PPPL-2097

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163 / 5-17 -84 85 (1)

PPPL--2097

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DR#0014-0

J-14681

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APRIL 1984

MASTER

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PRINCETON, NEW JERSEY

PREPARED FOR THE U.S. DEPARTMENT OF ENERGY,
UNDER CONTRACT DE-AC02-76-CHO-3073.

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ABSTRACT

The finite Larmor radius modification of the Suydam criterion involves a competition between stabilizing finite Larmor radius effects and destabilizing curvature. In the case of the toroidal calculation, corresponding to the Mercier criterion, ballooning effects from regions of unfavorable curvature must be taken into account. In the case of a model equilibrium, valid near the magnetic axis, a complete solution is obtained. Results indicate that the amount of finite Larmor radius stabilization needed to overcome the effects of unfavorable average curvature increases as a function of the toroidal ballooning parameter.

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I. INTRODUCTION

The Suydam criterion [1] governs the ideal MHD stability of flutelike modes which are highly localized in the vicinity of a rational surface. It depends on a competition between the stabilizing influence of shear and the destabilizing effect of unfavorable curvature. Kulsrud [2] and Stringer [3] have shown that finite Larmor radius (FLR) effects play an important role within an inner layer close to the rational surface and have a stabilizing influence on the ideal MHD modes. Their procedure involves a matching of an "inner" solution involving FLR radius terms to an "outer" ideal marginally stable solution with both solutions dependent on the parameter governing the Suydam criterion.

There is an analog of the Suydam criterion in an axisymmetric torus - namely the Mercier criterion [4]. This again involves modes which are localized about a rational surface but now are not exactly flutelike. In fact, they have a weak ballooning component which couples to the modulated toroidal curvature to produce an effective average curvature. The Mercier criterion involves a competition between the stabilizing effect of shear and this destabilizing unfavorable average curvature.

In this paper we investigate the effects of FLR on the Mercier criterion. Following the procedure of Refs. 2 and 3, we also consider an inner region where FLR effects are significant and an outer ideal region. By asymptotically matching the solutions from each region, a stability criterion is obtained. As before, the inner region involves only the average curvature but, as we shall see, the outer ideal region is additionally influenced by the presence of stronger ballooning effects not contained in the average curvature. This becomes more transparent if we use the ballooning transformation [5] which has the consequence that, in ballooning space, we are required to match "inner" ideal solutions to "outer" FLR modified solutions. In general, such a calculation will involve the numerical solution of the marginally stable ideal MHD ballooning equation. The asymptotic behavior of this solution can then be matched to that analytic solution of the outer FLR modified equation which is well behaved at infinity, leading to the stability criterion. Of course, we know [5] that the asymptotic form of the ideal marginal equation is a combination of two powerlike solutions with the value of each power dependent on the Mercier stability parameter. Indeed we can deduce the Mercier stability criterion itself merely from the requirement that neither of these solutions oscillate [5]. In the present calculation, however, we need to know the ratio of these two solutions in the asymptotic behavior. This requires numerical solution of the ideal ballooning equation with appropriate boundary conditions at the origin. Hence, ballooning effects will now modify the stability criterion.

Although we could provide a general formulation of this problem which would cover an arbitrary axisymmetric toroidal geometry, it is more interesting to consider an explicit example where the ideal ballooning equation possesses an analytic solution. Such a solution is possible for model equilibria [6-8] corresponding to a toroidal plasma with large aspect ratio, high poloidal beta, and low shear. In the following section we obtain an analytic stability criterion for this case.

II. THE FINITE LARMOR RADIUS MODIFIED EIGENVALUE EQUATION AND ITS SOLUTION

In ballooning space the eigenvalue equation describing the effects of FLR is obtained by the replacement $\omega^2 \to \omega(\omega - \omega_{*i})$ in the ideal MHD ballooning equation, where ω is the mode frequency and ω_{*i} is the ion diamagnetic drift frequency [9]. The ideal MHD ballooning equation is a second order differential equation defined on a poloidal anglelike coordinate θ where $-\infty < \theta < \infty$ rather than $-\pi < \theta < \pi$. The equation contains periodic coefficients which reflect the 2π periodicity of the equilibrium, but also contains secular terms arising from the presence of shear. Such an equation, in general, would require numerical solution. However, if one assumes low shear, it is then possible to define two length scales, one associated with the period of the equilibrium and a longer one associated with the shear through the secular terms. An averaging procedure over the shorter periodic scale can then be applied to obtain an equation containing only the longer secular scale, the properties of the periodic coefficients being subsumed into numerical coefficients in this equation. This technique has been applied to a high-β_p equilibrium (where β_p is the poloidal beta) in the vicinity of the magnetic axis [6–8].

We consider a model equation with the structure

$$\frac{d}{du}(1 + u^2)\frac{dF}{du} + \left[\frac{\lambda^2}{1+u^2} - \delta + \Omega(\Omega - \Omega_{*i})(1 + u^2)\right]F = 0 \qquad (1)$$

where we have introduced a stretched variable $u = s\theta$ defined in terms of the shear $s = (rdq/qdr)$, Ω and Ω_{*i} are the mode and ion diamagnetic frequencies normalized to the poloidal Alfvén frequency, and λ^2 is a parameter representing ballooning effects. Specifically, Ref. 8 suggests

$$\lambda^2 = \frac{\alpha^2}{s} - \frac{3}{32}\left(\frac{\alpha^2}{s}\right)^2 \qquad (2)$$

where the pressure parameter

$$\alpha = -\frac{2R}{B_0^2}\frac{dp}{dr}q^2$$

with R being the major radius, P the pressure, and B_0 the toroidal magnetic field. Unfavorable average curvature is represented by δ where, for example [7-10],

$$\delta = \frac{r\alpha}{Rs^2}\left(1 - \frac{1}{q^2}\right) + \frac{3}{128}\left(\frac{\alpha^2}{s}\right)^2 \tag{3}$$

contains a destabilizing term when $q < 1$ and a stabilizing term at finite pressure. When $\lambda^2 = 0$, Eq. (1) can be recognized as the Fourier transform of the eigenvalue equation in Refs. 2 and 3.

The Mercier criterion is associated with the solutions of this equation as $u \to \infty$ when $\Omega = 0^{(5)}$. The instability condition is that the asymptotic solutions

$$F \sim u^r \quad , \quad r_{\pm} = -\frac{1}{2} \pm \left(\frac{1}{4} + \delta\right)^{1/2} \tag{4}$$

oscillate; i.e.,

$$1 + 4\delta < 0 \quad . \tag{5}$$

When the plasma is stable according to the Mercier criterion, i.e., $\delta > -1/4$, there remains the possibility of ballooning instability. The marginal value of λ for such instability corresponds to that solution of Eq. (1) with $\Omega = 0$, which behaves as u^{r_-} as $u \to \infty$ (i.e., the "small" solution). In this case Eq. (1) has an exact solution of the form

$$F \sim \left(1 + u^2\right)^{-\lambda/2} \tag{6}$$

with

$$\lambda = \frac{1}{2}\left[1 + (1 + 4\delta)^{1/2}\right] \tag{7}$$

i.e., $r_- = -\lambda$. Thus, at the Mercier marginal point, ballooning instability exists when $\lambda > 1/2$.

We cannot solve Eq. (1) exactly, but if we consider, as in earlier studies [2,3], $\delta >> \Omega(\Omega - \Omega_{*i})$, then we can provide an eigenvalue condition by asymptotic matching of the solutions of simpler equations which are valid when $u^2 \sim 1$ and when

$$u^2 \sim \frac{\delta}{\Omega(\Omega-\Omega_{*i})} >> 1 \quad .$$

Thus, in the region $u^2 \sim 1$ we obtain the ideal equation

$$\frac{d}{du}\left[(1 + u^2)\frac{dF_0}{du}\right] + \left(\frac{\lambda^2}{1+u^2} - \delta\right)F_0 = 0 \tag{8}$$

and in the region

$$u^2 \sim \frac{\delta}{\Omega(\Omega-\Omega_{*i})} >> 1 \quad ,$$

the FLR modified equation

$$\frac{d}{du}\left(u^2\frac{dF_1}{du}\right) + \left[-\delta + \Omega(\Omega - \Omega_{*i})u^2\right]F_1 = 0 \quad . \tag{9}$$

The solution of Eq. (8) can be expressed in terms of Associated Legendre functions of imaginary argument. With the even boundary condition $dF_0/du = 0$

at u = 0,

$$F_0 = \frac{\pi}{2}\exp(i\pi\lambda)\left[i - \cot\frac{\pi}{2}\left(\nu + \lambda - \frac{1}{2}\right)\right]P^{\lambda}_{\nu-1/2}(iu) + \exp(i\pi\lambda)\,Q^{\lambda}_{\nu-1/2}(iu) \qquad (10)$$

where $\nu = 1/2\ (1 + 4\delta)^{1/2}$.

With regard to Eq. (9) we note that $\Omega(\Omega - \Omega_{*i}) = -(\Omega^2_{*i}/4)$ at marginal stability and remains negative definite even for a band of stable situations. We can therefore write

$$\Omega(\Omega - \Omega_{*i}) = \gamma^2 \quad . \qquad (11)$$

The solution of Eq. (9) can be obtained in terms of Bessel functions of imaginary argument [11]. With the boundary condition $F_1 \to 0$ as $u \to \infty$ we find the solution

$$F_1 = cu^{1/2}K_\nu(\gamma u) \quad . \qquad (12)$$

It is interesting to note that when $\lambda^2 = 0$, the solutions given in Eqs. (10) and (12) correspond to an inversion of the ideal and FLR regimes of the calculation in Refs. 2 and 3.

The eigenvalue condition follows from matching the solutions given in Eqs. (10) and (12) in their region of common validity, $1 \ll u^2 \ll \delta/\gamma^2$. Using the small argument limit of the Bessel functions, K_ν, we find that[11]

$$F^1 \sim c_1 u^{-\nu-1/2} + c_2 u^{\nu-1/2} \qquad (13)$$

where

$$\frac{c_1}{c_2} = -\left(\frac{2}{\gamma}\right)^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \quad , \tag{14}$$

and the asymptotic forms for the Associated Legendre functions lead to the behavior [11]

$$F_0 \sim c_1' u^{-\nu-1/2} + c_2' u^{\nu-1/2} \tag{15}$$

where

$$\frac{c_1'}{c_2'} = -2(2i)^{-2\nu} \frac{\Gamma[(1/2)+\lambda+\nu]\Gamma[(1/2)-\lambda+\nu]}{\nu\Gamma^2(\nu)} \left(\frac{\exp(-i\pi\lambda)}{1-\cot(\pi/2)[\lambda+\nu-(1/2)]}\right.$$

$$\left. - \frac{\cos \pi(\lambda+\nu)}{2 \sin \pi\nu}\right) \quad . \tag{16}$$

The matching condition yields the eigenvalue equation

$$\frac{c_1}{c_2} = \frac{c_1'}{c_2'} \quad . \tag{17}$$

Note that the marginal stability condition for ballooning modes given by Eq. (7) corresponds to the pole, $\lambda = (1/2) + \nu$, of the Γ function. Since we are interested in Mercier unstable situations, $1 + 4\delta < 0$, we may write $\nu = i\sigma/2$ with σ real. After some manipulations Eq. (17) can be written as

$$\gamma_n = 16 \exp\left\{\frac{2}{\sigma}\left[2 \arg \Gamma\left(1 + \frac{i\sigma}{2}\right) + \arg \Gamma\left(1 - \lambda + \frac{i\sigma}{2}\right) - \arg \Gamma(1 - 2\lambda + i\sigma)\right.\right.$$

$$\left.\left. + \frac{1}{2} \tan^{-1} \frac{\sinh(\sigma\pi/2)}{\cos \pi\lambda} - \frac{1}{2} \tan^{-1}\left(\tan\pi\lambda \tanh \frac{\sigma\pi}{2}\right) - n\pi\right]\right\} \equiv G_n(\sigma,\lambda) \quad . \tag{18}$$

The ambiguity in n may be removed by considering the Mercier marginal point, $\sigma = \lambda = 0$, where it can be shown [2] that $\gamma_n \to 0$ (i.e., no FLR stabilization is necessary). The most unstable choice consistent with $\lambda \to 0$ is $n = 1$. Since $\gamma_n = |\Omega_{*i}/2|$ at marginal stability, the stability criterion is

$$\left|\frac{\Omega_{*i}}{2}\right| = G_1(\sigma,\gamma) \quad . \tag{19}$$

Figure 1 shows plots of $G_1(\sigma,\lambda)$ against σ for a series of values of λ. Note here that $G_1(\sigma,0)$ reduces to $H_1(-4\delta)$ of Ref. 3. The validity of the asymptotic matching technique required $4\delta/\gamma^2 \gg 1$; i.e., $1 + \sigma^2 > \gamma^2$. Although there is no small parameter to ensure this condition, inspection of Fig. 1 indicates that it is reasonably well satisfied numerically.

We see from Fig. 1 that ballooning effects, which are proportional to λ, increase the amount of FLR stabilization required for a given unfavorable average curvature. The intercept at $\lambda = 1/2$ when $\sigma = 0$ corresponds to the presence of ideal ballooning instability even when there is no Mercier instability. The stabilizing influence of FLR on these ballooning modes has been reported in earlier work [12].

III. DISCUSSION AND CONCLUSIONS

We have considered the stabilization due to diamagnetic drift effects of a Mercier unstable toroidal equilibrium. To illustrate the salient features we have concentrated on a simple model equilibrium which exhibits unfavorable average curvature and, of course, regions of local unfavorable curvature leading to ballooning effects. The results generalize to a torus the cylindrical calculations of Kulsrud [2] and Stringer [3] at the expense of adding an additional parameter associated with ballooning. Clearly, Fig. 1

implies a greater degree of FLR stabilization is necessary in the torus to overcome the effects of unfavorable average curvature. In fact, even when the equilibrium is marginally Mercier unstable, FLR is necessary to stabilize the usual ballooning instability [12].

In terms of physical variables the stability criterion given by Eq. (19) expressed as

$$\ell > \frac{r_n r}{a_i R q^2} \left(\frac{2}{\beta}\right)^{1/2} G_1(\delta, \sigma) \tag{20}$$

where ℓ is the toroidal mode number, r_n is the density scale length, $a_i = (m_i T)^{1/2}/eB$ is the ion Larmor radius, and $\beta = 2P/B^2$ is the usual plasma pressure parameter.

Since a particular model equilibrium was employed in deriving Eq. (20), this analytic form for the FLR modified Mercier criterion should only be used to establish general qualitative trends for realistic situations. However, the techniques described in this paper can be applied in part to an arbitrary equilibrium to obtain more precise results. Specifically, in the asymptotic FLR region one can always obtain an averaged equation analogous to Eq. (9) based on the two scales available: the short periodic scale of the equilibrium and the long secular scale associated with shear in the asymptotic region [5]. An analytic solution of this equation, analogous to Eq. (12), must then be matched onto a solution of the marginal ideal ballooning equation. It should be noted that it will not usually be possible to produce an averaged equation corresponding to Eq. (8). This was feasible in the present study because of the small shear in the model equilibrium considered. For the more general cases it will be necessary to determine the ratio, $c_1^\prime/c_2^\prime$, from numerical solutions to the ideal ballooning equation which are in turn matched to an analytic form corresponding to Eq. (14).

In conclusion, we have taken into account the stabilizing finite gyroradius contribution from the ion diamagnetic drifts together with destabilizing ballooning effects in deriving a FLR modified Mercier criterion. An analytic form of this result is presented for a model equilibrium, and the numerical procedure necessary to generalize to more realistic equilibria is described.

ACKNOWLEDGMENT

This work was supported by United States Department of Energy Contract No. DE-AC02-76-CHO-3073.

FIGURE CAPTION

FIG. 1. Influence of the ballooning parameter, λ, on the degree of FLR stabilization, represented by γ, on the Mercier instability, characterized by σ. The $\lambda = 0$ curve reduces to the previous result of Ref. 3, and the general expression for γ is specified by Eqs. (18) and (19).

REFERENCES

[1] SUYDAM, B. R., in Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy (United Nations, Geneva, 1958) Vol. 31, p. 157.

[2] KULSRUD, R. M., Phys. Fluids 6 (1963) 904.

[3] STRINGER, T. E., Nucl. Fusion 15 (1975) 125.

[4] MERCIER, C., Nucl. Fusion Suppl. Pt 2 (1962) 801.

[5] CONNOR, J. W., HASTIE, R. J., TAYLOR, J. B., Proc. Roy. Soc. (London A365 (1979) 1.

[6] LORTZ, D., NUHRENBURG, J., Nucl. Fusion 19 (1979) 1207.

[7] STRAUSS, H., Phys. Fluids 24 (1981) 2004.

[8] COPPI, B., FERREIRA, A., RAMOS, J. J., Phys. Rev. Lett. 44 (1980) 990.

[9] TANG, W. M., CONNOR, J. W., HASTIE, R. J., Nucl. Fusion 20 (1980) 1439.

[10] MIKHAILOVSKII, A. B., Nucl. Fusion 14 (1974) 483.

[11] ERDELYI, A., Editor, Higher Transcendental Functions, McGraw-Hill, New York (1953) Vol. 1, p. 120 and Vol. 2, p. 1.

[12] TANG, W. M., CONNOR, J. W., WHITE, R. B., Nucl. Fusion 21 (1981) 891.

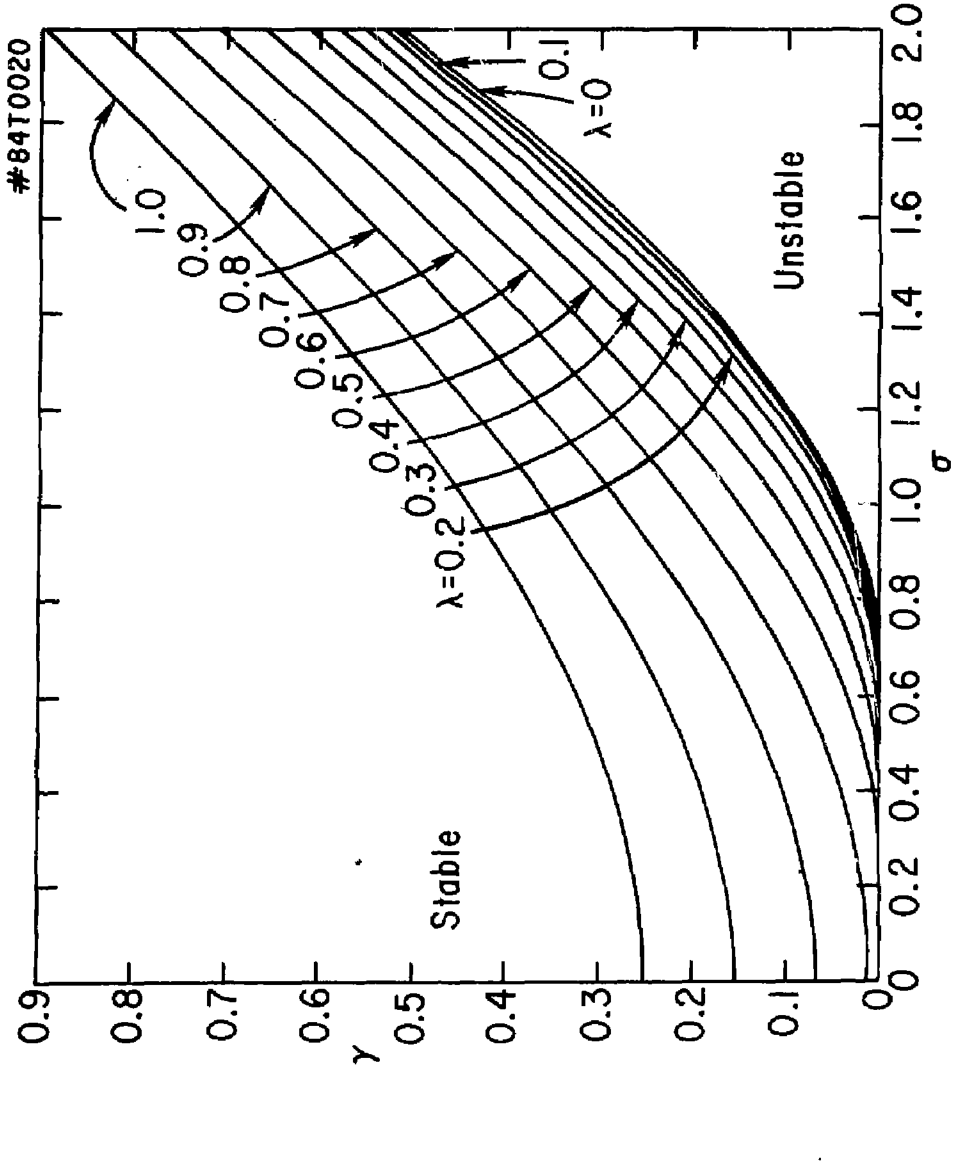
#84T0020
Stable
Unstable
λ=0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0.1
λ=0
γ
σ
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0

Fig. 1

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