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UNCERTAINTY COVARIANCES IN ROBOTICS APPLICATIONS

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ABSTRACT

The application of uncertainty covariance matrices in the analysis of robot trajectory errors is explored. First, relevant statistical concepts are reviewed briefly. Then, a simple, hypothetical robot model is considered to illustrate methods for error propagation and performance test data evaluation. The importance of including error correlations is emphasized.

INTRODUCTION

In developing technologies there is an ever-present need for better reliability. Safety and economics are the principal motivating factors in the nuclear industry. The science of Statistics provides elegant, rigorous methods for addressing this issue, but they are inadequately exploited in many scientific and engineering applications. Scientists and engineers tend to be poorly versed in Statistics. Consequently, they are prone to either avoid error considerations or to devote only modest attention to the issue. Some common rules are frequently employed in estimating errors, but these are sometimes misused. Lately, workers in the nuclear data and reactor research fields have begun to appreciate the worth of comprehensive error assessment [1,2]. The literature on robotics does not seem to reflect as much concern for this subject even though the related topic of control methodology is widely researched. Robotics is a discipline which is well suited for the application of statistical methods. The fundamental physical processes are surely well understood, and testing under various conditions is usually possible, free from arbitrary constraints such as, e.g., the severe restrictions governing nuclear reactor operations.

This presentation aims to acquaint the reader with certain ideas which are relevant to the consideration of errors in robotics. Therefore the discussion emphasizes concepts, simplified to encourage their assimilation, instead of dealing with specific robotic applications. The first section contains a brief outline of some basic statistical facts. Next, an hypothetical robot device is described. This instrument is sufficiently complex to illustrate several technical features, yet it is still simple enough so that understanding it requires no special knowledge. The robot is used to examine useful procedures such as monte-carlo simulation, error propagation and performance data evaluation. Since the formalism incorporates complete covariance matrices rather than just the diagonal errors, the crucial but often overlooked consequences of error correlations evolve in a natural way.

MATHEMATICAL CONCEPTS

The literature on Statistics and applied statistical methods is very extensive. Many textbooks treat the basic concepts adequately for the present purposes (e.g., Ref. 3 to 5). The challenge is to properly use the theory in a practical application such as robotics. A brief review of some important terms and ideas is in order:

Probability

Probability theory is the foundation of Statistics. Mathematicians prefer to commence from a set of axioms, but the more intuitive frequency definition is useful here. A sampling space (E) is the entirety of possible outcomes for sequential experiments or trials. Various subset spaces (A_i) can be defined. The probability of E, namely P(E), is unity while other probabilities $P(A_i)$ are ≤ 1. For a large number of trials n, $P(A_i)$ is approximated by the ratio n_i/n, where n_i is the number of trials where A_i occurs. Unless n approaches infinity, $P(A_i)$

is ambiguously defined, a fact which explains the mathematicians' preference for axiomatic approaches.

$P(A_j|A_i)$ is the conditional probability of A_j given that A_i has occurred. This is defined as $P(A_iA_j)/P(A_i)$, where A_iA_j indicates occurrence of both A_i and A_j. A_i and A_j are said to be independent if $P(A_j|A_i) = P(A_j)$, or equivalently if $P(A_iA_j)$ is the product $P(A_i)P(A_j)$. Suppose that all the considered A_i are exhaustive, i.e., taken together they form E, and that furthermore they are mutually exclusive. Then if B can occur only in combination with one of the A_i, Bayes theorem states that

$$P(A_i|B) = P(A_i)P(B|A_i)/\sum_j P(A_j)P(B|A_j) \quad .$$

This theorem has some important consequences for data evaluation and its interpretation is the subject of much controversy in the field of Statistics.

Random Variables and Probability Distributions

For present purposes a random variable is considered to be a physical parameter, e.g., a robot degree of freedom, which is governed by the laws of Physics and statistics. Random variables can either assume discrete or continuous (emphasized here) values. A trial is an action which produces a distinct value for the random variable. Probability density functions govern the outcome of trials; e.g., the probability $P(x<x_a)$ is given by $\int_{-\infty}^{x_a} f(x)dx$, where $f(x)$ is the probability density function for x in the sample space $-\infty < x < +\infty$. Multivariate distributions are defined similarly; e.g., the probability $P(x_1<x_{a1},\ldots,x_m<x_{am})$ is given by $\int_{-\infty}^{x_{a1}}dx_1\ldots\int_{-\infty}^{x_{am}}dx_m\ f(x_1,\ldots,x_m)$. If the x_i are independently distributed then

$$f(x_1,\ldots,x_m) = f_1(x_1)\ldots f_m(x_m) \quad .$$

Another useful fact is that functions of random variables behave as random variables.

Moments of Probability Distributions

Usually probability density functions are not explicitly known and cannot be readily deduced. However, certain useful moments can be realistically estimated from experimentation and their knowledge is often sufficient. The expectation value $\langle x\rangle$ of x, given by $\int_{-\infty}^{\infty} xf(x)dx$, is especially important. The main objective of experimentation is determination of parameter expected values. Error assessment, however, requires knowledge of higher-order moments. Define the quantity μ_ℓ given by $\int_{-\infty}^{\infty} (x-\langle x\rangle)^\ell \cdot f(x)dx$ as the ℓ^{th} moment about the mean. Since probability distributions are normalized, $\mu_0 = 1$. If $f(x)$ is symmetric about x, all odd moments vanish. The second moment μ_2 is very important. It is known as the variance in x, var(x). The square root of the variance is the standard deviation σ, commonly called the error in x. Higher-order moments provide further information about distribution shapes.

Moments are also defined for multivariate distributions. Analogous to the single-variable case, $\langle x_i\rangle$ is given by the formula $\int_{-\infty}^{\infty} dx_1\ldots\int_{-\infty}^{\infty} dx_m\ x_if(x_1,\ldots,x_m)$. From the multitude of possibilities for higher-order moments, the quantities $\int_{-\infty}^{\infty} dx_1\ldots \int_{-\infty}^{\infty} dx_m \cdot (x_i-\langle x_i\rangle)\ (x_j-\langle x_j\rangle)f(x_1,\ldots,x_m)$ are of greatest significance. When i=j, this yields the variances $var(x_i)$, while for $i\neq j$, one obtains the covariances $cov(x_i,x_j)$. The correlation coefficient $cor(x_i,x_j)$ is given by $cov(x_i,x_j)/[var(x_i)var(x_j)]^{1/2}$.

Gaussian Distributions

Many different types of distributions occur in Statistics. Often one is interested in the analysis of physical phenomena which depend in a complex way on many unexplored fundamental processes. The famous central limit theorem implies that any parameter which evolves from such complex origins tends to behave according to the Gaussian distribution, given by $f(x)=(2\pi\sigma)^{-1/2}\exp[-(x-\langle x\rangle)^2/2\sigma^2]$ with variance σ^2. Therefore, discussion will be limited to Gaussian functions here, although other distributions are not excluded a priori from robotics applications.

The Law of Error Propagation

Suppose a sequence $y_1,\ldots,y_m$ of random variables is derived from a distinct set $x_1,\ldots,x_n$ according to equations of the form $y_i=y_i(x_1,\ldots,x_n)$. Furthermore, suppose that the multivariate distribution function is not explicitly known, but that a variance-covariance matrix $\bar{C}_x$ for the x_i (designated collectively by $\vec{x}$) has been estimated. It is obviously necessary to have a method of estimating the variance-covariance (or simply "covariance") matrix $\bar{C}_y$ for $\vec{y}$. It can be shown that if (for

all y_i) the terms beyond first order in the Taylors series expansions, $y_i = y_i(\langle\vec{x}\rangle) + \sum_{j=1}^{n} (\partial y_i/\partial x_j)_{\vec{x}=\langle\vec{x}\rangle}(x_j - \langle x_j\rangle)$ + higher-order terms, are negligible, then, in matrix notion ("+" designates transpose), $\bar{C}_y = \bar{T}\,\bar{C}_x\,\bar{T}^+$. $\bar{T}$ is a matrix of partial derivatives $\partial y_i/\partial x_j$ (evaluated at the mean $\langle\vec{x}\rangle$) commonly called the transformation matrix. If m=n, and the transformation from $\vec{x}$ to $\vec{y}$ represents a change of basis in n-dimensional space, then the determinant of $\bar{T}$ is called the Jacobian of the transformation. The matrix formula for $\bar{C}_y$ in terms of $\bar{T}$ and $\bar{C}_x$ is known as the law of error propagation. It is valid when the variances of $\vec{x}$ are not too large. This rule is extremely valuable for robotics error analysis. It is emphasized that error propagation analyses must be performed utilizing complete variance-covariance information. In later sections examples will be given which demonstrate the extent of discrepancy which can arise if covariances are neglected.

Sampling and Estimators

The preceding discussion provides no clues as to how one acquires knowledge about the statistical behavior of processes of practical interest. In fact, one rarely has adequate knowledge of the requisite probability distribution functions in advance. However, one may for one reason or other have a strong basis to expect a certain form for a distribution, e.g., a Poisson distribution for radioactive decay or a Gaussian distribution for a robot degree of freedom, etc. Then, the entire distribution can be deduced provided that the essential parameters can be estimated. The process of estimating such parameters by experimentation with a finite subset of a larger (usually assumed infinite) population is known as sampling. This exercise is at the heart of applied statistics. The ensemble of random parameters resulting from a finite series of trials is called a sample. Any functional combination of the components of this sample is called a statistic. Statistics so defined are random variables. Statistics constructed to provide estimates of certain parameters are called estimators. For example, an n-fold sampling of a random variable x produces the set $\{x_1,\ldots,x_n\}$. The statistic $\bar{x}$ defined by $(\sum_{i=1}^{n} x_i)/n$ is called the sample mean, and it is an estimator of the population mean $\langle x\rangle$. An estimator is unbiased if for any sample size n, the expected value of the estimator equals the expected value of the basic parameter from the population, e.g., $\langle\bar{x}\rangle = \langle x\rangle$ for all n above. An estimator is consistent provided that its variance approaches zero in the limit of large samples. Another important estimator is $s^2 = [\sum_{i=1}^{n}(x_i - \bar{x})^2]/(n-1)$ where $\bar{x}$ is given above. This is an unbiased and consistent estimator for the population variance. If the fundamental distribution is Gaussian, knowledge of the mean and variance uniquely establishes the entire distribution.

It is desirable to have some way of deducing the quality of results estimated from a sample. Various statistical tests on samples exist for this purpose. The most widely used is the χ^2 (chi-square) test commonly associated with Gaussian distributions. If, e.g., σ^2 is the population variance and $\bar{x}$ and s^2 are defined as above, then the statistic fs^2/σ^2 follows the integrated distribution function $F(\chi^2) = (\int_0^{\chi^2} u^{\lambda-1}e^{-1/2u}du)/[2^{\lambda}\Gamma(\lambda)]$, with $\lambda = f/2$ (f denotes the degrees of freedom) and Γ being the standard gamma function found in mathematical tables. Tables of $F(\chi^2)$ are available. Essentially one evaluates this statistic and compares the result with $F(\chi^2)$ from a chi-square table for degrees of freedom f=n-1, where n is the sample size. The quantity $W = 1 - F(\chi^2)$ is a measure of confidence in the result. W near unity implies high confidence.

In addition to being unbiased and consistent, good estimators should be "sufficient". Sufficient estimators are essentially those which have minimum variance. The generation of sufficient estimators is a complex area of Statistics. One general method is the maximization of likelihood functions, but this topic will not be pursued. Suppose, however, that n experiments, each with somewhat different accuracy, yield values $x_1,\ldots,x_n$ with corresponding variances $\sigma_1^2,\ldots,\sigma_n^2$. If each measurement is governed by a Gaussian distribution, then it is possible to show that minimization of the statistic $\sum_{i=1}^{n}[(x_i - \bar{x})^2]/\sigma_i^2$ yields the maximum-likelihood estimator for $\bar{x}$. This is the basic idea of the common least-squares method. The Gauss-Markov theorem insures that of all the possible unbiased estimators, the least-squares solution has the smallest variance. In very general terms, the minimization of $[\vec{x} - \vec{G}(\vec{\theta})]^+ \bar{C}_x [\vec{x} - \vec{G}(\vec{\theta})]$ will lead to the minimum variance solution for a parameter set $\vec{\theta}$ and its covariance matrix $\bar{C}_\theta$, given a sampling $\vec{x}$ with population covariance matrix $\bar{C}_x$ and mean values $\vec{G}$. Often this leads to nonlinear problems where approximation techniques are required.

Bayes theorem is the basis for such an approximation method. One starts with the body of all a priori information, including covariances. New sampling results, with covariance information, are used in adjusting the prior knowledge by (hopefully) small amounts, thereby conveniently merging old and new information to provide a revised best estimate. Since small differences are often involved in this adjustment procedure, expansions can be used to linearize the least-squares estimator for more convenient analysis. This is discussed in a later section.

Monte-Carlo Simulation

Knowledge of the behavior of a random variable is completely embodied in its distribution function, f(x). Given f(x), statistical sampling procedures can be simulated on a computer provided that x itself can be sampled randomly. Computer algorithms which produce series of numbers R in the range (0,1) with a high degree of randomness are available. Then, e.g., if x is defined in the range (x_ℓ, x_h), random values can be generated by $x_i = x_\ell + (x_h - x_\ell)R_i$, and the expectation value of h(x) can be calculated, e.g., from the expression $(1/n)\sum_{i=1}^{n} h(x_i)f(x_i)$, when n is suitably large. There are many possible variations of this so-called monte-carlo method [6], one of which will be demonstrated for a robotics application later in this paper.

APPLICATION OF THE CONCEPTS TO AN HYPOTHETICAL ROBOT

Several of the mathematical concepts outlined in the preceding section will be illustrated in a robotics context in the present section.

Properties of the Robot

Consider the simple robotic device pictured in Fig. 1. Five degrees of freedom, which can be represented by the internal parameters of the robot $\vec{a}=(a_1,\ldots,a_5)$, define the spatial motion of the device. The tip of the robot arm at P possesses no explicit manipulation device. Therefore, it is assumed that the robot functions as a stylus, applicator or transporter of a very small quantity of material (such as a liquid drop). The performance of the robot is judged by the accuracy with which it follows a programmed path and delivers the pointer to a designated position in space, designated by Cartesian coordinates $\vec{b}=(b_1,b_2,b_3)$.

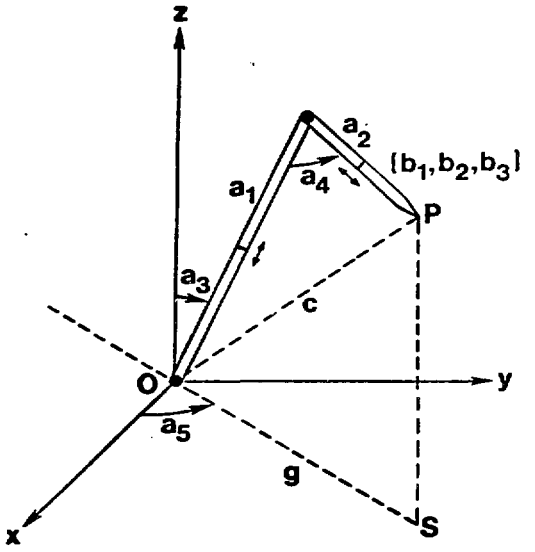

FIGURE 1: SCHEMATIC DRAWING OF AN HYPOTHETICAL ROBOT USED TO ILLUSTRATE SOME CONCEPTS OF UNCERTAINTY ANALYSIS.

Assume that each a_i is actuated and controlled independently of the other a-parameters. The time evolution $a_i(t)$ from t_o to t_f, for each of the five parameters defining the robot, is governed by mechanical equations of motion. Furthermore, it is assumed that sensors provide feedback to the coordinate actuators so that forces and torques are applied as needed to maintain the evolution of each coordinate $a_i(t)$ "reasonably" close to a programmed path defined by $a_{io}(t)$. Discussion of the methods for achieving this goal is beyond the scope of this paper. It is assumed that each $a_i(t)$ is normally (Gaussian) distributed about the design expectation $a_{io}(t)$, thus $p_i(a_i) = (2\pi\sigma_{ai})^{-1/2} \exp[-(a_i - a_{io})^2/2\sigma_{ai}^2]$ with $\sigma_{ai} = \sigma_{ai}(t)$ denoting the square root of the variance for a_i corresponding to the particular design-path point indicated by the time parameter t. The uncertainty in the status of a robot parameter at some point (time t) along the path is probably very likely to behave in a Gaussian manner because the current position is the result of numerous corrections initiated by the control circuitry as well as by perturbations due to play in linkages, etc. These are exactly the conditions under which the central limit theorem is expected to be applicable, thereby indicating near normal behavior

for the a_i. The performance measured in Cartesian space is of greatest interest. A relationship exists between the b_j and the a_i, i.e., $b_j=b_j(a_1,\ldots,a_5)$, or $\vec{b}=\vec{b}(\vec{a})$. In particular, from Fig. 1,

$$b_1 = g\cos a_5 \quad ,$$
$$b_2 = g\sin a_5 \quad ,$$
$$b_3 = a_1\cos a_3 - a_2\cos(a_4-a_5)$$
$$g = a_1\sin a_3 + a_2\sin(a_4-a_3) \quad .$$

To simplify the analysis, it is assumed that all standard deviations for the a_i are time independent. They are arbitrarily assumed to be: σ_{a1}= 0.5 cm, σ_{a2}=0.3 cm, σ_{a3}=1°, σ_{a4} = 1.5° and σ_{a5}=0.5°. These values are sufficiently small so that the law of error propagation defined previously is applicable for present purposes. In practice, the parameter variances for an actual robot would probably be relatively much smaller. Specific details of the programmed paths $a_{oi}(t)$ are of little concern for demonstration purposes, therefore it is assumed that

$$a_{io}(t) = a_{ioo} + (a_{iof}-a_{ioo})\,(t-t_o)/(t_f-t_o),$$

indicating linear progression from initial positions a_{ioo} to final positions a_{iof} as t advances from t_o to t_f. Absolute time is also of little importance to the analysis so the dimensionless variable $T=(t-t_o)/(t_f-t_o)$ is substituted to yield

$$a_{io}(T) = a_{ioo} + (a_{iof}-a_{ioo})T \quad ,$$

with $0 \leq T \leq 1$. Values for a_{ioo} and a_{iof} which are employed in this analysis appear in Table 1.

Propagation of Errors to the Work Space Reference Frame

The programmed workspace path $b_o(T)$ is readily evaluated using the preceding formulas. Table 2 exhibits representative values for the b_{io} as well as for two other derived quantities, $c_o = (b_{1o}^2 + b_{2o}^2 + b_{3o}^2)^{1/2}$ and $g_o = (b_{1o}^2 + b_{2o}^2)^{1/2}$.

The covariance matrix for $\vec{b}_o$ is derived from an application of the law of error propagation, embodied in the matrix formula

$$\bar{C}_b = \bar{T}_{ab}\bar{C}_a\bar{T}_{ab}^{+},$$

where the transformation matrices $\bar{T}_{ab}$ contain elements $\partial b_i/\partial a_i$ evaluated along the programmed path

TABLE 1

PRIMARY ROBOT PARAMETERS

i	a_{ioo}	a_{iof}
1	75 cm	127 cm
2	30 cm	64 cm
3	5°	37°
4	105°	169°
5	25°	71°

TABLE 2

DERIVED ROBOT PARAMETERS

T	b_{1o}[a]	b_{2o}[a]	b_{3o}[a]	c_o[a]	g_o[a]
0	32.70	15.25	79.92	87.69	36.08
0.1	38.22	21.71	87.01	97.48	43.96
0.2	43.16	29.33	94.11	107.61	52.18
0.3	47.31	38.04	101.16	117.98	60.71
0.4	50.48	47.74	108.11	128.51	69.48
0.5	52.49	58.29	114.90	139.12	78.44
0.6	53.16	69.53	121.46	149.71	87.53
0.7	52.38	81.28	127.73	160.20	96.69
0.8	50.03	93.30	133.66	170.51	105.86
0.9	46.03	105.37	139.92	180.54	114.98
1	40.37	117.24	144.25	190.22	123.99

[a] Dimensions in cm.

$\vec{a}_o$. Here $\bar{C}_a$ is a diagonal, time-independent matrix since the a_i are assumed to be independent with fixed variances σ_{ai}. Similarly,

$$\sigma_c^2 = \bar{T}_{bc}\,\bar{C}_b\,\bar{T}_{bc}^{+} \quad ,$$

$$\sigma_g^2 = \bar{T}_{bg}\,\bar{C}_b\,\bar{T}_{bg}^{+} \quad ,$$

where $\bar{T}_{bc}$ has elements $\partial c/\partial b_i$ while $\bar{T}_{bg}$ has elements $\partial g/\partial b_i$. It is instructive to write the formula for σ_c^2 in detail, namely

$$\sigma_c^2 = \sum_{i=1}^{3}(\partial c/\partial b_i)^2\sigma_{bi}^2 + \sum_{i\neq j}^{3}(\partial c/\partial b_i)(\partial c/\partial b_j)\sigma_{bi}\sigma_{bj}\mathrm{cor}(b_i,b_j).$$

The first sum represents the uncorrelated contribution while the second sum contains terms which exist whenever $\bar{C}_b$ is not diagonal. Neglect of these terms can lead to a serious error in obtaining the variance σ_c^2 for c (or for g as well). The analysis represented by these formulas is accomplished using a computer. Results for representative points along the robot path (various T) appear in Table 3.

TABLE 3

UNCERTAINTIES IN THE COORDINATES OF P

T	σ^a_{b1}	σ^a_{b2}	σ^a_{b3}	cor(b_1,b_2)	cor(b_1,b_3)	cor(b_2,b_3)
0	1.31	0.67	1.12	0.86	-0.59	-0.54
0.1	1.37	0.84	1.25	0.85	-0.65	-0.60
0.2	1.42	1.02	1.39	0.84	-0.70	-0.66
0.3	1.46	1.22	1.53	0.84	-0.74	-0.72
0.4	1.49	1.42	1.68	0.83	-0.77	-0.76
0.5	1.50	1.63	1.83	0.81	-0.79	-0.81
0.6	1.49	1.84	1.97	0.78	-0.80	-0.84
0.7	1.47	2.05	2.12	0.75	-0.79	-0.87
0.8	1.44	2.26	2.26	0.70	-0.76	-0.90
0.9	1.40	2.46	2.40	0.64	-0.71	-0.92
1	1.37	2.65	2.54	0.56	-0.63	-0.94

[a]Dimensions in cm.

Clearly the errors of $\vec{b}$ are substantially correlated even though the errors of $\vec{a}$ are not. These correlations are obviously of geometric origin. Their existence may surprise the reader who will, no doubt, recognize that $\vec{b}$ is an orthogonal representation for the robot workspace! More will be said on this point later in the paper. All the cor(b_1,b_2) values are positive. This means that b_1 and b_2 tend to "wander" together (toward larger or smaller values). On the other hand cor(b_1,b_3) and cor(b_2,b_3) are negative, indicating opposite wandering of these respective parameters (anti-correlation).

The effect of the correlations (or anticorrelations) on other derived parameters, e.g. on c and g, is governed by magnitudes and signs of the corresponding transformation matrix elements. This is demonstrated in Table 4. Neglect of correlations leads to overestimation of σ_c while σ_g is analogously underestimated.

TABLE 4

UNCERTAINTIES IN c AND g

T	σ^a_c		σ^a_g	
0	0.82	(1.13)	1.44	(1.22)
0.1	0.84	(1.25)	1.56	(1.26)
0.2	0.85	(1.37)	1.69	(1.31)
0.3	0.85	(1.49)	1.83	(1.37)
0.4	0.84	(1.62)	1.96	(1.46)
0.5	0.82	(1.75)	2.10	(1.57)
0.6	0.79	(1.89)	2.24	(1.72)
0.7	0.75	(2.04)	2.38	(1.90)
0.8	0.71	(2.20)	2.51	(2.10)
0.9	0.66	(2.37)	2.65	(2.32)
1	0.62	(2.54)	2.78	(2.55)

[a]Dimensions in cm. Values in parentheses result when off-diagonal elements in C_b are ignored.

Generation of Probability Profiles

The preceding examination of the robot's workspace path and associated errors is useful but it may not reveal quite as much as might be desired. In particular, a "feel" for the "shape" of the region of uncertainty associated with the trajectory $\vec{b}_o(T)$ is missing. However, if the variances for the primary $a_i(T)$ are reasonably well known, and if the fundamental probability distributions about $\vec{a}_o(T)$ are considered to be known, then monte-carlo simulation can be used to construct probability profiles for any number of desired spatial "sections". Such analyses may prove to be extremely valuable in robotics applications for they would allow determination of probabilities for catastrophic scenarios, e.g., collision of the robot with a barrier. Surely, one does not wish to exercise the possibilities offered by this method everywhere along the path (all T). Examination at a few key positions (selected T) should be sufficient.

For demonstration purposes the region of workspace path near T = 0.5 is selected here. Using a computer with a random number generating routine, a sample of 400 vectors $\vec{a}_k$ was generated by sampling normal distributions about $\vec{a}_o$. A corresponding set of vectors $\vec{b}_k$ was then generated and difference vectors $\Delta\vec{b}_k = \vec{b}_k - \vec{b}_o$ were deduced. In summary, each monte-carlo trial produced a triad of values, (Δb_{1k},Δb_{2k},Δb_{3k}), and these were plotted by computer, pairwise. The results appear in Fig. 2. This simulation provides an extremely good picture of what could be expected from several attempts to position the robot pointer at this specific point on the path (assuming that relative time T is without error). The density of points is greatest near the programmed path and tapers off away from it. The distribution is elongated (like a cigar) and tilted. This tilt is a characteristic of correlated variables. Notice in Fig. 2 (a) that most points fall in the first and third quadrants. This corresponds to the positive correlation between b_1 and b_2. In contrast, Figs. 2 (b) and (c) show points primarily in the second and fourth quadrants. This is characteristic of anticorrelation, an effect known to apply for b_1 and b_3 and for b_2 and b_3 (Table 3). If one were to represent a critical barrier for the

robot in terms of a forbidden region projected onto the section coordinates of Fig. 2, one could readily deduce the probability for a catastrophic collision by determining the ratio of the number of trials ending up in the forbidden region to the total number of trials (frequency approach to probability). This possibility will not be pursued further here.

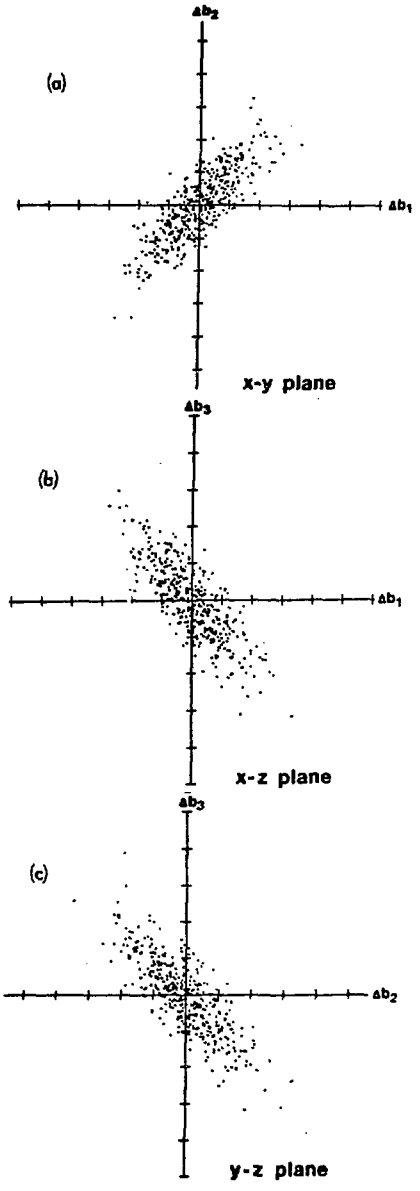

FIGURE 2: RESULTS FROM 400 MONTE-CARLO TRAILS TO PLACE THE ROBOT POINTER AT A DESIRED POSITION IN SPACE. POINTS ARE PROJECTIONS OF INDIVIDUAL HISTORY RESULTS ONTO COMPONENT PLANES OF A CARTESIAN COORDINATE SYSTEM WHICH IS ORIENTED AS SHOWN IN FIG. 1, BUT IS DISPLACED TO THE POINT OF REFERENCE. ALL AXES HAVE THE SAME SCALE. TIC MARKS ALONG EACH AXIS REPRESENT UNITS OF RESPECTIVE STANDARD DEVIATION.

Time Uncertainty Effects

It is instructive to explore the errors which can be traced solely to the parameter T. For simplicity we consider this problem separately from the preceding investigation, keeping in mind that the true situation involves a combination of these effects.

The $a_{io}(T)$ are assumed to depend linearly on T, as before. However, the $a_i(T)$ are no longer considered to be independently distributed about the corresponding $a_{io}(T)$ since the source of uncertainty, namely T, is common to all these robot primary degrees of freedom. Most likely, the distribution is normal with constant variance σ_T^2 for T from 0 to 1. However, for present purposes no knowledge of the T-distribution beyond the variance is required. The covariance matrix $\bar{C}_a$ is obtained from the matrix equation

$$\bar{C}_a = \bar{T}_{Ta} C_T \bar{T}_{Ta}^{+} \quad .$$

Since $\bar{C}_T$ is a very simple matrix, namely the constant σ_T^2, the matrix $\bar{C}_a$ is easily calculated. It turns out to be time independent since the transformation matrix $\bar{T}_{Ta}$ is also time independent. The elements of $\bar{C}_a$ are given by the formula

$$(\bar{C}_a)_{ij} = (a_{iof} - a_{ioo})\sigma_T^2(a_{jof} - a_{joo}) \quad .$$

The errors in the a_i are thus $(a_{iof} - a_{io})\sigma_T$, and the correlations $cor(a_i, a_j)$ all equal unity, as might be expected.

The error propagation analyses for $\vec{b}$, c and g proceed just as before. The errors in these derived results are also 100% correlated since they can all be traced to the error in the single common parameter T. For the demonstration robot it is assumed that $\sigma_T = 0.01$. Then, $\sigma_{a1} = 0.52$ cm, $\sigma_{a2} = 0.34$ cm, $\sigma_{a3} = 0.32°$, $\sigma_{a4} = 0.64°$ and $\sigma_{a5} = 0.46°$. Substitution of these values into the formulas yields the results presented in Tables 5 and 6. The results obtained when all correlations are arbitrarily ignored are also presented for comparison.

As before, consideration of correlations leads to results which differ vastly from those where correlations are ignored.

TABLE 5

UNCERTAINTIES IN THE COORDINATES OF P

T	σ^a_{b1}		σ^a_{b2}		σ^a_{b3}	
0	0.57	(0.53)	0.59	(0.35)	0.71	(0.65)
0.1	0.53	(0.55)	0.70	(0.43)	0.71	(0.68)
0.2	0.46	(0.58)	0.82	(0.50)	0.71	(0.71)
0.3	0.37	(0.61)	0.92	(0.57)	0.70	(0.75)
0.4	0.26	(0.65)	1.01	(0.64)	0.69	(0.78)
0.5	0.14	(0.70)	1.09	(0.71)	0.67	(0.82)
0.6	0.004	(0.75)	1.15	(0.78)	0.64	(0.86)
0.7	0.16	(0.80)	1.19	(0.84)	0.61	(0.89)
0.8	0.32	(0.87)	1.21	(0.90)	0.57	(0.93)
0.9	0.48	(0.93)	1.20	(0.96)	0.53	(0.96)
1	0.65	(1.00)	1.17	(1.02)	0.48	(0.99)

[a]Dimensions in cm. Values in parentheses are obtained if all correlations are ignored for error propagation calculations.

TABLE 6

UNCERTAINTIES IN c AND g

T	σ^a_c		σ^a_g	
0	0.96	(0.63)	0.77	(0.50)
0.1	1.00	(0.65)	0.81	(0.52)
0.2	1.03	(0.68)	0 84	(0.55)
0.3	1.05	(0.71)	0.87	(0.60)
0.4	1.06	(0.75)	0.89	(0.65)
0.5	1.06	(0.78)	0.90	(0.70)
0.6	1.05	(0.83)	0.91	(0.77)
0.7	1.04	(0.87)	0.92	(0.83)
0.8	1.02	(0.91)	0.91	(0.89)
0.9	0.99	(0.96)	0.91	(0.96)
1	0.94	(1.00)	0.89	(1.01)

[a]Dimensions in cm. Values in parentheses are obtained if all correlations are ignored for error propagation calculations.

Actually, it is incorrect to treat the errors due to T separately. Rigorous analysis of the simple robot problem presented here requires simultaneous consideration of both classes of error discussed above. It is not too hard to show that the correct form for the covariance matrix $\bar{C}_a$ which combines both types of error is

$$(\bar{C}_a)_{ij} = \delta_{ij}\sigma_{ai}^2 + (a_{iof}-a_{ioo})\sigma_T^2(a_{jof}-a_{joo}),$$

with $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$. This matrix is not diagonal, but no correlations will be as large as 100%. This matter will not be pursued further here.

Some Thoughts Concerning the Analysis of Robot Test Data

The preceding discussions have all involved error propagation calculations which required prior knowledge of uncertainties associated with the primary degrees of freedom a_i and the dimensionless "time" parameter T. This is certainly a perfectly reasonable course of action since some information of this sort will undoubtedly be available as a consequence of design studies, and perhaps from independent testing of separate components. Certainly this is not enough. The fully assembled robot must be tested to see how it performs as a system. Testing for uncertainties is but one of many aspects of a robot's performance which are likely to be investigated at this stage of it's development. This subject is a complicated one which cannot be treated in any detail in the space allotted here. Much development work is needed and this area offers interesting possibilities for future robotics research.

The naivest assumption which could be made is that no knowledge about the uncertainties is available, and that one must estimate $\vec{b}_o(T)$ and $\bar{C}_b(T)$, e.g., from repeated measurements of these parameters. Unbiased and consistent estimators for the b_{oi} and $(\bar{C}_b)_{ij}$ are, respectively,

$$\{b_{oi}\} = (1/n)\sum_{k=1}^{n} b_{ik},$$

$$\{(\bar{C}_b)_{ij}\} = [n/(n-1)]\,[(\sum_{k=1}^{n} b_{ik}b_{jk})/n - (\sum_{k=1}^{n} b_{ik})(\sum_{k=1}^{n} b_{jk})/n^2].$$

These estimators are evaluated using sampled values for all the b_i (there are n total samples). Formulas for the variances of these estimators can be derived, e.g., var $[\{b_{oi}\}] = \sigma_{bi}^2/n$ and var $[\{(\bar{C}_b)_{ii}\}] = 2\sigma_{bi}^4/(n-1)$, where σ_{bi} is the population variance. Clearly the uncertainties in these estimated quantities depend on the population variances which are not considered to be known. This presents the dilemma of not knowing how many samplings should be taken to achieve a desired level of confidence for the test results. For this reason, among others, this approach is generally of little practical value.

More sophisticated procedures for analyzing test data can be derived from least-squares methodology.

Let $\vec{\tau}$ represent a set of n measured test results with corresponding covariance matrix $\bar{C}_\tau$. Designate $\vec{p}$ as the m parameters $(p_1,\ldots,p_m)$ for which one desires the best-estimate values and their uncertainties (e.g., $\vec{p}$ might represent $\vec{b}$, or even single parameters such as c and g). Finally, assume that $\vec{\tau}$ and $\vec{p}$ are linked via a "design matrix" $\bar{D}$ so $\vec{\tau} \approx \bar{D}\vec{p}$. This relationship has been found to encompass many cases of practical interest. The method is discussed in detail in Refs. 7 and 8. In summary, one solves the problem by minimizing $R^2 = (\vec{\tau}-\bar{D}\vec{p})^+\bar{C}_\tau^{-1}(\vec{\tau}-\bar{D}\vec{p})$. R^2 follows the chi-square distribution for (n-m) degrees of freedom (n>m required), at its minimum. This procedure leads to the formulas $\vec{p} = \bar{C}_p\bar{D}^+\bar{C}_\tau^{-1}\vec{\tau}$ and $\bar{C}_p = (\bar{D}^+\bar{C}_\tau^{-1}\bar{D})^{-1}$ for the best-estimate parameters and their covariance matrix. The minimum value of R^2 (usually called χ^2) indicates the consistency of the test data. Large χ^2 (relative to n-m) might indicate possible errors in the tests. Also, the variances in the p_i, and corresponding scatter in the τ_k, could be well beyond the accuracies of the test measurements. Very small χ^2 would indicate that more accurate tests are needed to adequately characterize the robot.

This method is indeed far advanced beyond the naive one described earlier. A variety of testing techniques (with differing accuracies) can be used to generate the n samples $\tau_1,\ldots,\tau_n$. These measurements can be arbitrarily correlated. Furthermore, the τ_k values need not resemble the p_i-parameters one seeks, so long as a suitable design matrix can be produced.

This second approach is still limited in that prior design information is ignored. A more universal approach is required to merge a priori knowledge with information garned from tests, thereby generating global best-estimate robot parameters and their covariances. Fortunately, this is not too hard to do provided that one accepts a broad interpretation of Bayes theorem which states that prior knowledge of a system can be refined by the inclusion of new information which is assumed to be uncorrelated to the former. This procedure can be repeated ad infinicum. It follows directly that knowledge of the robot can be improved not only by laboratory tests on a design protoype but also by data accumulated later from production models in the field. This Baysian approach appeals to the pragmatic senses and is widely used in applied statistics even though pure statisticians still argue its validity on philosophical grounds.

Ref. 7 describes one manifestation of the Baysian approach to refining prior knowledge with new uncorrelated information. In summary, one assumes that $\vec{p}_o$ represents the current knowledge of a robot parameter set with relative (dimensionless) covariances matrix $\bar{C}_o$. When new test results are considered, one obtains a revised estimate $\vec{p}$ with relative covariance matrix $\bar{C}_1$. Let $\vec{x}_1$ consist of elements $(p_1-p_{o1})/p_{o1}$. The test data form a vector $\vec{\xi}$ with relative covariance matrix $\bar{C}_\xi$, and $\bar{D}_\xi$ is a design matrix depending on the character of the new information. Defining $\vec{y}_\xi$ as a vector with elements $(\xi_i-\xi_{oi})/\xi_{oi}$ ($\vec{\xi}_o$ is the value of $\vec{\xi}$ calculated using the a priori solution $\vec{p}_o$), one ultimately obtains the desired formulas $\vec{x}_1 = \bar{C}_1\bar{D}_\xi^+\bar{C}_\xi^{-1}\vec{y}_\xi$ and $\bar{C}_1^{-1} = \bar{C}_o^{-1} + \bar{D}_\xi^+\bar{C}_\xi^{-1}\bar{D}_\xi$. This procedure also involves the chi-square test which permits consistency of prior and new information to be investigated.

The material in this section is not illustrated with numerical examples owing to space limitation. However, the monte-carlo method could be used for simulation purposes. One could alter the distribution functions at will so they either agree with or differ noticeably from distributions based on design information. An exercise of this sort was recently pursued in some detail in a nuclear data application (see Ref. 9). There one readily sees the usefulness of the chi-square test in establishing consistency or inconsistency of various data sets considered in least-squares evaluation procedures.

CONCLUSION

The methods which have been described briefly in this paper should be more widely applied in the field of robotics. For example, it would be very worthwhile to investigate the link between control methodology and the estimation of probability distribution moments which are required in order to employ these statistical techniques. Research into the utilization of statistical analysis to optimize robot testing procedures would also be fruitful. Examination of details far beyond the scope of this paper would be required. Hopefully, the present paper will help to stimulate such efforts.

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