IC/83/186
INTERNAL REPORT
(Limited distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

OSCILLATIONS OF FIRST ORDER RETARDED AND ADVANCED FUNCTIONAL DIFFERENTIAL EQUATIONS*

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ABSTRACT

In this paper we consider first order linear and non-linear functional differential equations with continuous distributed retarded and advanced arguments. We give some good conditions under which every solution of these equations oscillates. We generalized and improved results of Ref. [1] ,[2] .

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November 1983

* To be submitted for publication

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I. INTRODUCTION

In this paper we consider first order linear and nonlinear functional differential equations with continuous distributed retarded arguments

$$y'(t)+\int_a^b p(t,\xi)\,y[g(t,\xi)]\,d\sigma(\xi)=0, \qquad (b>a) \tag{1.1}$$

$$y'(t)+\int_a^b f(t,\xi,y[g(t,\xi)])\,d\sigma(\xi)=0, \qquad (b>a) \tag{1.2}$$

and the advanced type

$$y'(t)-\int_a^b p(t,\xi)\,y[h(t,\xi)]\,d\sigma(\xi)=0, \quad (b>a) \tag{1.3}$$

$$y'(t)-\int_a^b f(t,\xi,y[h(t,\xi)])\,d\sigma(\xi)=0, \quad (b>a)\,. \tag{1.4}$$

Oscillations of these equations are studied.

Below we shall make the following assumptions:

(H_1) $p: R^+\times[a,b]\to R^+$ is continuous, $R^+=[0,+\infty)$;

(H_2) $\sigma:[a,b]\to R$ is a nondecreasing function;

(H_3) The integrals in equations are Stieltjes integrals;

(H_4) $g(t,\xi)\le t$, $\xi\in[a,b]$, $g: R^*\times[a,b]\to R^+$ is continuous, $R^*=[t_0,+\infty)$, $g(t,\xi)$ is a nondecreasing function with t or ξ respectively;

(H_5) $h(t,\xi)\ge t$, $\xi\in[a,b]$, $h: R^*\times[a,b]\to R^+$ is continuous, $h(t,\xi)$ is a nondecreasing function with t or ξ respectively;

(H_6) There exists function $\varphi(t,\xi)$ such that $\varphi(\varphi(t,\xi),\xi)=g(t,\xi)$, $\varphi(t,\xi)$ is a nondecreasing function with t or ξ respectively;

$$\lim_{t\to\infty}\min_{\xi\in[a,b]}\{\varphi(t,\xi)\}=+\infty$$

(H_7) There exists function $\psi(t,\xi)$ such that $\psi(\psi(t,\xi),\xi)=h(t,\xi)$, $\psi(t,\xi)$ is a nondecreasing function with t or ξ respectively;

$$\lim_{t\to\infty}\min_{\xi\in[a,b]}\{\psi(t,\xi)\}=+\infty\,.$$

Recently G. Ladas and I.P. Stavroulakis [1] obtained sufficient conditions under which all solutions of linear R.F.D.E. and A.F.D.E. with variable coefficients and constant delay are oscillatory. The following two lemmas are due to G. Ladas and I.P. Stavroulakis.

Lemma 1. Consider the differential equation with retarded arguments

$$y'(t)+\sum_{i=1}^n p_i(t)\,y(t-r_i)=0 \tag{1.5}$$

and assume that $p_i(t)\geq 0$; $r_i\geq 0$ and

$$\liminf_{t\to\infty}\int_{t-\frac{r_i}{2}}^t p_i(s)\,ds>0, \quad i=1,2\cdots n\,. \tag{1.6}$$

Then each one of the following conditions:

$$\liminf_{t\to\infty}\int_{t-r_i}^t p_i(s)\,ds>\frac{1}{e} \quad \text{for some } i,\ i=1,2\cdots n \tag{1.7}$$

$$\liminf_{t\to\infty}\int_{t-r}^t \sum_{i=1}^n p_i(s)\,ds>\frac{1}{e}\,,$$

where

$$r=\min\{r_1,r_2\cdots r_n\} \tag{1.8}$$

$$\left[\prod_{i=1}^n\left(\sum_{j=1}^n \liminf_{t\to\infty}\int_{t-r_j}^t p_i(s)\,ds\right)\right]^{1/n}>\frac{1}{e} \tag{1.9}$$

or

$$\frac{1}{n}\sum_{i,j=1}^n\left\{\left[\liminf_{t\to\infty}\int_{t-r_j}^t p_i(s)\,ds\right]\cdot\left[\liminf_{t\to\infty}\int_{t-r_i}^t p_j(s)\,ds\right]\right\}^{1/2}>\frac{1}{e} \tag{1.10}$$

implies that every solution of (1.5) oscillates.

Lemma 2. Consider the differential equation with advanced arguments

$$y'(t)-\sum_{i=1}^n p_i(t)\,y(t+r_i)=0 \tag{1.11}$$

and assume that $p_i(t) \geq 0$, $r_i \geq 0$ and

$$\liminf_{t\to\infty} \int_t^{t+\frac{r_i}{2}} P_i(s)\,ds > 0, \quad i = 1, 2, \cdots n \tag{1.12}$$

Then each one of the following conditions

$$\liminf_{t\to\infty} \int_t^{t+r_i} P_i(s)\,ds > \frac{1}{e} \quad \text{for some } i, \ i = 1, 2, \cdots n \tag{1.13}$$

$$\liminf_{t\to\infty} \int_t^{t+r} \sum_{i=1}^{n} P_i(s)\,ds > \frac{1}{e} \quad \text{where } r = \min\{r_1, r_2 \cdots r_n\} \tag{1.14}$$

$$\Big[\prod_{i=1}^{n} \Big(\sum_{j=1}^{n} \liminf_{t\to\infty} \int_t^{t+r_j} P_i(s)\,ds\Big)\Big]^{1/n} > \frac{1}{e} \tag{1.15}$$

or

$$\frac{1}{n} \sum_{i,j=1}^{n} \Big[\Big(\liminf_{t\to\infty} \int_t^{t+r_j} P_i(s)\,ds\Big)\Big(\liminf_{t\to\infty} \int_t^{t+r_i} P_j(s)\,ds\Big)\Big]^{1/2} > \frac{1}{e} \tag{1.16}$$

implies that every solution of (1.11) oscillates.

The author of Ref.[2] obtained some sufficient conditions under which all solutions of (1.1) and (1.2) are oscillatory.

In this paper we shall generalize and improve these results for very general linear and nonlinear equations (1.1)-(1.4) with continuous distributed retarded or advanced arguments.

In section II we obtain sufficient conditions under which all solutions of R.F.D.E. (1.1) and (1.2) oscillate. In section III we study A.F.D.E. (1.3) and (1.4). In section IV we give some examples to show that our results are very good and useful.

II. OSCILLATORY CRITERIONS FOR R.F.D.E. (1.1) AND (1.2)

We define $a_i (i = 0, 1, 2, .., n)$ such that $a = a_0 < a_1 < a_2 < \dots < a_{n-1} < a_n = b$,

and $\sigma(\xi)$ is continuous in a_i $(i = 0, 1, 2, \dots, n)$. Also set

$$I_i = [a_{i-1}, a_i]$$
$$\sigma_i(\xi) = \begin{cases} \sigma(\xi), & \xi \in I_i \\ \sigma(a_{i-1}), & \xi < a_{i-1} \\ \sigma(a_i), & \xi > a_i \end{cases} \tag{2.1}$$

$$P_i(s) = \int_a^b p(s,\xi)\,d\sigma_i(\xi), \tag{2.2$_1$}$$

$$P(s) = \int_a^b p(s,\xi)\,d\sigma(\xi). \tag{2.2$_2$}$$

Suppose the following conditions

(C_0)
$$\liminf_{t\to\infty} \int_{\varphi(t,\xi^*)}^{t} \int_a^{\xi^*+0} p(s,\xi)\,d\sigma(\xi)\,ds > 0 \tag{2.3}$$

for each $\xi^* \in [a,b]$, for which when

$\xi \in [a, \xi^*+\delta]$, then $\sigma(\xi) \not\equiv \text{const}$.

and when

$\xi \in [\xi^*-\delta, \xi^*+\delta']$

then $\sigma(\xi) \not\equiv$ const for each sufficient small $\delta > 0$, $\delta' > 0$.

(C_1)
$$\liminf_{t\to\infty} \int_{g(t,b)}^{t} P(s)\,ds > \frac{1}{e}\ ; \tag{2.4}$$

(C_2)
$$\liminf_{t\to\infty} \int_{g(t,a_i)}^{t} P_i(s)\,ds > \frac{1}{e}\ ; \tag{2.5}$$

for some i, i = 1, 2, 3 ... n;

(C_3)
$$\int_a^b \log\Big[\int_a^b \liminf_{t\to\infty} \int_{g(t,\xi)}^{t} p(s,\eta)\,ds\,d\sigma(\xi)\Big]\,d\sigma(\eta) + \int_a^b d\sigma(\eta) > 0 \tag{2.6}$$

(C_4)

$$\int_a^b\int_a^b \Big[\liminf_{t\to\infty}\int_{g(t,\eta)}^{t} p(s,\xi)\,ds\Big]^{1/2}\Big[\liminf_{t\to\infty}\int_{g(t,\xi)}^{t} p(s,\eta)\,ds\Big]^{1/2} d\sigma(\xi)\,d\sigma(\eta) > \frac{1}{e} \qquad (2.7)$$

Theorem 1. Conditions (2.3) and (2.4) imply that

$$y'(t)+\int_a^b p(t,\xi)\,y[g(t,\xi)]\,d\sigma(\xi)\le 0 \qquad (1.1)_1$$

has no eventually positive solution and

$$y'(t)+\int_a^b p(t,\xi)\,y[g(t,\xi)]\,d\sigma(\xi)\ge 0 \qquad (1.1)_2$$

has no eventually negative solutions.

Also if condition (2.3) is satisfied, then each one of (2.4)-(2.7) implies that every solution of (1.1) oscillates.

Proof. Here we only present details of the proof when (2.3) and each one of (2.4)-(2.7) is satisfied. Another two results can be treated by a similar fashion in Ref.[2].

Part 1. The fact that conditions (2.3) and (2.4) imply that all solutions of Eq. (1.1) oscillate is due to Ref.[2].

Part 2. The proof of sufficient condition (2.4) is obvious. We only notice that from Eq. (1.1) we obtain

$$\int_a^b p(s,\xi)\,y[g(s,\xi)]\,d\sigma(\xi)=\sum_{i=1}^{n}\int_a^b p(s,\xi)\,y[g(s,\xi)]\,d\sigma_i(\xi), \qquad (2.8)$$

$$y'(t)+\int_a^b p(s,\xi)\,y[g(s,\xi)\,d\sigma_i(\xi)\le 0 \qquad (2.9)$$

for the i for which (2.5) holds.

As in [2], we can prove that every solution of (1.1) oscillates.

Part 3. We present the proof when the condition (2.6) is satisfied. To this end suppose that there exists a solution $y(t)$ of (1.1) such that for t_0 sufficiently large,

$$y(t)>0, \quad t>t_0 .$$

Choose a $t_1>t_0$ such that $y[g(t,\xi)]>0$, for $t>t_1$ and thus, from (1.1), $y'(t)<0$ for $t>t_1$. Next choose $t_2>t_1$ such that $y(t)<y[g(t,\xi)]$, for $t>t_2$. Set

$$W(t,\xi)=\frac{y[g(t,\xi)]}{y(t)}, \qquad t>t_2 .$$

Define

$$l(\xi)=\liminf_{t\to\infty} W(t,\xi),$$

and also assume that all of them are finite or for some $\xi^*\in[a,b]$, $l(\xi^*)$ is infinite.

Case 1. All of $l(\xi)$, $\xi\in[a,b]$ are finite. Dividing both sides of (1.1) by $y(t)$, we obtain

$$\frac{y'(t)}{y(t)}+\int_a^b p(t,\xi)\,W(t,\xi)\,d\sigma(\xi)=0 . \qquad (2.10)$$

Integrating both sides of (2.10) from $g(t,\eta)$ to t, we find

$$\log y(t)-\log y[g(t,\eta)]+\int_{g(t,\eta)}^{t}\Big[\int_a^b p(s,\xi)\frac{y[g(s,\xi)]}{y(s)}\,d\sigma(\xi)\Big]ds=0 \qquad (2.11)$$

i.e.

$$\log W(t,\eta)-\int_{g(t,\eta)}^{t}\Big[\int_a^b p(s,\xi)\,W(s,\xi)\,d\sigma(\xi)\Big]ds=0 . \qquad (2.12)$$

Taking limit interiors on both sides of the above equation for $t\to\infty$, we obtain

$$\log l(\eta)=\liminf_{t\to\infty}\int_{g(t,\eta)}^{t}\Big[\int_a^b p(s,\xi)\,W(s,\xi)\,d\sigma(\xi)\Big]ds$$
$$=\liminf_{t\to\infty}\int_a^b\int_{g(t,\eta)}^{t} p(s,\xi)\,W(s,\xi)\,ds\,d\sigma(\xi)$$
$$\ge\int_a^b\liminf_{t\to\infty}\int_{g(t,\eta)}^{t} p(s,\xi)\,W(s,\xi)\,ds\,d\sigma(\xi)$$

$$= \int_a^b \liminf_{t\to\infty} \left\{ W(t^*(t),\xi) \int_{g(t,\eta)}^t p(s,\xi)\, ds \right\} d\sigma(\xi)$$

$$\geq \int_a^b \Big[\liminf_{t\to\infty} W(t,\xi)\Big] \cdot \Big[\liminf_{t\to\infty} \int_{g(t,\eta)}^t p(s,\xi)\, ds\Big]\, d\sigma(\xi)$$

$$= \int_a^b \ell(\xi) \cdot \Big[\liminf_{t\to\infty} \int_{g(t,\eta)}^t p(s,\xi)\, ds\Big]\, d\sigma(\xi).$$

So

$$\int_a^b \log \ell(\eta)\, d\eta \geq \int_a^b \left\{ \int_a^b \ell(\xi)\Big[\liminf_{t\to\infty} \int_{g(t,\eta)}^t p(s,\xi)\, ds\Big] d\sigma(\xi)\right\} d\sigma(\eta)$$

$$= \int_a^b \ell(\eta) \left\{ \int_a^b \liminf_{t\to\infty} \int_{g(t,\xi)}^t p(s,\eta)\, ds\, d\sigma(\xi)\right\} d\sigma(\eta).$$

Set the functional

$$I(\ell) = \int_a^b \left\{ \log \ell(\eta) - \ell(\eta)\Big[\int_a^b \liminf_{t\to\infty} \int_{g(t,\xi)}^t p(s,\eta)\, ds\, d\sigma(\xi)\Big]\right\} d\sigma(\eta) \qquad (2.13)$$

we obtain $I(\ell) \geq 0$.

Also set

$$F(\ell(\eta)) = \log \ell(\eta) - \ell(\eta)\Big[\int_a^b \liminf_{t\to\infty} \int_{g(t,\xi)}^t p(s,\eta)\, ds\, d\sigma(\xi)\Big]$$

$$\frac{dF(\ell)}{d\ell} = 0$$

i.e. $$\frac{1}{\ell(\eta)} - \int_a^b \liminf_{t\to\infty} \int_{g(t,\xi)}^t p(s,\eta)\, ds\, d\sigma(\xi) = 0$$

So if

$$\ell(\eta) = \left\{ \int_a^b \liminf_{t\to\infty} \int_{g(t,\xi)}^t p(s,\eta)\, ds\, d\sigma(\xi)\right\}^{-1} \equiv \ell^*(\eta)$$

then $I(\ell)$ has a maximum because

$$\frac{d^2F(\ell)}{d\ell^2} = -\frac{1}{\ell^2} < 0, \quad \text{That is}$$

$$I(\ell^*) = \int_a^b \left\{ -\log\Big[\int_a^b \liminf_{t\to\infty} \int_{g(t,\xi)}^t p(s,\eta)\, ds\, d\sigma(\xi)\Big] - 1 \right\} d\sigma(\eta)$$

$$\geq 0$$

Set $$P^*(\eta) = \int_a^b \liminf_{t\to\infty} \int_{g(t,\xi)}^t p(s,\eta)\, ds\, d\sigma(\xi)$$

i.e. $$\int_a^b \log P^*(\eta)\, d\sigma(\eta) + \int_a^b d\sigma(\eta) \leq 0$$

which contradicts (2.6).

Case 2. For some $\xi^* \in [a,b]$, $\ell(\xi^*)$ is infinite. From equation (1.1) we have

$$y(t) - y[\varphi(t,\xi^*)] + \int_{\varphi(t,\xi^*)}^t \int_a^{\xi^*+0} p(s,\xi)\, y[g(s,\xi)]\, d\sigma(\xi)\, ds \leq 0.$$

Dividing both sides of the above inequality first by $y(t)$ and then by $y[\varphi(t,\xi^*)]$ we obtain, respectively

$$1 - \frac{y[\varphi(t,\xi^*)]}{y(t)} + \int_{\varphi(t,\xi^*)}^t \int_a^{\xi^*+0} p(s,\xi) \frac{y[g(s,\xi)]}{y(t)}\, d\sigma(\xi)\, ds \leq 0 \qquad (2.14)$$

and

$$\frac{y(t)}{y[\varphi(t,\xi^*)]} - 1 + \int_{\varphi(t,\xi^*)}^t \int_a^{\xi^*+0} p(s,\xi) \frac{y[g(s,\xi)]}{y[\varphi(t,\xi^*)]}\, d\sigma(\xi)\, ds \leq 0 \qquad (2.15)$$

From (2.14) and $\ell(\xi^*)$ being infinite, we obtain

$$\frac{y[\varphi(t,\xi^*)]}{y(t)} \geq 1 + \frac{y[g(t,\xi^*)]}{y(t)} \int_{\varphi(t,\xi^*)}^t \int_a^{\xi^*+0} p(s,\xi)\, d\sigma(\xi)\, ds,$$

$$\liminf_{t\to\infty} \frac{y[\varphi(t,\xi^*)]}{y(t)} \geq 1 + \left\{\liminf_{t\to\infty} \frac{y[g(t,\xi^*)]}{y(t)}\right\} \left\{\liminf_{t\to\infty} \int_{\varphi(t,\xi^*)}^t \int_a^{\xi^*+0} p(s,\xi)\, d\sigma(\xi)\, ds\right\}$$

so

$$\liminf_{t\to\infty} \frac{y[\varphi(t,\xi^*)]}{y(t)} = +\infty$$

From (2.15) and $\ell(\xi^*)$ being infinite, we obtain

$$\int_{\varphi(t,\xi^*)}^{t}\int_a^{\xi^*+0} P(s,\xi)\frac{y[g(s,\xi)]}{y[\varphi(t,\xi^*)]}\,d\sigma(\xi)\,ds \le 1-\frac{y(t)}{y[\varphi(t,\xi^*)]},$$

$$\frac{y[g(t,\xi^*)]}{y[\varphi(t,\xi^*)]}\cdot\int_{\varphi(t,\xi^*)}^{t}\int_a^{\xi^*+0}P(s,\xi)\,d\sigma(\xi)\,ds\le 1-\frac{y(t)}{y[\varphi(t,\xi^*)]},$$

$$\left\{\liminf_{t\to\infty}\frac{y[g(t,\xi^*)]}{y[\varphi(t,\xi^*)]}\right\}\cdot\left\{\liminf_{t\to\infty}\int_{\varphi(t,\xi^*)}^{t}\int_a^{\xi^*+0}P(s,\xi)\,d\sigma(\xi)\,ds\right\}$$

$$\le 1-\left\{\liminf_{t\to\infty}\left[\frac{y[\varphi(t,\xi^*)]}{y(t)}\right]\right\}^{-1}$$

Because $$\liminf_{t\to\infty}\frac{y[g(t,\xi^*)]}{y[\varphi(t,\xi^*)]}=\liminf_{t\to\infty}\frac{y[\varphi(\varphi(t,\xi^*),\xi^*)]}{y[\varphi(t,\xi^*)]}=+\infty$$

the above last inequality leads to a contradiction.

Part 4. We present the proof when the condition (2.7) is satisfied. Otherwise there exists a solution $y(t)$ of (1.1) such that for t_0 sufficiently large $y(t) > 0$, $t > t_0$. As in part 2 of the proof of this Theorem we can define $W(t,\eta)$ and $\ell(\eta)$, and also obtain (suppose $\ell(\eta)$ is finite for all $\eta\in[a,b]$)

$$\log \ell(\eta)=\liminf_{t\to\infty}\int_{g(t,\eta)}^{t}\left[\int_a^b P(s,\xi)W(s,\xi)\,d\sigma(\xi)\right]ds$$

Integrating both sides of above equation from a to b in η, we obtain

$$\int_a^b \log \ell(\eta)\,d\sigma(\eta)=\int_a^b\left\{\liminf_{t\to\infty}\int_{g(t,\eta)}^{t}\left[\int_a^b P(s,\xi)W(s,\xi)\,d\sigma(\xi)\right]ds\right\}d\sigma(\eta)$$

As in part 3 of the proof of this Theorem we obtain

$$\int_a^b \log \ell(\eta)\,d\sigma(\eta)\ge\int_a^b\left\{\int_a^b \ell(\xi)\left[\liminf_{t\to\infty}\int_{g(t,\eta)}^{t}P(s,\xi)\,ds\right]d\sigma(\xi)\right\}d\sigma(\eta)$$

Define

$$H(\xi,\eta)=\liminf_{t\to\infty}\int_{g(t,\eta)}^{t}P(s,\xi)\,ds .$$

We have

$$\int_a^b\frac{\log \ell(\eta)}{\ell(\eta)}\,d\sigma(\eta)\ge\int_a^b\int_a^b\frac{\ell(\xi)H(\xi,\eta)}{\ell(\eta)}\,d\sigma(\xi)\,d\sigma(\eta)$$

$$=\frac{1}{2}\int_a^b\int_a^b\left[\frac{\ell(\xi)H(\xi,\eta)}{\ell(\eta)}+\frac{\ell(\eta)H(\eta,\xi)}{\ell(\xi)}\right]d\sigma(\xi)\,d\sigma(\eta)$$

$$\ge\int_a^b\int_a^b\sqrt{\frac{\ell(\xi)H(\xi,\eta)}{\ell(\eta)}\cdot\frac{\ell(\eta)H(\eta,\xi)}{\ell(\xi)}}\,d\sigma(\xi)\,d\sigma(\eta)$$

$$=\int_a^b\int_a^b\sqrt{H(\xi,\eta)H(\eta,\xi)}\,d\sigma(\xi)\,d\sigma(\eta) .$$

Notice that the functional

$$I(\ell)=\int_a^b\left[\frac{\log \ell(\eta)}{\ell(\eta)}-\frac{1}{e}\right]d\sigma(\eta)$$

has a maximum 0 when $\ell(\eta)\equiv e$, because if $\ell(\eta)\equiv e$ then

$$\frac{d}{d\ell}\left[\frac{\log \ell}{\ell}-\frac{1}{e}\right]\Big|_{\ell=e}=\frac{1-\log \ell}{\ell^2}\Big|_{\ell=e}=0,$$

$$\frac{d^2}{d\ell^2}\left[\frac{\log \ell}{\ell}-\frac{1}{e}\right]\Big|_{\ell=e}=\frac{-3+2\log \ell}{\ell^3}\Big|_{\ell=e}=-\frac{1}{e^3}<0$$

So, $I(\ell)\le 0$, i.e.

$$\int_a^b\frac{\log \ell(\eta)}{\ell(\eta)}\,d\sigma(\eta)\le\frac{1}{e}\int_a^b d\sigma(\eta)$$

So we have

$$\frac{1}{e}\int_a^b d\sigma(\eta) \geq \int_a^b \int_a^b \sqrt{H(\xi,\eta)H(\eta,\xi)}\, d\sigma(\xi)\, d\sigma(\eta) .$$

This last inequality leads to a contradiction with (2.7). As in Part 3, when for some ξ^*, $l(\xi^*) = +\infty$, we are also led to a contradiction. Therefore, the proof of the theorem is complete.

Remark 1. Notice that in condition (2.3) ξ^* must satisfy some condition, otherwise (2.3) is not satisfied.

Remark 2. If (1.1) is as (1.5), then conditions (C_0)-(C_4) change into (1.6)-(1.7) respectively.

If $|f(t,\xi,v)| \geq P(t,\xi)|v|$ $\quad (|v| \leq C_0,$ C_0 is some constant) and $f(t,\xi,0) \equiv 0$, then non-linear function $f(t,\xi,v)$ is "bounded linear".

Theorem 2. If the condition (2.3) and one of the conditions of (2.4)-(2.7) are satisfied, also $f(t,\xi,v)$ of (1.2) is "bounded linear", then every solution of (1.2) oscillates.

Proof. Otherwise there exists nonoscillatory solution $y(t) > 0$, one can obtain

$$0 = y'(t) + \int_a^b f(t,\xi,y[g(t,\xi)])\, d\sigma(\xi)$$
$$\geq y'(t) + \int_a^b P(t,\xi)\, y[g(t,\xi)]\, d\sigma(\xi) .$$

By the results of Theorem 1 we can obtain $(1.1)_1$ has no eventually positive solution and lead to contradictions. Also if there exists nonoscillatory solution $y(t) < 0$, then we can obtain

$$0 = y'(t) + \int_a^b f(t,\xi,y[g(t,\xi)])\, d\sigma(\xi)$$
$$\leq y'(t) + \int_a^b P(t,\xi)\, y[g(t,\xi)]\, d\sigma(\xi)$$

By a similar reason it will lead to contradictions. The proof of Theorem 2 is complete.

III. OSCILLATORY CRITERIONS FOR A.F.D.E.(1.3) and (1.4).

As in Section II, we define a_i, $\sigma_i(\xi)$, $P_i(\xi)$, $P(s)$. $(i = 1,2,\ldots n)$. Set

$$(C_0)' \quad \liminf_{t\to\infty} \int_t^{g(t,\xi^*)} \int_{\xi^*-}^{b} P(s,\xi)\, d\sigma(\xi)\, ds \tag{3.1}$$

for each $\xi^* \in (a,b)$, for which when $\xi \in [\xi^*-\delta, b]$, then $\sigma(\xi) \not\equiv$ const. and when $\xi \in [\xi^* - \delta, \xi^* + \delta']$ then $\sigma(\xi) \not\equiv$ const. for each sufficient small $\delta > 0, \delta' > 0$;

$$(C_1)' \quad \liminf_{t\to\infty} \int_t^{h(t,a)} P(s)\, ds > \frac{1}{e} \tag{3.2}$$

$$(C_2)' \quad \liminf_{t\to\infty} \int_t^{h(t,a_i)} P_i(s)\, ds > \frac{1}{e} \tag{3.3}$$

for some i, i = 1, 2, ... n

$$(C_3)' \quad \int_a^b \log\Big[\int_a^b \liminf_{t\to\infty} \int_t^{h(t,\xi)} P(s,\eta)\, ds\, d\sigma(\xi)\Big]\, d\sigma(\eta) + \int_a^b d\sigma(\eta) > 0 \tag{3.4}$$

$$(C_4)' \quad \int_a^b \int_a^b \Big[\liminf_{t\to\infty} \int_t^{h(t,\eta)} P(s,\xi)\, ds\Big]^{1/2} \cdot \Big[\liminf_{t\to\infty} \int_t^{h(t,\xi)} P(s,\eta)\, ds\Big]^{1/2} d\sigma(\xi)\, d\sigma(\eta) > \frac{1}{e} \tag{3.5}$$

Theorem 3. If the condition (3.1) is satisfied, then the condition (3.2) implies that

$$y'(t) - \int_a^b P(t,\xi)\, y[h(t,\xi)]\, d\sigma(\xi) \geq 0 \qquad (1.3)_1$$

has no eventually negative solutions. Also each one of (3.2)-(3.5) implies that every solution of (3.1) oscillates.

Proof. We only prove when conditions (3.1) and (3.4) are satisfied. Otherwise there exists a nonoscillatory positive solution $y(t) > 0$, $(t > t_0)$. Then $y'(t) > 0$ for $t > t_0$. Hence $y[h(t,\xi)] \geq y(t)$, for $t > t_0$. Set

$$z(t,\xi) = \frac{y[h(t,\xi)]}{y(t)}, \quad \text{for } t > t_0,\ \xi \in [a,b] \qquad (3.6)$$

and

$$\lambda(\xi) = \liminf_{t\to\infty} z(t,\xi), \quad \xi \in (a,b]. \qquad (3.7)$$

Then $z(t,\xi) > 1$, and $\lambda(\xi) \geq 1$. Dividing both sides of (1.3) by $y(t)$, for $t > t_0$ we obtain

$$\frac{y'(t)}{y(t)} - \int_a^b p(t,\xi)\, z(t,\xi)\, d\sigma(\xi) = 0$$

Integrating the last equation from t to $h(t,\eta)$ for $\eta \in [a,b]$, we have

$$\log y[h(t,\eta)] - \log y(t) = \int_t^{h(t,\eta)} \int_a^b p(s,\xi)\, z(s,\xi)\, d\sigma(\xi)\, ds \qquad (3.8)$$

We now consider the following two cases:

Case 1. $\lambda(\xi) < \infty$ for each $\xi \in [a,b]$. Then (3.8) yields

$$\log z(t,\eta) = \int_t^{h(t,\eta)} \int_a^b p(s,\xi)\, z(s,\xi)\, d\sigma(\xi)\, ds.$$

Taking limit interiors on both sides of the above equation for $t \to \infty$, we obtain

$$\log \lambda(\eta) \geq \int_a^b \lambda(\xi) \left[\liminf_{t\to\infty} \int_t^{h(t,\eta)} p(s,\xi)\, ds \right] d\sigma(\xi).$$

As in Theorem 1, we can obtain

$$\int_a^b \log\left[\int_a^b \liminf_{t\to\infty} \int_t^{h(t,\xi)} p(s,\eta)\, ds\, d\sigma(\xi) \right] d\sigma(\eta) + \int_a^b d\sigma(\eta) \leq 0$$

This inequation contradicts with (3.4).

Case 2. $\lambda(\xi^*) = \infty$ for some $\xi^* \in [a,b]$. From the equation (1.3) we have

$$\frac{y[\psi(t,\xi^*)]}{y(t)} - 1 - \frac{y[h(t,\xi^*)]}{y(t)} \int_t^{\psi(t,\xi^*)} \int_{\xi^*-0}^b p(s,\xi)\, d\sigma(\xi)\, ds \geq 0$$

As in case 2 of Theorem 1, we prove

$$\liminf_{t\to\infty} \frac{y[\psi(t,\xi^*)]}{y(t)} = +\infty$$

Also we have

$$1 - \frac{y(t)}{y[\psi(t,\xi^*)]} - \frac{y[h(t,\xi^*)]}{y[\psi(t,\xi^*)]} \int_t^{\psi(t,\xi^*)} \int_{\xi^*-0}^b p(s,\xi)\, d\sigma(\xi)\, ds \geq 0.$$

This inequality can lead to contradiction. Theorem 3 is proved.

Theorem 4. If the condition (3.1) and one of (3.2)-(3.5) are satisfied, also $f(t, \xi, V)$ of (1.4) is "bounded linear", then every solution of (1.4) oscillates.

Proof. As in Theorem 2, otherwise there exists nonoscillatory solution $y(t) > 0$, one can obtain

$$0 = y'(t) - \int_a^b f(t,\xi, y[h(t,\xi)])\, d\sigma(\xi) \leq y'(t) - \int_a^b p(t,\xi)\, y[h(t,\xi)]\, d\sigma(\xi).$$

By the results of Theorem 3, we find that $(1.3)_1$ has no eventually positive solution and leads to contradictions. If there exists nonoscillatory solution $y(t) < 0$, we can obtain

$$0 = y'(t) - \int_a^b f(t,\xi, y[h(t,\xi)])\, d\sigma(\xi) \geq y'(t) - \int_a^b p(s,\xi)\, y[h(t,\xi)]\, d\sigma(\xi).$$

By the results of Theorem 3, we find $(1.3)_2$ has no eventually negative solutions and leads to contradictions. The proof of Theorem 4 is complete.

IV. SOME EXAMPLES

Example 1.

$$y'(t) + \int_{-2}^{-1} p\, y(t+\xi)\, d\xi = 0. \qquad (4.1)$$

This example has been presented in Ref.[2]. It is shown that if $p > \frac{1}{e}$, then each solution of (4.2) oscillates; if $p \leq \frac{1}{2}\,\frac{1}{e}$ then there exists nonoscillatory solution, but when $\frac{1}{2}\,\frac{1}{e} < p \leq \frac{1}{e}$, in [2] it is not pointed that each solution of (4.1) oscillates.

(I) We can now use condition (2.5) to obtain this result. Notice that if we set $a_0 = -2$, $a_1 = T$, $a_2 = -1$, then (2.5) is

$$\liminf_{t\to\infty}\int_{t+a_1}^{t} P\,ds = a_1 p = Tp > \frac{1}{e}$$

Set $p = k\,\frac{1}{e}$. So $1 \geqslant k > \frac{1}{2}$, $1 \leqslant \frac{1}{k} < 2$, that is if we take T such that $2 > T > \frac{1}{k} \geqslant 1$, then the condition (2.5) is satisfied, so each solution of (4.1) oscillates.

(II). Note that condition (2.6) is

$$\int_{-2}^{-1}\left\{\log\left[\int_{-2}^{-1} P\cdot(-\xi)\,d\xi\right]+1\right\}d\eta > 0$$

that is

$$\log\left(\frac{3}{2}p\right)+1>0\ ,\quad p>\frac{2}{3}e^{-1}$$

It is pointed that if $\frac{2}{3}e^{-1} < p \leqslant e^{-1}$, then oscillations of (4.1) cannot be obtained by (2.6).

(III). Note the condition (2.7) is

$$\int_{-2}^{-1}\int_{-2}^{-1}[P\cdot(-\eta)]^{1/2}[P\cdot(-\xi)]^{1/2}\,d\xi\,d\eta > \frac{1}{e}$$

that is

$$p\cdot\left[\int_{-2}^{-1}\sqrt{-\xi}\,d\xi\right]^2 = \frac{4}{9}(9-4\sqrt{2})p > \frac{1}{e}$$

$$p > \frac{1}{\frac{4}{9}(9-4\sqrt{2})}e^{-1} \approx \frac{1}{1.48}e^{-1} > \frac{1}{2}e^{-1}$$

So if $e^{-1} \geqslant p > \frac{1}{\frac{4}{9}(9-4\sqrt{2})}e^{-1}$, then each solution of (4.1) oscillates. But if

$e^{-1}\frac{1}{\frac{4}{9}(9-4\sqrt{2})} \geqslant p > e^{-1}$, then oscillations of (4.2) cannot be obtained by (2.7).

Example 2.

$$y'(t)+\int_{-3}^{-2}\frac{3}{2e}(\xi+2)y(t+\xi)\,d\xi+\int_{-2}^{-1}\left(-\frac{2}{3e}\right)(\xi+2)y(t+\xi)\,d\xi = 0 \tag{4.2}$$

that is

$$P(t,\xi)=\begin{cases}\frac{3}{2e}(\xi+2)\ , & \xi\in[-2,-1]\ ,\\ -\frac{2}{3e}(\xi+2)\ , & \xi\in[-3,-2]\ ,\end{cases}$$

$$a_0 = a = -3,\quad a_1 = -2,\quad a_2 = b = -1,\quad \sigma(\xi) = \xi .$$

(I). The condition (2.5) is not satisfied. Notice that

$$\liminf_{t\to\infty}\int_{t-2}^{t}\left[\int_{-3}^{-2}-\frac{2}{3e}(\xi+2)\,d\xi\right]ds = \frac{2}{3e} < \frac{1}{e}$$

$$\liminf_{t\to\infty}\int_{t-1}^{t}\left[\int_{-2}^{-1}\frac{3}{2e}(\xi+2)\,d\xi\right]ds = \frac{3}{4e} < \frac{1}{e}$$

(II). The condition (2.4) is satisfied. Notice that

$$\liminf_{t\to\infty}\int_{t-1}^{t}\left[\int_{-3}^{-2}-\frac{1}{3e}(\xi+2)\,d\xi+\int_{-2}^{-1}\frac{3}{2e}(\xi+2)\,d\xi\right]ds = \frac{13}{12e} > \frac{1}{e}$$

(III). The condition (2.6) is satisfied. Notice that

$$\int_{-3}^{-2}\log\left[\int_{-3}^{-1}(-\xi)\left(-\frac{2}{3e}\right)(\eta+2)\,d\xi\right]d\eta+\int_{-3}^{-1}(-\xi)\frac{3}{2e}(\eta+2)\,d\xi\,]\,d\eta$$

$$+\int_{-3}^{-1}d\eta = \int_0^1\log[16\eta(1-\eta)]\,d\eta = \int_0^1\log 16\,d\eta = \log 16 > 0$$

(IV). The condition (2.7) is satisfied. Notice that

$$I_1 = \int_{-3}^{-2}\int_{-3}^{-2}\left[(-\eta)\left(-\frac{2}{3e}\right)(\xi+2)\right]^{1/2}\left[(-\xi)\left(-\frac{2}{3e}\right)(\eta+2)\right]^{1/2}d\xi\,d\eta$$

$$= \frac{2}{3e}\left[\int_0^1\sqrt{(\eta-3)(\eta-1)}\,d\eta\right]^2 = \frac{2}{3e}\left[\sqrt{3}-\frac{1}{2}\ln(2-\sqrt{3})\right]^2;$$

$$I_2 = \int_{-2}^{-1}\int_{-2}^{-1}\left[(-\eta)\frac{3}{2e}(\xi+2)\right]^{1/2}\left[(-\xi)\left(\frac{3}{2e}(\eta+2)\right)\right]^{1/2}d\xi\,d\eta$$

$$= \frac{3}{2e}\left[\int_{-2}^{-1}\sqrt{(-\xi)(\xi+2)}\,d\xi\right]^2 = \frac{3}{2e}\left(\frac{\pi}{2}\right)^2$$

$$I_3 = 2\int_{-2}^{-1}\int_{-3}^{-2}\left[(-\eta)\left(-\frac{2}{3e}\right)(\xi+2)\right]^{1/2}\left[(-\xi)\frac{3}{2e}(\eta+2)\right]^{1/2}d\xi\,d\eta$$

$$= \frac{2}{e}\left[\sqrt{3}-\frac{1}{2}\ln(2-\sqrt{3})\right]\cdot\frac{\pi}{2}$$

$$I = \frac{1}{3}\left[(\sqrt{3}-\frac{1}{2}\ln(2-\sqrt{3})+\frac{3\pi}{4}\right]^2 > 1$$

Example 3.

$$y'(t)+\int_{-3}^{-2}\frac{1}{2e}y(t+\zeta)d\zeta+\int_{-2}^{-1}\frac{2}{5e}(\zeta+2+\frac{1}{2e})y(t+\zeta)d\zeta=0 \qquad (4.3)$$

(I). The condition (2.7) is satisfied. Notice that

$$I>I_1=\frac{1}{2}\int_{-3}^{-2}\int_{-3}^{-2}[(-\eta)\frac{1}{2e}]^{1/2}[(-\zeta)\frac{1}{2e}]^{1/2}d\zeta d\eta\approx\frac{2}{9}\times 6.44e^{-1}>e^{-1}$$

(II). The condition (2.4) is not satisfied. Notice that

$$\liminf_{t\to\infty}\int_{t-1}^{t}[\int_{-3}^{-2}\frac{1}{2e}d\zeta+\int_{-2}^{-1}\frac{2}{5e}(\zeta+2+\frac{1}{2e})d\zeta]ds=\frac{7}{10e}<\frac{1}{e}$$

(III). The condition (2.5) is not satisfied. Notice that

$$\liminf_{t\to\infty}\int_{t-1}^{t}\int_{-2}^{-1}\frac{2}{5e}(\zeta+2+\frac{1}{2e})d\zeta ds<\frac{1}{e}$$

$$\liminf_{t\to\infty}\int_{t-2}^{t}\int_{-3}^{-2}\frac{1}{2e}d\zeta ds=\frac{1}{e}\ ;$$

(IV). The condition (2.6) is satisfied. Notice that

$$\int_{-3}^{-2}\log[\int_{-3}^{-1}(-\zeta)\frac{1}{2e}d\zeta]d\eta+\int_{-2}^{-1}\log[\int_{-3}^{-1}(-\zeta)\frac{2}{5e}(\eta+2+\frac{1}{2e})d\zeta]d\eta$$

$$+\int_{-3}^{-1}d\eta=\log\frac{16}{5}+\int_{0}^{1}\log(\rho+\frac{1}{2e})d\rho>\log\frac{16}{5}-1>0$$

From Examples 1, 2 and 3 we can easily obtain examples of advanced type equations.

Example 4.

$$y'(t)-\int_{1}^{2}py(t+\zeta)d\zeta=0 \qquad (4.4)$$

If $p>\frac{1}{e}$ then the condition (3.3) is satisfied and each solution of (4.4) oscillates; if $\frac{1}{2}e^{-1}<p\leqslant e^{-1}$ then the condition (3.2) is satisfied and each solution of (4.4) oscillates but the condition (3.3) is not satisfied. If $\frac{2}{3}e^{-1}<p\leqslant e^{-1}$, then the condition (3.4) is not satisfied and the oscillation of (4.4) cannot be obtained by (3.4). If $\frac{1}{\frac{4}{9}(9-4\sqrt{2})}e^{-1}\geqslant p>e^{-1}$, then the condition (3.5) is not satisfied.

Example 5.

$$y'(t)-\int_{2}^{3}\frac{3}{2e}(\zeta-2)y(t+\zeta)d\zeta-\int_{1}^{2}(-\frac{2}{7e})(\zeta-2)y(t+\zeta)d\zeta=0 \qquad (4.5)$$

The condition (3.3) is not satisfied. But conditions (3.2), (3.4), (3.5) are all satisfied and each solution of (4.5) oscillates.

Example 6.

$$y'(t)-\int_{2}^{3}\frac{1}{2e}y(t+\zeta)d\zeta-\int_{1}^{2}\frac{2}{5e}(\zeta-2+\frac{1}{2e})y(t+\zeta)d\zeta=0 \qquad (4.6)$$

Conditions (3.2), (3.3) are not satisfied. But conditions (3.4), (3.5) are satisfied. So each solution of (4.6) oscillates.

ACKNOWLEDGMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. He is also grateful to Professor G. Vidossich for his encouragement.

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