}{2\beta},\frac{\alpha t\beta}{2}\right) + I_1(\alpha t)+\beta^{-1}I_0(\alpha t)\right\}\end{aligned} \tag{15}$$

where we have made use of the results [16]

$$\sum_{r=1}^{\infty} z^\kappa I_\kappa(y) = \exp\left\{\frac{y}{2}(z+z^{-1})\right\} J\left(\frac{y}{2z},\frac{yz}{2}\right)$$

and $J(u,v) = 1 - e^{-v}\int_0^u e^{-w}\, I_0(2(wv)^{1/2})\,dw$.

It is further shown [16] that if

$$\xi = 2(uv)^{1/2} \quad , \quad \eta = (v/u)^{1/2} \quad , \quad z = (v^{1/2}-u^{1/2})^2$$

then for $\eta < 1$, z/ξ small as $\xi \to \infty$ (which is the case in hand) we have

$$J(u,v) \sim \exp\left[-(z+\xi)\right] I_0(\xi)/2.$$

From (15) we may then deduce that, as $t \to \infty$

$$\langle x^2\rangle(t) \sim (\alpha t/2)\exp(-\lambda t)\left[(\beta+\beta^{-1})I_0(\alpha t) + 2I_1(\alpha t)\right].$$

Finally, on using (11), we deduce that

$$\langle x^2\rangle(t) \sim \left[\alpha(\beta+\beta^{-1}+2)(t/\pi)^{1/2}/2\right]\exp\{-(\lambda-\alpha)t\} . \tag{16}$$

For $\beta = 1$ and hence $\alpha = \lambda$, the result in (16) of course reduces to that given previously in (12).

III. RANDOMISED RANDOM WALK: RENEWAL PROCESS TRANSITIONS

In this section we shall generalize the results obtained in the previous section to the case when the superimposed random walk is symmetric but with transitions occurring according to a stationary renewal process.

Let $\phi(t)$ be the interval probability density function (pdf) of the renewal process and $\phi_n(t)$ be the pdf of the time at which the n^{th} event occurs. To be precise, $\phi_n(t)\,dt = \Pr$ [given an event at time $t = 0$, the n^{th} event occurs in the time interval $(t, t+dt)$]. Obviously, $\phi_n(t)$ satisfy the following recurrence:

$$\phi_0(t) = \delta(t)$$
$$\phi_1(t) = \phi(t)$$
$$\vdots$$
$$\phi_n(t) = \int_0^t \phi_{n-1}(t-\tau)\,\phi(\tau)\,d\tau \quad , \quad n \geq 1 .$$

We define further $Q(n,t)$ as the probability that, given an event at time $t = 0$, the interval $(0,t)$ contains exactly n events. This is a probability distribution in n $\left[\sum_n Q(n,t) = 1\right]$ for all t. Note also that by this definition, an event is assumed to have occurred at time $t = 0$ and the semi-closed interval $[0,t)$ actually contains $(n+1)$ events.

By standard renewal arguments, it is easily found that

$$Q(n,t) = \int_0^t \phi_n(t-t_1)\int_0^{t_1}\phi_B(t_1-\tau)\,d\tau \tag{17}$$

where $\phi_B(t)$ is the backward recurrence time function of the renewal process [14]. Under the assumption of stationarity it may be shown that

$$\phi_B(t) = \int_t^\infty \phi(\tau)\,d\tau$$

and

$$\phi_B^*(s) = \left[1-\phi^*(s)\right]/s$$

where the Laplace transform of a function $f(t)$ is defined as

$$f^*(s) = \int_0^\infty e^{-st} f(t)\,dt .$$

Thus from (17)

$$Q^*(n,s) = \left[1-\phi^*(s)\right]\left[\phi^*(s)\right]^n/s . \tag{18}$$

We would like to mention in passing that if we adopt a slightly different definition of $R(n,t)$ as the probability that the interval $(0,t)$ contains exactly n events (note that in this definition the beginning and ending time instants do not coincide with the occurrence of an event), then we would have [17),18)]

$$R(n,t) = \int_0^t \phi_B(t-t_2)\,dt_2 \int_0^{t_2}\phi_n(t_1)\,\phi_F(t_2-t_1)\,dt_1 \quad , \quad n \geq 1$$

$$= 1 - \int_0^t \phi_F(t_1)\,dt_1 . \tag{19}$$

Here $\phi_F(t)$ is the forward recurrence time function [19)] and

$$\phi_F(t) = \mu\int_t^\infty \phi(\tau)\,d\tau$$

$$\phi_F^*(s) = \mu\left[1-\phi^*(s)\right]/s$$

where μ, the mean rate of the renewal process, is given by

$$\mu = \left[\int_0^\infty t\,\phi(t)\,dt\right]^{-1} = -\left[\phi^{*\prime}(0)\right]^{-1} .$$

Thus from (19)

$$R^*(n,s) = \mu\left[1-\phi^*(s)\right]^2\left[\phi^*(s)\right]^{n-1}/s^2 \quad , \quad n \geq 1$$

$$= 1/s - \mu\left[1-\phi^*(s)\right]/s^2 \quad , \quad n = 0 . \tag{20}$$

The corresponding formula in (4) may now be written (in the double Fourier-Laplace transform domain) as

$$\tilde{P}^*(u,s) = \sum_{n=0}^{\infty} Q^*(n,s)\left[\sigma_1(u)\right]^n$$

$$= s^{-1}\left[1-\phi^*(s)\right]/\left[1-\phi^*(s)\,\sigma_1(u)\right] \tag{21}$$

on using the result in (18). Or, if we use the result in (20) we would have

$$\tilde{P}_1^*(u,s) = \sum_{n=0}^{\infty} R^*(n,s)\left[\sigma_1(u)\right]^n$$

$$= s^{-1} - s^{-2}\left[1-\phi^*(s)\right]\left[1-\sigma_1(u)\right]/\left[1-\phi^*(s)\,\sigma_1(u)\right]. \tag{22}$$

A result similar to (21) has been previously given by Montroll and West [20].

It is interesting to note that for the special case of Poisson transitions considered in the previous section, the interval pdf is of course exponential with parameter ($\lambda = \mu$)

$$\phi(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$
$$= 0, \quad t < 0.$$

Here $\phi^*(s) = \lambda/(\lambda+s)$ and it is easily derived from both (21) and (22) that

$$\tilde{P}^*(u,s) = \tilde{P}_1^*(u,s) = \left[s+\lambda\,\sigma_1(u)\right]^{-1}. \tag{23}$$

The fact that for the Poisson transition case both P and P_1 yield the same expression is due to the "lack of memory" of the underlying exponential pdf, in that the situations of starting with an event or at an arbitrary time are equivalent; other renewal processes would yield different results in these cases.

On inverting the Laplace transform in (23) we have

$$\tilde{P}(u,t) = \exp\left\{-\lambda t\left[1-\cos u - i(p_1-q_1)\sin u\right]\right\}.$$

Further progress can be made in inverting the Fourier transform by noting that [20]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iu\nu}\exp\left(a\cos u + b\sin u\right)du$$

$$= \left[(a+ib)/(a^2+b^2)^{1/2}\right]^{-\nu} I_\nu\left[(a^2+b^2)^{1/2}\right]$$

and finally

$$P(\nu,t) = e^{-\lambda t}\,(p_1/q_1)^{\nu/2}\, I_\nu\left[2\lambda t(p_1q_1)^{1/2}\right]$$

the same result we obtained previously.

We shall now consider the case when $\sigma_1(u) = \cos u$ (implying a symmetric superimposed walk) and note the inverse transform relation [21]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iu\nu}(1-z\cos u)^{-1}du = y^{|\nu|}(1-z^2)^{-1/2}$$

where $y = [1-(1-z^2)^{1/2}]/z$. Thus (21) gives

$$P^*(\nu,s) = \frac{1}{s}\left[\frac{1-\phi^*(s)}{1+\phi^*(s)}\right]^{1/2}\left\{\frac{1-\left[1-\phi^{*2}(s)\right]^{1/2}}{\phi^*(s)}\right\}^{|\nu|}. \tag{24}$$

From (3) we have the double transform result

$$\tilde{W}^*(u,s) = \sum_\nu P^*(\nu,s)\left[\sigma_0(u)\right]^{|\nu|}$$

$$= \frac{1}{s}\left[\frac{1-\phi^*(s)}{1+\phi^*(s)}\right]^{1/2}\cdot\frac{\phi^*(s)+\sigma_0(u)\left\{1-\left[1-\phi^{*2}(s)\right]^{1/2}\right\}}{\phi^*(s)-\sigma_0(u)\left\{1-\left[1-\phi^{*2}(s)\right]^{1/2}\right\}}. \tag{25}$$

The expression in (25) then gives us explicitly the solution, in transform form, of the probability distribution $W(x,t)$ for the case of a symmetric randomised random walk where transitions occur in renewal instants and superimposed on an asymmetric random walk with structure factor $\sigma_0(u)$ given in (1).

The mean and second order moment can be readily obtained, abeit labouriously from (25). Thus we find

$$\langle x\rangle^*(s) = s^{-1}(p_0-q_0)\phi^*(s)/[1-\phi^{*2}(s)]^{1/2}. \tag{26}$$

For example, when $\phi^*(s) = \lambda/(\lambda+s)$, then

$$\langle x\rangle^*(s) = \lambda(p_0-q_0)\, s^{-3/2}(s+2\lambda)^{-1/2}$$

and standard Laplace inversion table gives

$$\langle x\rangle(t) = \lambda(p_0-q_0)\, e^{-\lambda t}\left[(1+2\lambda t)\, I_0(\lambda t) + 2\lambda t\, I_1(\lambda t)\right].$$

Writing (26) in the form

$$\langle x\rangle^*(s) = (p_0-q_0)\cdot\frac{\phi^*(s)}{[1+\phi^*(s)]^{1/2}}\cdot\frac{1}{s^{3/2}[1-\phi^*(s)]/s} \tag{27}$$

and noting that $\phi^*(s) \sim 1 - \mu^{-1}s + O(s^2)$ as $s \to 0$, and we have

$$\langle x\rangle^*(s) \sim \left[(p_0-q_0)/2^{1/2}\right]\left(\mu/s^3\right)^{1/2}. \tag{28}$$

By applying the Tauberian theorem of Hardy-Littelwood-Karamata [22], we see that as $t \to \infty$

$$\langle x\rangle(t) \sim (p_0-q_0)\left(\frac{2\mu t}{\pi}\right)^{1/2}. \tag{29}$$

Thus, the time dependent mean is non-zero if the basic random walk is asymmetric. Furthermore, it may be shown that

$$\langle x^2\rangle^*(s) = \frac{\phi^*(s)}{s\{[1-\phi^*(s)][1+\phi^*(s)]\}^{1/2}} + \frac{2(p_0-q_0)^2\phi^*(s)\{1-[1-\phi^{*2}(s)]^{1/2}\}}{s[1-\phi^*(s)]\{\phi^*(s)+1-[1-\phi^{*2}(s)]^{1/2}\}}. \tag{30}$$

By analogous arguments in arriving at (28), we find that

$$\langle x^2\rangle^*(s) \sim \left(\frac{\mu}{2s^3}\right)^{1/2} + (p_0-q_0)^2\cdot\frac{\mu}{s^2}.$$

Thus, it is seen that as $t \to \infty$

$$\begin{aligned}\langle x^2\rangle(t) &\sim (p_0-q_0)^2\,\mu t \qquad &&\text{if } p_0 \neq q_0\\ &\sim \left(\frac{2\mu t}{\pi}\right)^{1/2} &&\text{if } p_0 = q_0.\end{aligned} \tag{31}$$

It is rather interesting to note from (31) that the time-dependent mean-square displacements of a particle undergoing randomised random walk on a random walk critically depend on the bias in the step transition probabilities of the basic random walk. The asymmetric case gives rise to a diffusion-like long time behaviour while in the symmetric case, a square-root law is predicted.

IV. CONCLUDING REMARKS

In this paper we have discussed the model proposed by Kehr and Kunter concerning the random walk on a random walk in the case when the superimposed random walk is randomised in time according to the occurrence of some stochastic point processes. Certain generalizations are made with regard to the structure of the basic and superimposed random walks in that unequal forward and backward step transition probabilities are incorporated in one or both of the processes.

In the case when transitions in the superimposed random walk are randomised in accordance with a Poisson process, fairly explicit results have been obtained. In particular, from the closed form expressions for the moments we are able to deduce quite simply their asymptotic behaviour which agrees with that given by Kehr and Kunter using other means. Finally, the general case when transitions take place according to a stationary renewal process is discussed where the general solution of the probability $W(x,t)$ has been obtained explicitly in the transform domain for the case when the superimposed random walk is symmetric; the basic random walk may be asymmetric.

Of particular interest is the conclusion that when the basic random walk is symmetric the asymptotic behaviour for large time of mean-square displacement is still given by

$$\langle x^2\rangle(t) \sim \left(\frac{2\mu t}{\pi}\right)^{1/2}$$

where now μ is the mean rate of the underlying renewal point process. However, if asymmetry is introduced into the basic random walk, one finds the following asymptotic large-time behaviour

$$\langle x\rangle(t) \sim (p_0-q_0)\left(\frac{2\mu t}{\pi}\right)^{1/2}$$

$$\langle x^2\rangle(t) \sim (p_0-q_0)^2 \mu t .$$

In analysing the phenomenon of self-diffusion in a one-dimensional lattice gas, Kehr and Kunter have introduced a "correlated" random walk by modifying the basic random walk so that the probabilities p_f for forward steps and p_b for backward steps are different. These are related to the concentration c of particles via the relations $p_f = 1 - c$ and $p_b = c$. This is precisely the asymmetric basic random walk case we discussed in Sec.III. They have argued, on intuitive ground that properties in their correlated random-walk model are deducible from those for the non-correlated (symmetric) random walk model by the inclusion of a linear correlation factor

$$f = \frac{1+p_f-p_b}{1-p_f+p_b} = \frac{1-c}{c} .$$

Thus they suggest that asymptotic time-dependent mean-square displacement of a particle performing continuous time random walk on the modified basic chain is given by their Eqs.(43), viz.

$$\langle x^2\rangle(t) = \frac{2(1-c)}{c}\left(\frac{\Gamma t}{\pi}\right)^{1/2} .$$

This, when compared with their Eq.(34), is seen to be of the same form except for the scaling factor of f.

On the basis of the detailed analysis in the previous section and the results enunciated in (29) and (31), we believe the above conclusion is incorrect. A cursory look at the general solution in (25) would convince one that the occurrence of the structure factor $\sigma_0(u)$ has substantially modified the form of the solution and is definitely more than just introducing a scaling correlation factor of the type proposed.

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