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ABSTRACT

The spatially symmetric electrovac problem is investigated in the General Relativity, with common, geodesic and shearing eigenrays. It is shown that all these solutions are Ernst counterparts of the corresponding vacuum solutions.

АННОТАЦИЯ

Изучены пространство-симметричные решения уравнения Эйнштейна-Максвелла с совпадающими геодезическими собственными лучами со сдвигом. Показано, что эти решения являются парами Эрнста соответствующих вакуумных решений.

KIVONAT

Az általános relativitáselméletben vizsgáljuk a térszerűen szimmetrikus, közös geodetikus nyíró sajátsugarakkal rendelkező elektrovákuummegoldásokat. Az adódik, hogy e megoldások mind a megfelelő vákuummegoldások Ernst-párjai.

1. INTRODUCTION

The spin coefficient technique has led to many new solutions of the Einstein equation of General Relativity, Without assuming any symmetry, the 4-dimensional Newman-Penrose equations can be solved for geodesic rays, and the Kerr solution is among these solutions, belonging to a special subclass, where the shear of the rays vanishes too [1], [2], [3]. When the space-time possesses a non-null Killing symmetry, the problem is essentially 3-dimensional, and after a decomposition

$$ds^2 = f(dy + \omega_r dx^r)^2 - f^{-1} g_{rs} dx^r dx^s \qquad (1.1)$$

$$i = 1,2,3$$

it can be reformulated in a 3-dimensional background space or space-time, whose metric tensor is $g_{ik}(x^m)$ [4], [5] Then the eigenrays play a role analogous to that of the rays in 4-dimensions. (The definition of eigenrays can be found in Refs. 4 and 5; they are projections of rays if the rays are shearfree.) Being the dimensionality of the background space smaller, more cases can be analytically handled there, in fact, all the classes $\kappa\sigma=0$ have been integrated in vacuum both for stationarity and for space-like symmetry. Since the class $\kappa=\sigma=0$ is known from the 4-dimensional calculation, this means $\sigma\neq\kappa=0$ and $\kappa\neq\sigma=0$ as new solutions. (When $\sigma\neq0$, the eigenrays do not coincide with the projections of the shears, so, even if they are geodesic, the rays are not.) Unfortunately, none of these new classes contains any generalization of the Kerr metric [6-9].

In the presence of material fields the problem is more complicated, but it is interesting that for stationary, rigidly rotating dust the same $\kappa\sigma=0$ classes have been integrated [10]. With pressure the integration has been successful only for $\kappa=\sigma=0$ [11].

The electrovac case shows some structural similarities with the vacuum problem. Nevertheless, the equations are more complicated, and generally their integration is still not performed. However, there is a special case, i.e. when the gravitational field $\underline{G}$ (produced by the decomposition process) and the electromagnetic field $\underline{H}$ have common eigenrays, which can be handled in the same way as the vacuum. The condition that the eigenrays of the two field coincide can be formulated algebraically as [12]

$$(\underline{G}\times\underline{H})^2 = 0 \tag{1.2}$$

Here $\underline{G}$ and $\underline{H}$ are defined in a rather complicated way, the actual forms will be discussed in Sect. 2.

For stationarity the integration was successful in each case when the vacuum equations could be integrated [7], [12], and the class $\kappa=\sigma=0$ contains the electrified generalization of the Kerr solution [13]. There is a strong tendency to get the Ernst counterparts of the vacuum solutions, nevertheless, only when the strength of the $\underline{G}$ field dominates that of the $\underline{H}$ field [7], [12].

For space-like symmetry the $\sigma=0$, $\kappa\neq0$ solutions are known, containing only Ernst counterparts of the vacuum metrics [9]. So there remains the case $\sigma\neq0$, $\kappa=0$ as promising for integration.

Here we show that this class contains also Ernst counterparts only. Having done this, the process of integrating the $\kappa\sigma=0$, $(\underline{G}\times\underline{H})^2=0$ metrics essentially ends, except producing explicit forms for special line elements, if necessary. Since the Kerr and Kerr-Newman solutions are isolated among the $\kappa\sigma=0$ metrics, obviously the $\kappa\neq0$, $\sigma\neq0$ class should be investigated in order to get asymptotically flat solutions, however, until now constructive methods are not known for handling this class.

2. THE FIELD EQUATIONS FOR ELECTROVAC

Consider the 3+1 decomposition for stationary space-times as in eq. (1.1). A complex vector field $\underline{G}$

$$\underline{G} = \frac{1}{2f}(\nabla f - if^2 \nabla x \underline{\omega}) \tag{2.1}$$

can be introduced, instead of the derivatives of f and $\underline{\omega}$. From the electromagnetic potential A^μ a complex three-vector

$$\underline{H} = \frac{1}{\sqrt{|f|}}[\nabla A_o + i(\nabla x \underline{A} - \underline{\omega} x \nabla A_o)f] \tag{2.2}$$

can be constructed, and then the sourcefree Einstein-Maxwell equations contain f, $\underline{\omega}$, A_o and $\underline{A}$ only in these combinations [12], [14], [15]. In fact, the field equations have the form

$$(\nabla - \underline{G})\underline{G} = \bar{\underline{H}}\underline{H} \pm \bar{\underline{G}}\underline{G} \tag{2.3a}$$

$$\nabla x \underline{G} = \bar{\underline{H}} x \underline{H} \pm \bar{\underline{G}} x \underline{G} \tag{2.3b}$$

$$(\nabla - \underline{G})\underline{H} = \frac{1}{2}(\underline{G} - \bar{\underline{G}})H \tag{2.3c}$$

$$\nabla x \underline{H} = -\frac{1}{2}(\underline{G} + \bar{\underline{G}}) x \underline{H} \tag{2.3d}$$

$$\underline{\underline{R}} = -\underline{G}o\underline{G} - \underline{G}o\bar{\underline{G}} \pm \underline{H}o\bar{\underline{H}} \pm \bar{\underline{H}}o\underline{H} \tag{2.3e}$$

where $\pm$ stands for sgn (f). Here all the tensorial operations are meant with respect to the metric g_{ik} of the background space. Now, eqs. (2.3b,d) are integrability conditions for some scalars B,φ:

$$\begin{aligned} B_{,i} &= \epsilon_{ikl}(A^{k;l} - \omega^k A_o{}^{;l}) f\sqrt{g} \\ \varphi_{,i} &= \epsilon_{ikl}\omega^{k;l} f^2 \sqrt{g} + 2(BA_{o,i} - A_o B_{,i}) \end{aligned} \tag{2.4}$$

so one can define two complex scalars [12], [14], [15]

$$\varepsilon = f - \phi\bar{\phi} + i\varphi \qquad (2.5)$$
$$\phi = A_o + iB$$

by means of which the remaining of eqs. (2.3) get the form

$$(\mathrm{Re}\varepsilon + \phi\bar{\phi})\Delta\varepsilon = (\nabla\varepsilon + 2\bar{\phi}\nabla\phi)\nabla\varepsilon$$

$$(\mathrm{Re}\varepsilon + \phi\bar{\phi})\Delta\phi = (\nabla\varepsilon + 2\bar{\phi}\nabla\phi)\nabla\phi \qquad (2.6)$$

$$R_{ik} = -\frac{1}{2}(\mathrm{Re}\varepsilon + \phi\bar{\phi})^{-2}\mathrm{Re}(\varepsilon_{,i}\bar{\varepsilon}_{,\underline{k}} + 2\phi\bar{\phi}_{,i}\varepsilon_{,k} +$$
$$+ 2\phi\bar{\phi}_{,k}\varepsilon_{,i} - 4(\mathrm{Re}\varepsilon)\phi_{,i}\phi_{,k})$$

The form of these formulae is independent of the sign of $f=K_\mu K^\mu$; the $\pm$ signs in eqs. (2.3) are required because the definition of $\underline{H}$ contains a square root.

Eqs. (2.6) are called Ernst equations, because they were found first by Ernst for the stationary axisymmetric problem [16]. They remain valid even if there is only one symmetry [14], but for two symmetries the metric in them is flat (in cylindric coordinates), while now it should be calculated from the last of eq. (2.6).

Now, it is useful to introduce new field quantities instead of ε and φ as

$$\varepsilon = \frac{\xi-1}{\xi+1} \qquad (2.7)$$
$$\phi = \frac{q}{\xi+1}$$

Then the first two of eqs. (2.6) get the form

$$(\xi\bar{\xi} + q\bar{q} - 1)\Delta\xi = 2(\bar{\xi}\nabla\xi + \bar{q}\nabla q)\nabla\xi \qquad (2.8)$$
$$(\xi\bar{\xi} + q\bar{q} - 1)\Delta q = 2(\bar{\xi}\nabla\xi + \bar{q}\nabla q)\nabla q$$

Let us assume that, for some reason, $\underline{G} = \alpha\underline{H}$, and none of them vanishes. This means that

$$q = q(\xi) \tag{2.9}$$

Substituting this into the second of eqs. (2.8) one gets $q_{,\xi\xi}=0$, i.e.

$$q = C + K(\xi + 1) \tag{2.10}$$

C and K being constant. But the second term yield only a constant in the potential ϕ, which can be removed, according to eqs. (2.4-5). Thus there remains

$$q = q_o = \text{const.} \tag{2.11}$$

and then the first of eqs. (2.8) becomes [12]

$$(\xi\bar{\xi} + q_o q_o - 1)\Delta\xi = 2\cdot\bar{\xi}(\nabla\xi)^2 \tag{2.12}$$

If $|q_o| < 1$, the substitution

$$\xi = \sqrt{1 - q_o\bar{q}_o}\,\eta \tag{2.13}$$

leads to the vacuum Ernst equation for η. Thus the $\underline{G}=\alpha\underline{H}$, $|q_o|<1$ metrics are called the Ernst counterparts of the corresponding vacuum solutions. They can be generated in an almost trivial way from the vacuum metrics, since only f and φ get new expressions, g_{ik} remains unchanged.

If $|q_o|\geq 1$, such a generation is possible, but not from the vacuum solutions. So $|q_o|\geq 1$ metrics are not Ernst counterparts of the vacuum line elements. In fact, for stationarity, some such solutions are explicitly known [7], [12].

The case $|q_o|\geq 1$ is not possible for spacelike symmetry, because then $f = \text{Re}\varepsilon + \phi\bar{\phi}$ would not be negative (cf. eq. (2.7)). So for $f<0$ all the $\underline{G} = \alpha\underline{H}$ metrics are Ernst counterparts of vacuum solutions.

3. THE SPIN COEFFICIENT EQUATIONS

The complete set of field equations in tensorial form is given in eq. (2.3). Hence we can proceed as in Ref. 5, by introducing a complex basic vactor triad. The result is a system of spin coefficient equations (cf. Ref. 15). If the real vector of the triad is chosen a tangent to the eigenray congruence, then one component of $\underline{G}$, G_-, is 0. In contrast to the stationary case, now there are some exceptional cases when eigenrays do not exist [5], [15]. However, here we assume that eigenrays do exist, both for the $\underline{G}$ and for the $\underline{H}$ fields. After choosing the triad suitably, the spin coefficient ϵ can be made 0 by permitted triad rotation [5], [15].

According to our fundamental assumption, $\underline{G}$ and $\underline{H}$ possess common eigenrays, so there exists such a triad gauge that

$$\epsilon = G_- = H_- = 0 \tag{3.1}$$

and, since in the investigated class the eigenrays are geodesic and shearing,

$$\sigma \neq \kappa = 0 \tag{3.2}$$

Then the nontrivial equations are as follow (cf. Ref. 15):

$$G\rho = -\rho^2 - \sigma\bar{\sigma} - \gamma^2 \tag{3.3a}$$

$$D\sigma = -(\rho+\bar{\rho})\sigma \tag{3.3b}$$

$$D\tau = -\rho\tau + \bar{\sigma}\bar{\tau} - G_o\bar{G}_- - H_o\bar{H}_- \tag{3.3c}$$

$$\delta\rho - \bar{\delta}\sigma = 2\sigma\tau - \bar{G}_oG_+ - \bar{H}_oH_+ \tag{3.3d}$$

$$\delta\tau + \bar{\delta}\bar{\tau} = -2\tau\bar{\tau} - \sigma\bar{\sigma} + \rho\bar{\rho} - \gamma^2 - G_+\bar{G}_- - H_+\bar{H}_- \tag{3.3e}$$

$$DG_o = (-2\bar{\rho} + G_o)G_o - \gamma^2 \tag{3.3f}$$

$$DH_o = (-2\bar{\rho} + \frac{3}{2} G_o - \frac{1}{2} \bar{G}_o) H_o \qquad (3.3g)$$

$$\delta G_o - DG_+ = (\bar{\rho} + \bar{G}_o) G_+ + \bar{H}_o H_+ \qquad (3.3h)$$

$$\delta H_o - DH_+ = (\bar{\rho} + \frac{1}{2} G_o + \frac{1}{2} \bar{G}_o) H_+ - \frac{1}{2} G_+ H_o \qquad (3.3i)$$

$$\bar{\delta} G_o = \sigma G_+ - \bar{G}_- G_o - \bar{H}_- H_o \qquad (3.3k)$$

$$\bar{\delta} H_o = \bar{\sigma} H_+ - \frac{1}{2} \bar{G}_- H_o \qquad (3.3l)$$

$$\bar{\delta} G_+ = (\rho - \bar{\rho}) G_o - \tau G_+ - \bar{G}_- G_+ - \bar{H}_- H_+ \qquad (3.3m)$$

$$\bar{\delta} H_+ = (\rho - \bar{\rho}) H_o - (\tau + \frac{1}{2} \bar{G}_-) H_+ \qquad (3.3n)$$

where, as a shorthand notation,

$$\gamma^2 = G_o \bar{G}_o + H_o \bar{H}_o \qquad (3.4)$$

If $H_o = 0$, then, from eq. (3.3l), $H_+ = 0$ too, which is the vacuum case not investigated here (cf. Ref. 8). Thus $H_o \neq 0$. But then $\gamma^2 \neq 0$, and, from eq. (3.3f), $G_o \neq 0$ too.

The differential operators D, δ and $\bar{\delta}$ commute as [5]

$$D\delta - \delta D + \bar{\rho}\delta + \sigma\bar{\delta} = 0 \qquad (3.5a)$$

$$\delta\bar{\delta} - \bar{\delta}\delta - \tau\delta + \bar{\tau}\bar{\delta} + (\rho - \bar{\rho}) D = 0 \qquad (3.5b)$$

From eq. (3.3b)

$$D(\sigma/\bar{\sigma}) = 0 \qquad (3.6)$$

Such a phase factor can be removed from σ by means fo the remaining triad rotations [5], [15], so from here σ is real and positive, and the triad is completely fixed.

4. THE PROOF OF THE PROPORTIONALITY BETWEEN $\underline{G}$ AND $\underline{H}$

The steps follow here are some amalgam of the calculations in Refs. 6, 8 and 12, so it is needless to go into details of the identical steps. First we apply the commutator (3.5a) on $\ln G_o$:

$$\delta(\ln(G_o\sigma)) = G_+ - 2\bar{\tau} \tag{4.1}$$

just as in the vacuum case [8]. Applying it on $\ln H_o$ the result is

$$\delta(\ln(H_o\sigma)) = \frac{1}{2} G_+ - 2\bar{\tau} + \frac{H_+G_o}{H_o} \tag{4.2}$$

and then the propagation laws for γ are

$$D\gamma = -(\rho + \bar{\rho})\gamma \tag{4.3a}$$

$$\delta\gamma^2 = -(2\bar{\tau} + \delta\ln\sigma)\gamma^2 + (G_o\bar{G}_- + H_o\bar{H}_-)\sigma \tag{4.3b}$$

Taking the mixed derivatives of γ^2 one gets

$$\gamma^2(3\bar{\delta}\rho + \bar{\delta}\bar{\rho} + 2\delta\sigma) + \sigma\delta\gamma^2 = 0 \tag{4.4}$$

again as in Refs. 6, 8 and 12. Hence the steps of Ref. 6 can be repeated, with the redefinition of $\delta_{\pm}$ according to Ref. 12, arriving again at

$$\delta\sigma = \delta\gamma = \delta\rho = \bar{\delta}\rho = 0 \tag{4.5}$$

The only nonvanishing component of $\underline{G}\times\underline{H}$ is

$$X = G_oH_+ - H_oG_+ \tag{4.6}$$

Eqs. (3.3) yield the propagation laws for X as

$$DX = \frac{3}{2}(-2\bar{\rho} + G_o - \bar{G}_o)X$$
$$\bar{\delta}X = -(\tau + \frac{3}{2}\bar{G}_-)X \tag{4.7}$$

From eqs. (3.3d), (4.3b) and (4.5)

$$(\sigma^2 - \gamma^2)\tau = 0$$
$$2\gamma\tau = \bar{G}_o G_+ + \bar{H}_o H_+ \qquad (4.8)$$

whence

$$X\bar{X} = \gamma^2(G_+\bar{G}_- + H_+\bar{H}_- - 4\tau\bar{\tau}) \qquad (4.9)$$

If $\tau=0$, eqs. (4.8-9) immediately yield $X = 0$. If not, $\sigma^2 = \gamma^2$, and, from the $\bar{\delta}$ derivative of eq. (3.3d) one gets

$$\bar{\delta}\tau = -3\tau^2 + \frac{1}{2}(\rho - \bar{\rho})\sigma \qquad (4.10)$$

Acting on this equation by D,

$$X\bar{X} + \gamma^2(\sigma^2 + \gamma^2 - \bar{\rho}^2) = 0 \qquad (4.11)$$

Hence ρ is real. Now, taking the mixed derivatives of eq. (4.7):

$$\delta \ln X = \frac{3}{2} G_+ - 7\bar{\tau} \qquad (4.12)$$

Applying now the commutator (3.5b) on ln X, the result is

$$\gamma^2 + \sigma^2 = \rho^2 \qquad (4.13)$$

But then, compared this to eq. (4.11)

$$X = 0$$

that is, $\underline{G}$ and $\underline{H}$ are proportional vectors.

5. THE SOLUTIONS

We showed in Sect. 2 that in the space-like symmetric case if $\underline{G}$ and $\underline{H}$ are proportional vectors, then the solution is an Ernst counterpart of the corresponding vacuum solution. The vacuum solutions are given in Ref. 8, and there is no reason to explicitly list the counterparts here, being the generation process is trivial [12]. As it can be seen by investigating the vacuum solutions, there are 3 subclasses. The first possesses 3 Killing vectors with the commutation

$$[K_1,K_2] = -2\gamma^o Q K_3$$
$$[K_1,K_3] = 0 \qquad (5.1)$$
$$[K_2,K_3] = 0$$

where γ^o and Q are constant parameters of the solution. By redefining the Killing vectors the right hand side coefficient in eq. (5.1) can be made 1, if it is not 0. Such a symmetry group does not seem to imply obvious physical meaning, except the case $Q=0$, when the vacuum solution is the Kasner Universe.

In the second case there are two commuting space-like Killing vectors. Until now, no physical interpretation has been found for these solutions.

The same is true for the third case, which possesses only one spatial Killing vector, which was originally assumed for the decomposition.

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