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NONOSCILLATORY BEHAVIOUR CAUSED BY RETARDED AND ADVANCED ARGUMENTS FOR EVEN-ORDER FUNCTIONAL DIFFERENTIAL INEQUALITIES AND EQUATIONS

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ABSTRACT

In this paper we study the nonoscillatory behaviour of even order functional differential inequalities and equations with continuous distributed retarded and advanced arguments. For arguments we give some conditions under which these inequalities only have nonoscillatory solutions of degree 0 or n.

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1. INTRODUCTION

In this paper we are concerned with nonoscillatory behaviour caused by retarded and advanced arguments for the following differential inequalities and equations;

$$\left\{y^{(n)}(t)-p\,y[g(t)]\right\} \operatorname{sgn} y[g(t)] \geqslant 0, \tag{1.1}$$

$$\left\{y^{(n)}(t)-p\,y[h(t)]\right\} \operatorname{sgn} y[h(t)] \geqslant 0. \tag{1.2}$$

$$\left\{y^{(n)}(t)-p\int_a^b\cdots\int_a^b y[g[\cdots[[g(t,\xi_1)],\xi_2],\cdots],\xi_n]\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right\}\cdot$$
$$\cdot \operatorname{sgn}\left\{\int_a^b\cdots\int_a^b y[g[\cdots[[g(t,\xi_1)],\xi_2],\cdots],\xi_n]\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right\} \geqslant 0, \tag{1.3}$$

$$\left\{y^{(n)}(t)-p\int_a^b\cdots\int_a^b y[h[\cdots[[h(t,\xi_1)],\xi_2],\cdots],\xi_n]\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right\}\cdot$$
$$\cdot \operatorname{sgn}\left\{\int_a^b\cdots\int_a^b y[h[\cdots[[h(t,\xi_1)],\xi_2],\cdots],\xi_n]\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right\} \geqslant 0, \tag{1.4}$$

where n is even.

Below the following conditions are assumed to hold:

(1) $p>0$, $g(t)<t$, $h(t)>t$, $g(t)$ and $h(t)$ are continuous, $g(t,\xi)\leqslant t$, $h(t,\xi)\geqslant t$, $\lim_{t\to\infty} g(t)=\infty$, $\lim_{t\to\infty} g(t,\xi)=\infty$ for $\xi\in[a,b]$;

(2) $\sigma=[a,b]\to R$ is a nondecreasing function;

(3) integrals in (1.3) and (1.4) are Stieltjes integrals;

(4) $g(t,\xi)$ and $h(t,\xi)$ are continuous nondecreasing functions with t or ξ respectively;

(5) there exists $\varphi(t,\xi)$, $\chi(t,\xi)$ such that $\varphi(\varphi(t,\xi),\xi)=g(t,\xi)$, $\psi(\psi(t,\xi),\xi)=h(t,\xi)$, $\lim_{t\to\infty}\varphi(t,\xi)=\lim_{t\to\infty}(t,\xi)=\infty$, $\varphi(t,\xi)$ and $\psi(t,\xi)$ are nondecreasing functions with t or ξ respectively.

<u>Definition 1</u>: If $y(t)$ is a nonoscillatory solution of (1.1)—(1.4), also there are an even integer $\ell\in\{0,2,\ldots,n\}$ and a number $t_1>0$ such

that

$$\begin{aligned} &y(t)y^{(i)}(t)>0, \quad \text{on } [t_1,\infty) \quad \text{for } 0\leqslant i\leqslant \ell \\ &(-i)^{i-\ell}y(t)y^{(i)}(t)>0 \quad \text{on } [t_1,\infty) \quad \text{for } \ell+1\leqslant i\leqslant n \end{aligned} \tag{1.5}$$

then such a $y(t)$ is said to be a nonoscillatory solution of degree ℓ, and the set of all solutions of someone of (1.1) — (1.4) is denoted by $\mathcal{N}_\ell^-$.

Suppose the set of all nonoscillatory solutions of someone of (1.1)— (1.4) is denoted by $\mathcal{N}^-$

<u>Lemma 1</u>: We have

$$\mathcal{N}=\mathcal{N}_0\cup\mathcal{N}_1\cup\cdots\cup\mathcal{N}_n \tag{1.6}$$

The proof of this Lemma is similar to the proof in [2].

In this paper we give some sufficient conditions under which $\mathcal{N}_0^-=\phi$ (i.e. $\mathcal{N}^-=\mathcal{N}_n^-$) for (1.1), (1.3) and $\mathcal{N}_n^-=\phi$ (i.e. $\mathcal{N}^-=\mathcal{N}_0^-$) for (1.2), (1.4). For variatory classes of $g(t)$, $g(t,\xi)$ or $h(t)$, $h(t,\xi)$ we obtain variatory nonoscillatory criteria for (1.1)——(1.4). We generalized results in [1]. On the other hand, we also give some nonoscillatory criteria for general superlinear inequalities:

$$\left\{y^{(n)}(t)-f(y[g(t)])\right\} \operatorname{sgn} y[g(t)] \geqslant 0, \tag{1.7}$$

$$\left\{y^{(n)}(t)-f(y[h(t)])\right\} \operatorname{sgn} y[h(t)] \geqslant 0, \tag{1.8}$$

$$\left\{y^{(n)}(t)-\int_a^b\cdots\int_a^b f(y[g[g[\cdots[[g(t,\xi_1)],\xi_2],\cdots],\xi_n])\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right\}\cdot$$
$$\cdot \operatorname{sgn}\left\{\int_a^b\cdots\int_a^b f(y[g[\cdots[[g(t,\xi_1)],\xi_2],\cdots],\xi_n])\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right\} \geqslant 0, \tag{1.7'}$$

$$\left\{y^{(n)}(t)-\int_a^b\cdots\int_a^b f(y[h[h[\cdots[[h(t,\xi_1)],\xi_2],\cdots],\xi_n])\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right\}\cdot$$
$$\cdot \operatorname{sgn}\left\{\int_a^b\cdots\int_a^b f(y[h[h[\cdots[[h(t,\xi_1)],\xi_2],\cdots],\xi_n])\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right\} \geqslant 0, \tag{1.8'}$$

Definition 2: Nonlinear function $f(x)$ is said to be superlinear if

$$\lim_{|x|\to\infty} \frac{|f(x)|}{|x|} = \infty , \tag{1.9}$$

We shall make use of the following results of one-order inequalities which were obtained by the author in [3].

Set

$$\left\{ z'(t) + \int_a^b p(t,\xi) z[g(t,\xi)]\, d\sigma(\xi) \right\} \operatorname{sgn} \int_a^b z[g(t,\xi)]\, d\sigma(\xi) \le 0 . \tag{1.10}$$

$$\left\{ z'(t) - \int_a^b p(t,\xi) z[h(t,\xi)]\, d\sigma(\xi) \right\} \operatorname{sgn} \int_a^b z[h(t,\xi)]\, d\sigma(\xi) \geqslant 0 , \tag{1.11}$$

$$P(t) = \int_a^b p(t,\xi)\, d\sigma(\xi) . \tag{1.12}$$

(H_1) $$\liminf_{t\to\infty} \int_{g(t,b)}^{t} P(s)\,ds > 1/e . \tag{1.13}$$

(H_2) $$\liminf_{t\to\infty} \int_{\varphi(t,b)}^{t} P(s)\,ds > 0 , \tag{1.14}$$

(H_1') $$\liminf_{t\to\infty} \int_{t}^{h(t,a)} P(s)\,ds > 1/e . \tag{1.15}$$

(H_2') $$\liminf_{t\to\infty} \int_{t}^{\psi(t,a)} P(s)\,ds > 0 . \tag{1.16}$$

Lemma 2. If conditions (H_1) and (H_2) are satisfied, then (1.10) has no nonoscillatory solution.

Lemma 3. If conditions (H_1') and (H_2') are satisfied, then (1.11) has no nonoscillatory solution.

2. RETARDED INEQUALITIES AND EQUATIONS

Here we are interested in the situation in which there is no nonoscillatory solution of degree 0 of (1.1) or (1.3) $(\mathcal{N}_0 = \phi)$ or all non-

oscillatory solutions of (1.1) or (1.3) are of degree n $(\mathcal{N} = \mathcal{N}_n)$.

Theorem 1: If $g(t) \le kt$, for some $k \in (0,1)$ and all $t > T$, then there is no nonoscillatory solution of degree 0 of (1.1) $(\mathcal{N}_0 = \phi)$.

Proof. Case 1: In the case $g(t) = kt$ for some $k \in (0,1)$ and all $t \ge T$. Let $y(t)$ be a nonoscillatory positive solution of degree 0 of (1.1). So $y(t) > 0$, $y'(t) < 0$, $y''(t) > 0$, ..., $y^{(n)}(t) > 0$, for $t \ge t_0$. Set

$$\begin{aligned} k &= k^{*^n}, \quad p = p^{*^n} k^{-\frac{n(n-1)}{2}} , \\ z(t) &= y^{(n-1)}(t) - p^* u_{n-2}\, y^{(n-2)}(k^* t) + p^{*^2} u_{n-3}\, y^{(n-3)}(k^{*^2} t) - \\ &\quad \cdots + p^{*^{n-2}} u_1\, y'(k^{*^{n-2}} t) - p^{*^{n-1}} u_0\, y(k^{*^{n-1}} t) . \end{aligned} \tag{2.1}$$

We have

$$\begin{aligned} z'(t) &= y^{(n)}(t) - p^* u_{n-2} k^* y^{(n-1)}(k^* t) + p^{*^2} u_{n-3} k^{*^2} y^{(n-2)}(k^{*^2} t) - \\ &\quad \cdots + p^{*^{n-2}} k^{*^{n-2}} y''(k^{*^{n-2}} t) - p^{*^{n-1}} u_0 k^{*^{n-1}} y'(k^{*^{n-1}} t) . \end{aligned}$$

We choose positive constants $U_0, U_1, \ldots, U_{n-2}$, such that

$$\begin{cases} 1 - k^* u_{n-2} = 0 , \\ k^{*^2} u_{n-3} - u_{n-2} = 0 , \\ \cdots\cdots \\ k^{*^{n-2}} u_1 - u_2 = 0 . \\ u_1 - k^{*^{n-1}} u_0 = 0 \end{cases}$$

that is, $U_{n-2} = 1/k^*$, $U_{n-3} = 1/k^{*^3}$, ..., $U_0 = 1/k^{*^{\frac{n(n-1)}{2}}}$.

Notice

$$y^{(n)}(t) - p\, y[kt] \geqslant 0 . \tag{2.2}$$

and

$$z'(t)+p^{*}z[k^{*}t]=y^{(n)}(t)-p^{*^{n}}k^{-\frac{n(n-1)}{2}}y[k^{*^{n}}t]$$

$$=y^{(n)}(t)-p\,y[kt]\geqslant 0. \tag{2.3}$$

Then we can obtain that $z(t)$ is a negative solution of (2.3). But

$$\liminf_{t\to\infty} p(t-kt)=+\infty>1/e,$$

$$\liminf_{t\to\infty} p(t-\sqrt{k}\,t)=+\infty>0.$$

i.e. Conditions (H_1) and (H_2) are satisfied. So using Lemma 2 we can see that there is no negative solution of (2.3). This led to a contradiction. The proof for $y(t)<0$ being similar.

Case 2. In the case $g(t)<kt$ for some $k\in(0,1)$ and all $t\geqslant 0$. If there is a positive solution of degree 0 $y(t)>0$ for (1.1), then $y'(t)<0$, i.e. $y[g(t)]>y[kt]$. So we have

$$0\leqslant y^{(n)}(t)-p\,y[g(t)]\leqslant y^{(n)}(t)-p\,y[kt].$$

From the proof in Case 1, we are also led to a contradiction. The proof for $y(t)<0$ being similar. The proof of Theorem 1 is complete.

Theorem 2. If $g(t)\leqslant t-\tau$ for some $\tau>0$ and all $t\geqslant t_0$, then the condition

$$\sqrt[n]{p}\,\frac{\tau}{n}\,e>1, \tag{2.4}$$

implies that (1.1) has no nonoscillatory solution of degree 0 ($\mathcal{N}_0=\phi$).

Proof. Case 1. If $g(t)=t-\tau$ for some $\tau>0$ and all $t\geqslant t_0$, then Kusano [1] has proved this result.

Case 2. If $g(t)<t-\tau$ for some $\tau>0$ and all $t\geqslant t_0$, then

we can suppose there is a nonoscillatory positive solution of degree 0, $y(t)>0$ of (1.1). So we have $y'(t)<0$, $y[g(t)]>y(t-\tau)$, and

$$0\leqslant y^{(n)}(t)-p\,y[g(t)]\leqslant y^{(n)}(t)-p\,y(t-\tau),$$

From the proof of Case 1 we can deduce that the condition (2.4) implies the inequality

$$y^{(n)}(t)-p\,y(t-\tau)\geqslant 0,$$

has no nonoscillatory positive solution. This led to a contradiction. The proof for $y(t)<0$ being similar. We proved this Theorem.

Remark: Using Theorems 1, 2, we have the following results. If $g(t)$ is taken forms as $t^{\nu}(0<\nu<1)$, $\ln t$, then for all p (1.1) has no nonoscillatory solution of degree 0. Also if $g(t)$ are taken as $t-|\sin t|$, $\frac{t^2}{t+1}$, then for some p, which satisfies $\sqrt[n]{p}\,\frac{\tau}{n}\,e>1$ (1.1) has no nonoscillatory solution of degree 0. But if $g(t)$ is taken as $t-e^{-t}$, $t-\frac{1}{t}$ then we need the condition as follows.

$$\limsup_{t\to\infty}\frac{p}{i!(n-i-1)!}\int_{g(t)}^{t}|s-g(t)|^{i}\,|g(t)-g(s)|^{n-i-1}ds>1.$$

for some $i\in\{0,1,\ldots,n-1\}$, to assume that (1.1) has no nonoscillatory solution of degree 0.

Theorem 3. For (1.3) $\mathcal{N}=\mathcal{N}_0\cup\mathcal{N}_n$. If in (1.3) one of following conditions is satisfied

(i) $g(t,\xi)\leqslant t-\xi$ $(b-a>0)$ for all $t\geqslant t_0$, $\xi\in[a,b]$, and $\sqrt[n]{p}\,\frac{a}{n}\,e>1$; (2.5)

(ii) $g(t,\xi)\leqslant kt\xi$ for some k, which satisfied $0<k\xi<1$, $\xi\in[a,b]$, and all $t\geqslant t_0$. (2.6)

then for (1.3) $\mathcal{N}_0=\phi$, $\mathcal{N}=\mathcal{N}_n$. For (2.2) $\mathcal{N}_0=\phi$; $\mathcal{N}=\mathcal{N}_n$.

Proof. (1) In the case (1.3). Otherwise, let $y(t)$ be a nonoscillatory solution of (1.3) of degree ℓ, $0<\ell<n$. We may assume that $y(t)>0$ for $t\geqslant t_0$.

Take t_2 such that

is bounded. In fact we may choose t_2 such that

$$\max_{\substack{\xi_i\in[a,b]\\ i=1,2,\cdots,n}} g[g[\cdots[[g[t,\xi_1]],\xi_2],\cdots],\xi_n] \geqslant t_1 .$$

For $t \geqslant t_2$ we have

$$y^{(\ell)}(t)-y^{(\ell)}(t_2)=\sum_{j=\ell+1}^{n-1}\frac{y^{(j)}(t_2)}{(j-\ell)!}(t-t_2)^{j-\ell}+\int_{t_2}^{t}\frac{(t-s)^{n-\ell-1}}{(n-\ell-1)!}y^{(n)}(s)ds$$

$$=\sum_{j=\ell+1}^{n-1}\frac{y^{(j)}(t_2)}{(j-\ell)!}(t-t_2)^{j-\ell}+\int_{t_2}^{t}\frac{(t-s)^{n-\ell-1}}{(n-\ell-1)!}\left\{\int_a^b\cdots\int_a^b p\cdot y[g[\cdots[[g[t,\xi_1]],\cdots],\xi_n]d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right\}ds$$

$$\geqslant\sum_{j=\ell+1}^{n-1}\frac{y^{(j)}(t_2)}{(j-\ell)!}(t-t_2)^{j-\ell}+p\left[\int_a^b\cdots\int_a^b d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right]y(t_1)\int_{t_2}^{t}\frac{(t-s)^{n-\ell-1}}{(n-\ell-1)!}ds$$

$$=\sum_{j=\ell+1}^{n-1}\frac{y^{(j)}(t_2)}{(j-\ell)!}(t-t_2)^{j-\ell}+p\left[\int_a^b\cdots\int_a^b d\sigma(\xi_1)\cdots d\sigma(\xi_n)\right]y(t_1)\cdot\frac{(t-t_2)^{n-\ell}}{(n-\ell)!} .$$

It follows that $y^{(\ell)}(t)\to\infty$ as $t\to\infty$. But from $y^{(\ell)}(t)>0$, $y^{(\ell+1)}(t)>0$, we have $y^{(\ell)}(t)\leqslant y^{(\ell)}(t_0)=\text{const}$.

This led to a contradiction. So we obtain $\mathcal{N}=\mathcal{N}_0\cup\mathcal{N}_\ell$

(2).1. Suppose condition (2.5) is satisfied. Firstly, suppose $g(t,\xi)=t-\xi$ for $\xi\in[a,b]$, $t\geqslant t_0$, $(b>a>0)$. Set

$$z(t)=y^{(n-1)}(t)-p^*\int_a^b y^{(n-2)}(t-\xi_1)d\sigma(\xi_1)+p^{*2}\int_a^b\int_a^b y^{(n-3)}(t-\xi_1-\xi_2)d\sigma(\xi_1)d\sigma(\xi_2)$$
$$-\cdots-p^{*\,n-1}\int_a^b\cdots\int_a^b y(t-\xi_1-\cdots-\xi_{n-1})d\sigma(\xi_1)\cdots d\sigma(\xi_{n-1}) .$$

We have

$$z'(t)+p^*\int_a^b z(t-\xi)d\xi = y^{(n)}(t)-p^{*n}\int_a^b\cdots\int_a^b y(t-\xi_1-\cdots-\xi_n)d\sigma(\xi_1)\cdots d\sigma(\xi_n) .$$

As in Theorem 2, if $y(t)$ is a nonoscillatory positive solution of degree 0 of

$$y^{(n)}(t)-p^{*n}\int_a^b\cdots\int_a^b y(t-\xi_1-\cdots-\xi_n)d\sigma(\xi_1)\cdots d\sigma(\xi_n)\geqslant 0 ,$$

then we have $y(t)>0$, $y'(t)<0$, $y''(t)>0$, $\ldots$, $y^{(n-1)}(t)<0$, $y^{(n)}(t)>0$, that is, $z(t)$ is a nonoscillatory negative solution of

$$z'(t)+p^*\int_a^b z(t-\xi)d\xi\geqslant 0 ,$$

Using the condition $n\sqrt{p}\,\frac{b}{n}\,e>1$ from Lemma 2, this fact led to a contradiction. Secondly, suppose $g(t,\xi)<t-\xi$ for $\xi\,[a,b]$; $t\geqslant t_0$, wa can obtain

$$y^{(n)}(t)-p\int_a^b\cdots\int_a^b y[g[g[\cdots[[g[t,\xi_1]],\xi_2],\cdots],\xi_n]d\sigma(\xi_1)\cdots d\sigma(\xi_n)$$
$$\geqslant y^{(n)}(t)-p\int_a^b\cdots\int_a^b y(t-\xi_1-\xi_2\cdots-\xi_n)d\sigma(\xi_1)\cdots d\sigma(\xi_n)\geqslant 0 ,$$

i.e. $y(t)>0,\ y'(t)>0,$

where $y(t)$ is a nonoscillatory positive solution of degree 0 of (1.3), i.e. $y(t)>0,\ y'(t)<0,\ y[g[\cdots[[g[t,\xi_1]],\xi_2],\cdots],\xi_n]>y(t-\xi_1-\xi_2-\cdots-\xi_n)$.
As in Theorem 2, we can easily obtain a contradiction.

2. Suppose condition (2.6) is satisfied. Firstly, suppose $g(t,\xi)=kt\xi$ for some k, which satisfied $0<k\xi<1$, $\xi\in[a,b]$, $b>a>0$, $t\geqslant t_0$. Set

$$z(t)=y^{(n-1)}(t)-p^*u_{n-2}\int_a^b y^{(n-2)}(k^*t\xi_1)d\sigma(\xi_1)+$$
$$+p^{*2}u_{n-3}\int_a^b\int_a^b y^{(n-3)}(k^{*2}t\xi_1\xi_2)d\sigma(\xi_1)d\sigma(\xi_2)-\cdots$$
$$-p^{*\,n-1}u_0\int_a^b\cdots\int_a^b y(k^{*\,n-1}t\xi_1\xi_2\cdots\xi_{n-1})d\sigma(\xi_1)\cdots d\sigma(\xi_{n-1}) .$$

where $p^*=\sqrt[n]{p}\,(k^*a)^{\frac{n-1}{2}}$, $k^*=\sqrt[n]{k}$.

So we have

$$\zeta'(t)=y^{(n)}(t)-p^{*}u_{n-2}k^{*}\int_a^b \xi_1 y^{(n-1)}(k^{*}t\xi_1)\,d\sigma(\xi_1)+$$
$$+p^{*2}u_{n-3}k^{*2}\int_a^b\int_a^b y^{(n-2)}(k^{*2}t\xi_1\xi_2)\xi_1\xi_2\,d\sigma(\xi_1)d\sigma(\xi_2)-\cdots$$
$$-p^{*n-1}u_0k^{*n-1}\int_a^b\cdots\int_a^b \xi_1\xi_2\cdots\xi_{n-1}y'(k^{*n-1}t\xi_1\xi_2\cdots\xi_{n-1})\,d\sigma(\xi_1)\cdots d\sigma(\xi_{n-1}),$$

$$\zeta'(t)+p^{*}\int_a^b \zeta(k^{*}t\xi)\,d\sigma(\xi)=$$
$$=y^{(n)}(t)+p^{*}\Big[\int_a^b(1-u_{n-2}k^{*}\xi_1)y^{(n-1)}(k^{*}t\xi_1)\,d\sigma(\xi_1)\Big]+$$
$$+p^{*2}\Big[\int_a^b\int_a^b(u_{n-3}k^{*2}\xi_1\xi_2-u_{n-2})y^{(n-2)}(k^{*2}t\xi_1\xi_2)\,d\sigma(\xi_1)d\sigma(\xi_2)\Big]+\cdots$$
$$+p^{*n-1}\Big[\int_a^b\cdots\int_a^b(u_1-u_0k^{*}\xi_1\cdots\xi_{n-1})y'(k^{*n-1}t\xi_1\cdots\xi_{n-1})\,d\sigma(\xi_1)\cdots d\sigma(\xi_{n-1})\Big]-$$
$$-p^{*n}u_0\Big[\int_a^b\cdots\int_a^b y(k^{*n}t\xi_1\cdots\xi_n)\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\Big].$$

If we choose U_{n-2}, U_{n-3}, ..., U_1, U_0 such that

$$u_{n-2}=(k^{*}a)^{-1},\quad u_{n-3}=(k^{*}a)^{-3},\ \ldots,\ u_0=(k^{*}a)^{-\frac{n(n-1)}{2}},$$

i.e.

$$1-u_{n-2}k^{*}\xi_1\le 0,\quad u_{n-3}k^{*2}\xi_1\xi_2-u_{n-2}\ge 0,\ \cdots,$$
$$u_1-u_0k^{*n-1}\xi_1\cdots\xi_{n-1}\le 0,\quad \text{for } \xi_i\in[a,b],\ i=1,2,\cdots,n,$$

then we have

$$\zeta'(t)+p^{*}\int_a^b \zeta(k^{*}t\xi)\,d\sigma(\xi)$$
$$\ge y^{(n)}(t)-p^{*n}(k^{*}a)^{-\frac{n(n-1)}{2}}\int_a^b\cdots\int_a^b y(k^{*n}t\xi_1\cdots\xi_n)\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)$$
$$=y^{(n)}(t)-p\int_a^b\cdots\int_a^b y(kt\xi_1\cdots\xi_n)\,d\sigma(\xi_1)\cdots d\sigma(\xi_n).$$

Suppose $y(t)$ is a positive solution of the following inequality:

$$y^{(n)}(t)-p\int_a^b\cdots\int_a^b y(kt\xi_1\cdots\xi_n)\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\ge 0.$$

Then we can obtain that $\zeta(t)$ is a positive solution of

$$\zeta'(t)+p^{*}\int_a^b \zeta(k^{*}t\xi)\,d\sigma(\xi)\ge 0.$$

From Lemma 2, we can obtain a contradiction. Secondly, suppose $g(t,\xi)<kt\xi$ for some k, which satisfies $0<k\xi<1$, $\xi\in[a,b]$, and all $t>t_0$, then we can easily obtain that

$$0\le y^{(n)}(t)-p\int_a^b\cdots\int_a^b y[g[g[\cdots[g(t,\xi_1)],\cdots],\xi_n]\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)$$
$$\le y^{(n)}(t)-p\int_a^b\cdots\int_a^b y(t-\xi_1-\xi_2-\cdots-\xi_n)\,d\sigma(\xi_1)\cdots d\sigma(\xi_n),$$

where $y(t)$ is a positive solution of degree 0 of (1.3), i.e.

$$y(t)>0,\quad y'(t)<0,\quad y[g[g[\cdots[g(t,\xi_1)],\xi_2],\cdots],\xi_n]\ge y(t-\xi_1-\cdots-\xi_n).$$

As in Theorem 1, we can easily obtain a contradiction.

(3). In the case (2.2). From Theorem 1, we obtain $\mathcal{N}_0=\phi$. So we have $\mathcal{N}=\mathcal{N}_n$. The proof of Theorem 3 is complete.

<u>Theorem 4</u>. For the superlinear inequality (1.7) or (1.7)' if (2.5) or (2.6) is satisfied, then $\mathcal{N}=\mathcal{N}_n$, i.e. $\mathcal{N}_0=\phi$.

Notice there exists a large enough $\lambda>0$ such that $f(y(g(t)))\ge\lambda y(g(t))$, we easily prove Theorem 4 for (1.7). Similarly for (1.7)' we have

$$f(y[g[g[\cdots[g(t,\xi_1)],\xi_2],\cdots],\xi_n])$$
$$\ge\lambda y[g[g[\cdots[g(t,\xi_1)],\xi_2],\cdots],\xi_n],$$

where $\lambda>0$ is a large enough number.

3. ADVANCED INEQUALITIES AND EQUATIONS

Now we turn to (1.2) and (1.4). Our object is to give some sufficient condition under which $\mathcal{N}_n=\phi$, $\mathcal{N}=\mathcal{N}_0$ for (1.2) or (1.4).

Theorem 5. If $h(t) \geq \omega t$ for some $\omega > 1$ and all $t \geq T$, then all nonoscillatory solutions of (1.2) are of degree 0. ($\mathcal{N} = \mathcal{N}_0$ i.e. $\mathcal{N}_n = \phi$).

Proof. (1) In case $h(t) = \omega t$, for some $\omega > 1$ and all $t \geq T$. Let $y(t)$ be a nonoscillatory positive solution of degree $\ell > 0$ of (1.2). Proceeding exactly as in the proof of Theorem 3, we see that $\ell = n$, that is, $y(t) > 0$, $y'(t) > 0, \ldots, y^{(n-1)}(t) > 0$ for all large t, say $t \geq t_1 \geq T$. Notice

$$y^{(n)}(t) - p\,y[\omega t] \geq 0. \tag{3.1}$$

Define

$$z(t) = y^{(n-1)}(t) + p^* u_{n-2}\, y^{(n-2)}(\omega^* t) + p^{*2} u_{n-3}\, y^{(n-3)}(\omega^{*2} t) + \cdots + p^{*n-1} u_0\, y(\omega^{*n-1} t) \geq 0,$$

where $p^* = \sqrt[n]{p}\,\omega^{-\frac{n(n-1)}{2}}$, $\omega^* = \sqrt[n]{\omega}$, u_i $(i = 0, 1, \ldots, n-2)$ are some constants.

Then we have

$$z'(t) = y^{(n)}(t) + p^* u_{n-2}\, y^{(n-1)}(\omega^* t) + p^{*2} u_{n-3}\, y^{(n-2)}(\omega^* t) + \cdots + p^{*n-1} u_0\, k^{*n-1} y'(\omega^{*n-1} t).$$

If we choose positive constants $U_0, U_1, \ldots, U_{n-2}$ which satisfy the following conditions

$$\begin{aligned} &\omega^* u_{n-2} - 1 = 0, \\ &\omega^{*2} u_{n-3} - u_{n-2} = 0, \\ &\cdots\cdots \\ &\omega^{*n-1} u_0 - u_1 = 0, \end{aligned} \tag{3.2}$$

that is $u_{n-2} = 1/\omega^*$, $u_{n-3} = 1/\omega^{*3}, \ldots, u_0 = \omega^{*\,-\frac{n(n-1)}{2}}$,

So $z(t)$ is a positive solution of (3.2). But

$$\liminf_{t \to +\infty} p(\omega t - t) = +\infty > 1/e,$$

$$\liminf_{t \to +\infty} p(\sqrt{\omega}\, t - t) = +\infty > 0,$$

that is, conditions (H_1') and (H_2') are satisfied.
So using Lemma 3 we can see that there is no a positive solution of (3.2). This led to a contradiction.

(2). In the case $h(t) > \omega t$ for some $\omega > 1$ and all $t \geq T$. If there is a positive solution $y(t) > 0$ of degreee $\ell > 0$ for (1.2), then $y'(t) > 0$ i.e. $y[h(t)] > y[\omega, t]$. So we have

$$0 \leq y^{(n)}(t) - p\,y[h(t)] \leq y^{(n)}(t) - p\,y[\omega t].$$

From the proof in the case $h(t) = \omega t$, we can obtain a contradiction. The proof of Theorem 5 is complete.

Theorem 6. If $h(t) \geq t + \tau$ for some $\tau > 0$ and all $t \geq T$ then the condition

$$\sqrt[n]{p}\,\frac{\tau}{n}\,e > 1 \tag{3.3}$$

implies that (1.2) has no nonoscillatory solution of degree n. ($\mathcal{N}_n = \phi$).

Proof. In the case $h(t) = t + \tau$ for some $\tau > 0$ and all $t \geq T$. Suppose (3.3) is satisfied. Otherwise, suppose there is a nonoscillatory positive solution $y(t) > 0$ of degree n of (1.2). So we have $y'(t) > 0$ i.e. $y[h(t)] > y(t - \tau)$ and

$$0 \leq y^{(n)}(t) - p\,y[h(t)] \leq y^{(n)}(t) - p\,y(t + \tau)$$

From the proof in case (1) of Theorem 5, we can deduce that condition (3.3) implies the inequality

$$y^{(n)}(t) - p\,y(t + \tau) \geq 0$$

has no nonoscillatory positive solution. This led to a contradiction. As in case 2 of Theorem 5, for $h(t)>t+\xi$ we are led to a contradiction. We proved Theorem 6.

Theorem 7: For (1.4), $\mathcal{N}=\mathcal{N}_0\cup\mathcal{N}_n$. If in (1.4) one of following conditions is satisfied

(i) $h(t,\xi)\geq t+\xi$ for all $t\geq T$, $\xi\in[a,b]$, $b>a$

and $n\sqrt[n]{p}\,\frac{a}{n}\,e>1$ (3.4)

(ii) $h(t,\xi)>kt\xi$ for some k, which satisfies

$k\xi>1$, $\xi\in[a,b]$, and all $t\geq T$;

then for (1.4) $\mathcal{N}_n=\phi$, $\mathcal{N}=\mathcal{N}_0$.

For (3.1), $\mathcal{N}_n=\phi$, $\mathcal{N}=\mathcal{N}_0$.

Proof. (1) In the case (1.4) notice that in the proof of Theorem 3 we only need that $\lim\limits_{t\to+\infty} g(t,\xi)=+\infty$, $\xi\in[a,b]$

If $g(t,\xi)$ is changed by $h(t,\xi)$, then results can also be proved.

(2) 1. In the case (1.4). Suppose condition (3.4) is satisfied. Firstly suppose $h(t,\xi)=t+\xi$ for $\xi\in[a,b]$ $t\geq T$, $b>a>0$.

Set

$$z(t)=y^{(n-1)}(t)+p^*\int_a^b y^{(n-2)}(t+\xi_1)\,d\sigma(\xi_1)+p^{*2}\int_a^b\int_a^b y^{(n-3)}(t+\xi_1+\xi_2)\,d\sigma(\xi_1)\,d\sigma(\xi_2)$$
$$+\cdots+p^{*\,n-1}\int_a^b\cdots\int_a^b y(t+\xi_1+\xi_2+\cdots+\xi_{n-1})\,d\sigma(\xi_1)\cdots d\sigma(\xi_{n-1}),$$

where $p^*=\sqrt[n]{p}>0$.

So we have

$$z'(t)=y^{(n)}(t)+p^*\int_a^b y^{(n-1)}(t+\xi_1)\,d\sigma(\xi_1)+p^{*2}\int_a^b\int_a^b y^{(n-2)}(t+\xi_1+\xi_2)\,d\sigma(\xi_1)\,d\sigma(\xi_2)$$
$$+\cdots+p^{*\,n-1}\int_a^b\cdots\int_a^b y'(t+\xi_1+\xi_2+\cdots+\xi_{n-1})\,d\sigma(\xi_1)\cdots d\sigma(\xi_{n-1}),$$

$$z'(t)-p^*\int_a^b z(t+\xi)\,d\xi=y^{(n)}(t)-p^{*n}\int_a^b\cdots\int_a^b y(t+\xi_1+\cdots+\xi_n)\,d\sigma(\xi_1)\cdots d\sigma(\xi_n).$$

As in Theorem 6, if $y(t)$ is a nonoscillatory positive solution of degree n of

$$y^{(n)}(t)-p\int_a^b\cdots\int_a^b y(t+\xi_1+\cdots+\xi_n)\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)\geq 0,$$

then we have $y(t)>0$, $y'(t)>0$, $y''(t)>0$, ... $y^{(n-1)}(t)>0$, $y^{(n)}(t)>0$, that is, $z(t)$ is a nonoscillatory positive solution of

$$z'(t)-p^*\int_a^b z(t+\xi)\,d\xi\geq 0.$$

Using the condition $n\sqrt[n]{p}\,\frac{a}{n}\,e>1$, from Lemma 3 this led to a contradiction.

Secondly, suppose $h(t,\xi)>t+\xi$ for $\xi\in[a,b]$, $b>a>0$, $t\geq T$.

Then we can easily obtain

$$0\leq y^{(n)}(t)-p\int_a^b\cdots\int_a^b y[h[h[h[\cdots[h[t,\xi_1],\xi_2],\cdots],\xi_n]\,d\sigma(\xi_1)\cdots d\sigma(\xi_n)$$
$$\leq y^{(n)}(t)-p\int_a^b\cdots\int_a^b y(t+\xi_1+\xi_2+\cdots+\xi_n)\,d\sigma(\xi_1)\cdots d\sigma(\xi_n),$$

where $y(t)$ is a nonoscillatory positive solution of degree n of (1.4) i.e. $y(t)>0$, $y'(t)>0$,... $y^{(n)}(t)>0$.

$$y[h[h[\cdots[h[t,\xi_1],\xi_2],\cdots],\xi_n]\geq y(t+\xi_1+\xi_2+\cdots+\xi_n)$$

As in Theorem 6, we can easily obtain a contradiction.

2. Suppose (3.5) is satisfied. Firstly, suppose $h(t,\xi)=\omega t\xi$ for some ω, which satisfies $\omega\xi>1$, $\xi\in[a,b]$, $t\geq T$. Set,

$$z(t)=y^{(n-1)}(t)+p^*u_{n-2}\int_a^b y^{(n-2)}(\omega^* t\xi_1)\,d\sigma(\xi_1)+p^{*2}u_{n-3}\int_a^b\int_a^b y^{(n-3)}(\omega^{*2} t\xi_1\xi_2)\,d\sigma(\xi_1)\,d\sigma(\xi_2)$$
$$+\cdots+p^{*\,n-2}u_1\int_a^b\cdots\int_a^b y(\omega^{*\,n-2}t\xi_1\cdots\xi_{n-2})\,d\sigma(\xi_1)\cdots d\sigma(\xi_{n-2})$$
$$+p^{*\,n-1}u_0\int_a^b\cdots\int_a^b y(\omega^{*\,n-1}t\xi_1\cdots\xi_{n-1})\,d\sigma(\xi_1)\cdots d\sigma(\xi_{n-1}),$$

where $p^*=\sqrt[n]{p}\,(\omega^* a)^{-\frac{n(n-1)}{2}}$, $\omega^*=\sqrt[n]{\omega}$, U_{n-2}, U_{n-3}, ... U_1, U_0 are positive constants, $y(t)$ is a nonoscillatory positive solution of degree n of (1.4) of $h(t,\xi)=\omega t\xi$, i.e. $y(t)>0$, $y'(t)>0$,... $y^{(n)}(t)>0$.

So we have

$$
\begin{aligned}
&z'(t) - p^* \int_a^b z(\omega^* t\, \xi)\, d\sigma(\xi) \\
&= y^{(n)}(t) + p^* \int_a^b (u_{n-2}\omega^* \xi_1 - 1)\, y^{(n-1)}(\omega^* t\, \xi_1)\, d\sigma(\xi_1) + \\
&\quad + p^{*2} \int_a^b \int_a^b (u_{n-3}\omega^{*2} \xi_1 \xi_2 - u_{n-2})\, y^{(n-2)}(\omega^* t\, \xi_1 \xi_2)\, d\sigma(\xi_1) d\sigma(\xi_2) + \cdots + \\
&\quad + p^{*n-1} \int_a^b \cdots \int_a^b (u_0 \omega^{*n-1} \xi_1 \xi_2 \cdots \xi_{n-1} - u_1)\, y'(\omega^{*n-1} t\, \xi_1 \cdots \xi_{n-1})\, d\sigma(\xi_1) \cdots d\sigma(\xi_{n-1}) \\
&\quad - p^{*n} u_0 \int_a^b \cdots \int_a^b y(\omega^{*n} t\, \xi_1 \cdots \xi_n)\, d\sigma(\xi_1) \cdots d\sigma(\xi_n).
\end{aligned}
$$

If we choose $U_{n-2}, U_{n-3}, \ldots, U_1, U_0$, which satisfies

$$
u_{n-2} = 1/\omega^* a, \quad u_{n-3} = 1/(\omega^* a)^3, \quad \cdots,
$$
$$
u_0 = (\omega^* a)^{-\frac{n(n-1)}{2}}
$$

i.e. $\quad u_{n-2}\omega^* \xi_1 - 1 \geq 0, \; u_{n-3}\omega^{*2} \xi_1 \xi_2 - u_{n-2} \geq 0, \cdots, \; u_0 \omega^{*n-1} \xi_1 \cdots \xi_{n-1} - u_1 \geq 0,$

then we have

$$
\begin{aligned}
z'(t) - p^* \int_a^b z(\omega^* t\, \xi)\, d\sigma(\xi) &\geq y^{(n)}(t) - p^{*n} (\omega^* a)^{-\frac{n(n-1)}{2}} \int_a^b \cdots \int_a^b y(\omega^{*n} t\, \xi_1 \cdots \xi_n)\, d\sigma(\xi_1) \cdots d\sigma(\xi_n) \\
&= y^{(n)}(t) - p \int_a^b \cdots \int_a^b y(\omega t\, \xi_1 \cdots \xi_n)\, d\sigma(\xi_1) \cdots d\sigma(\xi_n) \geq 0.
\end{aligned}
$$

So we deduce that $z(t)$ is a nonoscillatory positive solution of the following inequality.

$$
z'(t) - p^* \int_a^b z(\omega^* t\, \xi)\, d\sigma(\xi) \geq 0.
$$

Using Lemma 3, we can obtain a contradiction.

Secondly, suppose $h(t, \xi) > \omega t \xi$, for some ω, which satisfies $\omega \xi > 1$, for $\xi \in [a, b]$, and all $t \geq T$, then we can easily obtain that

$$
\begin{aligned}
0 &\leq y^{(n)}(t) - p \int_a^b \cdots \int_a^b y[h[h[\cdots[h[t, \xi_1]], \xi_2], \cdots], \xi_n]\, d\sigma(\xi_1) \cdots d\sigma(\xi_n) \\
&\leq y^{(n)}(t) - p \int_a^b \cdots \int_a^b y(\omega t\, \xi_1 \xi_2 \cdots \xi_n)\, d\sigma(\xi_1) \cdots d\sigma(\xi_n),
\end{aligned}
$$

where $y(t)$ is a positive solution of degree n of (1.4), i.e.

$$
y(t) > 0, \quad y'(t) > 0,
$$
$$
y[h[h[\cdots[h[t, \xi_1]], \xi_2], \cdots], \xi_n] > y(\omega t\, \xi_1 \cdots \xi_n).
$$

As in theorem 5, we can easily obtain a contradiction.

(3) In the case (1.2). The proof of the Theorem is obvious. Proof of Theorem 7 is complete.

Theorem 8. For the general superlinear inequality (1.8)', if (3.4) or (3.5) is satisfied, then we have $\mathcal{N}_n = \phi$, $\mathcal{N} = \mathcal{N}_0$.

The proof of Theorem 8 is similar to Theorem 4.

Remark. From Theorem 4 and Theorem 8 we can see that if deviating arguments are taken forms as t^{α} $(\alpha > 0, \alpha \neq 1)$, εnt; $t - |\sin t|$, $t^2/(t+1)$, e^t, $t^2/(t-1)$, then for superlinear inequality (2.4) $\mathcal{N}_0 = \phi$, $\mathcal{N} = \mathcal{N}_n$ and for (3.4) $\mathcal{N}_n = \phi$, $\mathcal{N} = \mathcal{N}_0$. Notice that for the superlinear case we do not need such conditions as (2.4), (3.3) to have $\mathcal{N}_n = \phi$, $\mathcal{N} = \mathcal{N}_0$

4. EQUATIONS WITH ADVANCED AND RETARDED ARGUMENTS.

In this section we are concerned with the differential equations with both retarded and advanced arguments,

$$
\begin{aligned}
y^{(n)}(t) = p \int_a^b \cdots \int_a^b y[g[g[\cdots[g[t, \xi_1]] \xi_2], \cdots], \xi_n]\, d\sigma_1(\xi_1) \cdots d\sigma_1(\xi_n) \\
+ q \int_a^b \cdots \int_a^b y[h[h[\cdots[h[t, \xi_1]], \xi_2], \cdots], \xi_n]\, d\sigma_2(\xi_1) \cdots d\sigma_2(\xi_n).
\end{aligned} \tag{4.1}
$$

where $p > 0$, $q > 0$, $\sigma_1(\xi)$ and $\sigma_2(\xi)$ are continuous nondecreasing functions.

Theorem 9. For the equation (4.1) $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n$. If one of following conditions is satisfied:

(i) $\quad g(t, \xi) < t - \xi$, for $\xi \in [a, b]$, $b > a > 0$, all $t \geq T$ and $\sqrt[n]{p}\, \frac{b}{n} e > 1$.

(ii) $g(t,\xi) < kt\xi$ for some k, which satisfied $k\xi \in (0,1)$, $\xi \in [a,b]$, and all $t \geq T$;

then for (4.1) $\mathcal{N} = \mathcal{N}_n$.

Also if one of the following conditions is satisfied:

(iii) $h(t,\xi) > t+\xi$, for $\xi \in [a,b]$, $b>a>0$, and all $t \geq T$, and $\sqrt[n]{p}\,\frac{a}{n}\,e > 1$;

(iv) $h(t,\xi) > \omega t \xi$ for some ω, which satisfies $\omega\xi > 1$, $\xi \in [a,b]$ and all $t \geq T$,

then for (4.1) $\mathcal{N} = \mathcal{N}_0$.

If conditions (i), (ii), conditions (iii) and (iv) are satisfied, then (4.1) is oscillatory.

Proof. Notice

$$y^{(n)}(t) - p\int_a^b \cdots \int_a^b y[g(g(\cdots[[g(t,\xi_1)],\xi_2],\cdots],\xi_n]\,d\sigma(\xi_1)\cdots d\sigma(\xi_n) \geq 0,$$

$$y^{(n)}(t) - q\int_a^b \cdots \int_a^b y[h(h(\cdots[[h(t,\xi_1)],\xi_2],\cdots],\xi_n]\,d\sigma(\xi_1)\cdots d\sigma(\xi_n) \geq 0.$$

Using Theorem 3 and Theorem 7 we can easily obtain results of Theorem 9 . The proof of Theorem 9 is complete. From Theorem 9 we can easily obtain results for

$$y^{(n)}(t) = p\,y[g(t)] + q\,y[h(t)], \tag{4.2}$$

where $p > 0$, $q > 0$. Here these results are omitted.

5. SOME EXAMPLES.

In the following examples suppose n is even

Example 1

$$\left\{y^{(n)}(t) - p\int_a^b \cdots \int_a^b y(\sqrt{n}\,t - \xi_1 - \cdots - \xi_n)\,d\xi_1 d\xi_2 \cdots d\xi_n\right\} \operatorname{Sgn}\left[\int_a^b \cdots \int_a^b y(\sqrt{n}\,t - \xi_1 - \cdots - \xi_n)\,d\xi_1 d\xi_2 \cdots d\xi_n\right] \geq 0, \tag{5.1}$$

where $p > 0$, $b > a > 0$. If $\sqrt[n]{p}\,\frac{R}{n}\,e > 1$, then for (5.1) $\mathcal{N} = \mathcal{N}_n$.

We consider the following inequality:

$$\left\{y^{(n)}(t) - p\int_a^b y(\sqrt{n}\,t - \xi)\,d\xi\right\} \operatorname{sgn}\left\{\int_a^b y(\sqrt{n}\,t - \xi)\,d\xi\right\} \geq 0. \tag{5.1'}$$

If $n = 2$, then we have

$$p^*\int_{a^*}^{b^*}\int_{a^*}^{b^*} y(\sqrt{2}\,t - \xi_1 - \xi_2)\,d\xi_1 d\xi_2 \leq p^*\int_a^b\int_a^b y(\sqrt{2}\,t - \xi)\,d\xi\, d\eta = p^*(b-a)\int_a^b y(\sqrt{2}\,t - \xi)\,d\xi$$

where $a^* = \frac{1}{\sqrt{2}}a$, $b^* = \frac{1}{\sqrt{2}}b$. For any n we have

$$p^*\int_{a^*}^{b^*} \cdots \int_{a^*}^{b^*} y(\sqrt{n}\,t - \xi_1 - \xi_2 - \cdots - \xi_n)\,d\xi_1 d\xi_2 \cdots d\xi_n \leq p^*(b-a)^{n-1}\int_a^b y(\sqrt{n}\,t - \xi)\,d\xi = p\int_a^b y(\sqrt{n}\,t - \xi)\,d\xi,$$

where $a^* = \frac{1}{\sqrt{n}}a$, $b^* = \frac{1}{\sqrt{n}}b$, $p^* = p\,(b-a)^{-(n-1)}$.

Notice that

$$y^{(n)}(t) - p^*\int_{a^*}^{b^*} \cdots \int_{a^*}^{b^*} y(\sqrt{n}\,t - \xi_1 - \cdots - \xi_n)\,d\xi_1 \cdots d\xi_n \geq y^{(n)}(t) - p\int_a^b y(\sqrt{n}\,t - \xi)\,d\xi.$$

So we can obtain that if $p > 0$, $b > a > 0$, $\sqrt[n]{p^*}\,\frac{a^*}{n}\,e > 1$, then for (5.1) $\mathcal{N} = \mathcal{N}_n$.

Example 2

$$\left\{y^{(n)}(t) - p\int_a^b \cdots \int_a^b y(kt\xi_1 \cdots \xi_n)\,d\sigma(\xi_1) \cdots d\sigma(\xi_n)\right\} \cdot \operatorname{sgn}\left\{\int_a^b \cdots \int_a^b y(kt\xi_1 \cdots \xi_n)\,d\sigma(\xi_1) \cdots d\sigma(\xi_n)\right\} \geq 0, \tag{5.2}$$

where $p > 0$, $b > a > 0$. If k is such that $kb < 1$, then for (5.2) $\mathcal{N} = \mathcal{N}_n$. If k is such that $ka > 1$, then for (5.2) $\mathcal{N} = \mathcal{N}_0$.

Example 3

$$\left\{y^{(n)}(t) - p\int_a^b \cdots \int_a^b y(t^2 + \xi_1 + \cdots + \xi_n)\,d\xi_1 \cdots d\xi_n\right\} \cdot \operatorname{sgn}\left\{\int_a^b \cdots \int_a^b y(t^2 + \xi_1 + \cdots + \xi_n)\,d\xi_1 \cdots d\xi_n\right\} \geq 0, \tag{5.3}$$

If $\sqrt[n]{p}\,\frac{a}{n}\,e > 1$, then for (5.3) $\mathcal{N} = \mathcal{N}_0$ ($p > 0$, $b > a > 0$).

Example 4

$$y^{(n)}(t) = p\int_{a_1}^{b_1}\cdots\int_{a_1}^{b_1} y[k_1 t\xi_1\cdots\xi_n]\,d\xi_1\cdots d\xi_n + q\int_{a_2}^{b_2}\cdots\int_{a_2}^{b_2} y[k_2 t\xi_1\cdots\xi_n]\,d\xi_1\cdots d\xi_n , \tag{5.4}$$

where $p > 0$, $q > 0$, $b_i > a_i > 0$ $(i=1,2)$. $0 < k_1 b_1 < 1$, $k_2 a_2 > 1$. Then (5.4) oscillates.

Example 5.

$$\begin{aligned} y^{(n)}(t) = {} & p\int_{a_1}^{b_1}\cdots\int_{a_1}^{b_1} f_1(y[t-\xi_1-\cdots-\xi_n])\,d\xi_1\cdots d\xi_n \\ & + q\int_{a_2}^{b_2}\cdots\int_{a_2}^{b_2} f_2(y[t+\xi_1+\cdots+\xi_n])\,d\xi_1\cdots d\xi_n , \end{aligned} \tag{5.5}$$

where f_1 and f_2 are superlinear. Then (5.5) oscillates.

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