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ABSTRACT

In U_4 theory of gravitation Cartan's contorsion is determined, among other extraordinary fields, by a pair of standard massless spin 2 self-interacting fields.

АННОТАЦИЯ

В теории гравитации U_4 конторзия Картана, наряду с несколькими другими полями с необычными свойствами, определена парой стандарных безмассовых самовоздействующих полей со спином два.

KIVONAT

A gravitáció U_4-es elméletében a Cartan-féle kontorzió, több egyéb szokatlan sajátosságu mező mellett, egy kettes spinü tömegnélküli önkölcsönható mezőpárral adott.

INTRODUCTION

At present the old Einstein-Cartan's theory of gravitation is intensively studied again (for a review cf. [1], for some new aspects see [2]). The theory assumes that the real space-time is an U_4 manifold, and the connection is given by

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{f}{2} K^i_{jk}, \quad f = \sqrt{32\pi G}, \tag{1}$$

where Γ^i_{jk} are the Christoffel symbols defined by the g_{ij} metric tensor in the usual manner and $K^{ijk} = -K^{jik}$ are the components of contorsion with respect to a coordinate basis. Here the gravitational constant G has the dimension of $(\text{length})^2$, because we use the natural system $\hbar=c=1$.

Cartan's contorsion is the gauge field of Lorentz group [1]. Nevertheless, the physical character of this field is still a fully open question. (The standard Yang-Mills gauge fields of internal symmetry groups are spin 1 fields; Einstein's gravity is a spin 2 field. But what about the contorsion?)

The purpose of this paper is to show that contorsion is determined in general case by two self-interacting massless spin 2 fields.

1. DECOMPOSITION OF THE CONTORSION TENSOR

Let $F^{ijk} = -F^{jik}$ be the components of an antisymmetric tensor in U_4 manifold. Then one has (see Appendix A.):

$$F^{ijk} = W^{k[i;j]} + \mu^{ijpm} Z^k{}_{p;m}, \quad W^{ki}{}_{;i} = Z^{ki}{}_{;i} = 0, \tag{1.1}$$

where a semicolon denotes partial derivatives and [ ] denotes antisymmetrization without the factor $\frac{1}{2}$. μ^{ijpm} are the components of the fully antisymmetric tensor ($\mu^{ijpm} = (-\det g_{ij})^{-1} e^{ijpm}$, $e^{0123} = -1$). Thus for the contorsion tensor

$$K^{ijk} = U^{k[i;j]} + \mu^{ijpm} V^k{}_{p;m}, \quad U^{ki}{}_{;i} = V^{ki}{}_{;i} = 0 \tag{1.2}$$

hold, which is in fact an infinite series of the form $\sum_{n=0}^{\infty} f^n.(\ldots)$. To show this it is enough to express

$$U_{ki;j} = U_{ki,j} - (\Gamma^m_{kj} + \frac{f}{2} K^m_{kj})U_{mi} - (\Gamma^m_{ij} + \frac{f}{2} K^m_{ij})U_{km}, \tag{1.3}$$

and substitute (1.2) into K_{ijk}. Then the new terms $f^2 K_{ijk}$ arise; substituting (1.2) into $f^2 K_{ijk}$ the new terms $f^3 K_{ijk}$ arise; etc. The same procedure is to be done for $V_{ij;k}$, too.

2. CONTORSION AS THE PAIR OF MASSLESS SPIN 2 FIELDS

In U_4 theory the Einstein's Lagrangian changes into [1]

$$\tilde{L} = -\frac{2}{f^2} g^{ik}(\tilde{\Gamma}^m_{ij} \tilde{\Gamma}^j_{mk} - \tilde{\Gamma}^j_{ik} \tilde{\Gamma}^j_{mj}) \tag{2.1}$$

Consider now a special case, when the following restrictions hold:

a. In (1.2) $U^{[ij]} = 0$ and $V^{[ij]} = 0$, i.e. U^{ij} and V^{ij} are symmetric.

b. U_4 is a Weitzenböck space T_4, i.e. the Riemannian part of curvature tensor is zero. Then the metric tensor has the form $g_{ij} = \eta_{ij} \equiv \mathrm{diag}(1,-1,-1,-1)$.

c. fK_{ijk} are infinitesimally small, the therefore in (1.2) the covariant derivatives may be substituted by partial derivatives.

The Lagrangian (2.1) takes the form

$$L_{(O)} = -\frac{1}{2}(K^{ijk}K_{jki} + K^{ij}{}_j K_i{}^k{}_k) = \frac{1}{2}(U^{ij,k}U_{ij,k} + 2U^{,i}U^k_{i,k} - 2U^{ij,k}U_{ik,j} - U^{,i}U_{,i}) + \frac{1}{2}(V^{ij,k}V_{ij,k} + 2V^{,i}V^k_{i,k} - 2V^{ij,k}V_{ik,j} - V^{,i}V_{,i}) = L_{(O)}(U) + L_{(O)}(V), \tag{2.2}$$

where we introduced the term

$$L_{(O)}(U) = \frac{1}{2}(U^{ij,k}U_{ij,k} + 2U^{,i}U^k_{i,k} - 2U^{ij,k}U_{ik,j} - U^{,i}U_{,i}), \quad U \equiv U^i_i, \tag{2.3}$$

and omitted the four-divergencies. Note now that the formulas

$$U^{ji}{}_{,j}U^{k}_{i,k} = U^{ji,k}U_{jk,i} + \text{four-divergence},$$
$$(2.4)$$
$$e^{ijkm}e_{iprs} = \delta^{j}_{p}\delta^{k}_{[r}\delta^{m}_{s]} + \delta^{j}_{r}\delta^{k}_{[s}\delta^{m}_{p]} + \delta^{j}_{s}\delta^{k}_{[p}\delta^{m}_{r]}$$

were used. Lagrangian (2.3) is the standard Lagrangian of a free massless spin 2 field [3]. Therefore, $L_{(0)}$ describes two free massless spin 2 fields. fields. Of course, in (2.2) - (2.4) we can use the relations

$$U^{ij}{}_{,j} = V^{ij}{}_{,j} = 0, \tag{2.5}$$

however, the relations are convenient for the demonstration of massless spin 2 character (compare with [3]).

The massless spin 2 fields, described by potentials U^{ij} and V^{ij} ("U-field" and "V-field"), should change under the infinitesimal gauge transformations as (see [3])

$$\bar{U}^{ij} = U^{ij} + A^{(i,j)}, \qquad \bar{V}^{ij} = V^{ij} + B^{(i,j)}, \tag{2.6}$$

where A^i and B^i are infinitesimally small components of four-vectors, and where ( ) denotes symmetrization without the factor $\frac{1}{2}$. Therefore, the contorsion should change under the infinitesimal gauge transformations as follows:

$$\bar{K}^{ijk} = \bar{U}^{k[i,j]} + e^{ijpm}\bar{V}^{k}{}_{p,m} = K^{ijk} + A^{[i,j]k} + e^{ijpm}B_{p,m}{}^{k} =$$
$$= K^{ijk} + \omega^{ij,k}, \qquad \omega^{ij} = -\omega^{ji}. \tag{2.7}$$

The infinitesimal ω^{ij} are the components of an arbitrarily chosen antisymmetric tensor. As it is well-known, under the local Lorentz-rotation of tetrad basis the components of contorsion change in accordance with (2.7) (see Appendix B).

We arrived at the result that in our special case, when the restrictions a.,b. and c. are fulfilled, Cartan's contorsion contains two massless spin 2 fields.

3. INTERACTION AND SELF-INTERACTION OF U- AND V-FIELD

Consider now a more general case. Suppose that the restrictions a. and b. of Chapter 2. are fulfilled, but fK^{ijk} are arbitrary. In this case the Lagrangian (2.1) takes the form

$$L = \sum_{n=0}^{\infty} f^{n} L_{(n)}, \tag{3.1}$$

where $L_{(0)}$ is given by (2.2). It is obvious that $L_{(n)}$ ($n \geq 1$) describe the interaction of U- and V-field. Interaction between the U- and V-field is given by the terms containing the products of potentials U^{ij} and V^{ij}; the self-interaction of U-field (V-field) is given by the terms containing the product of potentials $U^{ij}(V^{ij})$.

Note now that the interaction and self-interaction here are highly similar to the case of Einstein's gravity [4].

Consider now that the restrictions b. and c. of Chapter 2. are not fulfilled; i.e. $g_{ij} \neq \eta_{ij}$. In this case the Lagrangian (2.1) takes the form (3.1) again. Nevertheless, here we have three massless interacting and self-interacting spin 2 fields: Einstein's gravitational field ("graviton field"), U- and V-field.

Consider now the most general case, when the restrictions a., b. and c. of Chapter 2. are not fulfilled. In this case the contorsion contains the pair of the massless spin 2 fields again, because in (1.2) the components U^{ij} and V^{ij} may be decomposed into the symmetrical and antisymmetrical parts, and the symmetrical parts give U- and V-field again. Nevertheless, here are other fields, too. These are determined by

$$Q^{ij} = U^{[ij]}, \qquad R^{ij} = V^{[ij]}. \tag{3.2}$$

The decomposition of these components is straightforward. One has:

$$Q^{ij} = M^{[i;j]} + \frac{1}{\sqrt{2}} \mu^{ijpm} N_{[p;m]}, \quad M^{i}{}_{;i} = N^{i}{}_{;i} = 0 \,,$$
$$R^{ij} = P^{[i;j]} + \frac{1}{\sqrt{2}} \mu^{ijpm} S_{[p;m]}, \quad P^{i}{}_{;i} = S^{i}{}_{;i} = 0. \tag{3.3}$$

Thus in general case the contorsion contains the second derivatives of M^i, N^i, P^i and S^i, and therefore the relevant field equations for these components of vector fields are fourth order differential equations. In other words, these vector fields are extraordinary, because these can hardly be interpreted as standard spin 1 fields.

CONCLUSIONS

Today it is not clear yet that the gravitation in classical limit is or is not described by the Einstein-Cartan's theory. This question was not studied here. We ad hoc assumed that the contorsion was non-vanishing, and we examined the field character of contorsion. What we have shown is that,: contorsion always contains a pair of massless spin 2 fields, the gauge fields of Lorentz group. Nevertheless, as is the general case, there are other extraordinary vector fields, too.

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APPENDIX A. DECOMPOSITION OF AN ANTISYMMETRIC TENSOR

In the flat space-time an antisymmetric tensor - in general case- is defined by two vectors [5]. Here we show that the restriction of flat space--time is not essential.

Let $F^{ij} = -F^{ji}$ be the components of a tensor in the U_4 manifold. Then one has:

$$F^{ij} = V^{[i;j]} + \frac{1}{\sqrt{2}} \mu^{ijkm} W_{[k;m]},$$
$$H^{ij} = \frac{1}{\sqrt{2}} \mu^{ijkm} F_{km} = W^{[i;j]} + \frac{1}{\sqrt{2}} \mu^{ijkm} V_{[k;m]}. \tag{A.1}$$

Proof: If (A.1) hold, then the following relations are fulfilled:

$$F^{ij}{}_{;j} = V^{[i;j]}{}_{j} + \frac{1}{\sqrt{2}} \mu^{ijkm} W_{[k;m]j} ,$$
$$H^{ij}{}_{;j} = W^{[i;j]}{}_{j} + \frac{1}{\sqrt{2}} \mu^{ijkm} V_{[k;m]j} . \tag{A.2}$$

This is a system of eight second order linear hyperbolic equations. To solve this for V^i and W^i one has to have the relevant initial data on a Cauchy surface $\varphi=0$. Two equations in (A.2)-namely $F^{0j}{}_{;j}=\ldots$ and $H^{0j}{}_{;j}=\ldots$ - represent constraints on the initial data, because these contain no second derivatives of time. Let $V^{\alpha}|_{\varphi}(\alpha = 1,2,3)$, $W^{\alpha}|_{\varphi}$, $V^{\alpha,i}\, n_j|_{\varphi}$ ard $W^{\alpha,i}\, n_i|_{\varphi}$ be given, where n^i are the components of normal vector of $\varphi=0$ (without lose of generality these initial data can be vanishing). To have unambiguous solutions for V^i and W^i we still need two other restrictions (for example $V^i{}_{;i}=W^i{}_{;i}=0$). In fact we have a system highly similar to Einstein equations. This completes the proof.

In the choice of V^i and W^i we have the following gauge freedom:

$$\bar{V}^i = V^i + A^{;i} + B^i, \qquad \bar{W}^i = W^i + C^{;i} + D^i, \qquad B^i{}_{;i} = D^i{}_{;i} = 0 \tag{A.3}$$

where

$$-B^{i} = M^{i} + Q^{i}, \qquad -D^{i} = N^{i} + R^{i},$$

$$A^{;[ij]} = M^{[i;j]} + \frac{1}{\sqrt{2}} \mu^{ijkm} N_{[k;m]}, \qquad M^{i}{}_{;i} = N^{i}{}_{;i} = 0 ,$$

$$B^{;[ij]} = R^{[i;j]} + \frac{1}{\sqrt{2}} \mu^{ijkm} Q_{[k;m]}, \qquad R^{i}{}_{;} = Q^{i}{}_{;i} = 0. \qquad \text{(A.4)}$$

APPENDIX B. LOCAL LORENTZ-ROTATIONS

Let $e^{i}_{(a)}$ be the components of a-th tetrad vector. (Tetrad indices are in brackett.) Components of connection with respect to this tetrad basis are given by

$$\bar{\Gamma}_{(a)(b)(c)} = -\gamma_{(a)(b)(c)} + \frac{f}{2} K_{(a)(b)(c)}, \qquad \text{(B.1)}$$

where

$$\gamma_{(a)(b)(c)} = \frac{1}{2}(\lambda_{(a)(b)(c)} - \lambda_{(b)(a)(c)} - \lambda_{(c)(a)(b)},$$

$$\lambda_{(a)(b)(c)} = e_{(a)[i'j]} e^{i}_{(b)} e^{j}_{(c)} \qquad \text{(B.2)}$$

are well-known quantities. Now we introduce a new tetrad basis by local Lorentz rotation

$$\bar{e}^{i}_{(a)} = \Lambda^{(b)}{}_{(a)} e^{i}_{(b)}, \qquad \eta_{(a)(b)} = \Lambda^{(c)}{}_{(a)} \Lambda^{(d)}{}_{(b)} \eta_{(c)(d)} . \qquad \text{(B.3)}$$

In this basis the connection is given by

$$\bar{\bar{\Gamma}}_{(a)(b)(c)} = -\bar{\gamma}_{(a)(b)(c)} + \frac{f}{2} \bar{K}_{(a)(b)(c)} =$$

$$= (-\gamma_{(d)(g)(h)} + \frac{f}{2} K_{(d)(g)(h)}) \Lambda^{(d)}{}_{(a)} \Lambda^{(g)}{}_{(b)} \Lambda^{(h)}{}_{(c)}$$

$$- \Lambda_{(d)(a)'(g)} \Lambda^{(d)}{}_{(b)} \Lambda^{(g)}{}_{(c)} . \qquad \text{(B.4)}$$

If specially the restrictions a., b., c. of Chapter 2. are fulfilled, then $K_{(a)(b)(c)}$ are infinitesimal and $e^{i}_{(a)} = \delta^{i}_{a}$ may be chosen. After the local infinitesimal Lorentz rotations

$$\Lambda^{(a)}{}_{(b)} = \delta^a_b + \omega^{(a)}{}_{(b)}, \quad \omega^{(a)(b)} = -\omega^{(b)(a)}, \tag{B.5}$$

where $\omega^{(a)(b)}$ are infinitesimally small, one obtains

$$\bar{\bar{\Gamma}}_{(a)(b)(c)} = \frac{f}{2} K_{(a)(b)(c)} + \omega_{(a)(b)',(c)} . \tag{B.6}$$

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