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THE EQUATION OF STATE OF HOT DENSE MATTER

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Abstract.- The approaches which have been used so far to calculate the equation of state of hot dense matter are briefly reviewed.

1. INTRODUCTION

One of the important ingredients in the description of stellar collapse is the equation of state which gives the pressure as a function of density and temperature. This relation is required up to about nuclear density and up to temperatures of about ten MeV. Calculating the equation of state is a difficult problem because it involves the determination of the equilibrium state of a mixture of neutrons, protons, nuclei, electrons, neutrinos and photons, with a negligible amount of positrons. The main difficulty is that one needs the properties of hot exotic nuclei immersed into a nucleon vapor. In contrast, electron, neutrino and photon contributions to the pressure can be treated simply and accurately by using standard formulae for non interacting gases.

The purpose of the present paper is to give a brief review of some of the approaches which have been developed so far to work out the equation of state. This review is not claimed to be exhaustive. It will in fact be mainly devoted to the high density region between 10^{12} g/cm^3 and nuclear density 3×10^{14} g/cm^3, where a microscopic treatment of nucleon-nucleon interactions is desirable. Indeed, in this domain, the external nucleon gas begins to modify the properties of nuclei. In principle one should calculate the pressure as a function of both variables, density and temperature. However, as was pointed out by Bethe, Brown,

Applegate and Lattimer (1), entropy and lepton fraction Y_ℓ are nearly constant in stellar collapse beyond densities of the order of 10^{12} g/cm^3 because of neutrino trapping. Entropy is about unity per baryon, while the lepton fraction $Y_\ell = Y_e + Y_\nu$, where Y_ν is the number of neutrinos per baryon, is about 0.35 (2). The equation of state is thus needed only along the adiabat $S/A \simeq 1$, which reduces significantly the amount of numerical work.

In the first attempts to derive an equation of state at high density, the baryon contribution to the pressure was evaluated as that of a mixture of Boltzmann gases of nucleons and nuclei in statistical equilibrium. This approach is reviewed in the next section and will be referred to as macroscopic since it incorporates nucleon-nucleon interactions only through semiempirical formulae for nuclear masses and level densities. It is valid as long as mass formulae are expected to be reasonably accurate. However when nuclei become hot, corrections to liquid drop parameters, in particular surface energy coefficients, as well as distorsions of nuclei by external nucleons should be included. For this reason, various microscopic approaches have been developed. Among these, we will review in section 4-7 the bulk matter approximation (3, 4), the compressible liquid drop model (5, 6), the Thomas-Fermi (7, 8) and Hartree-Fock (9, 10) approximations. In section 8 a comparison between various approaches is given.

2. MACROSCOPIC APPROACH

This approach, suitable up to $\rho \leq 10^{13}$ g/cm^3, has been used and developed by several authors, in particular Sato (11), Arnett (12), Mazurek, Lattimer and Brown (13), and El Eid and Hillebrandt (14). It describes hot dense matter as a statistical equilibrium between nuclei with various neutron numbers N and proton numbers Z. If we label by a single index i the pair (N, Z), the number ρ_i of nuclei i per unit volume is given by

$$\rho_i = Z_i(\beta) \left(2\pi \frac{\hbar^2}{m_i} \beta \right)^{-3/2} \exp(\alpha_i + \beta B_i) \qquad (1)$$

In this formula β is the inverse temperature, m_i the mass of the nucleus i, B_i its binding energy, usually taken from the droplet model of Myers and Swiatecki (15). The quantity $Z_i(\beta)$ is the partition function of the nucleus i

$$Z_i(\beta) = \sum_{n \geq o} \exp[-\beta (E_n(i) - E_o(i))] \qquad (2)$$

where $E_n(i)$ is the energy of the n-th excited state of the nucleus i. The quantity α_i in equation 1 is the modified degeneracy parameter

$$\alpha_i = N\,\beta\,(\mu_n - V_n) + Z\,\beta\,(\mu_p - V_p) \qquad (3)$$

where μ_n and μ_p are the neutron and proton chemical potentials in the external uniform gas and V_n and V_p the corresponding potentials, which for a given nucleon-nucleon force, are known functions of the neutron and proton densities ρ_n and ρ_p. In practice the chemical potentials are adjusted iteratively in order to obtain the desired values of the baryon density ρ and the proton or lepton fraction Y_p or Y_ℓ. The total pressure is calculated as the sum of electron and nucleon gas contributions, and of the various contributions $\rho_i kT$ arising from nuclei.

To evaluate the partition functions in equation 1 it is convenient to use semiempirical formulae for the level densities $\rho(E)$ related to $Z(\beta)$ by

$$Z_i(\beta) = \int_0^\infty \rho(E)\,\exp(-\beta E)\,dE \qquad (4)$$

The simplest such formula in the one derived in the Fermi gas model (16)

$$\rho(E^*) = \frac{1}{12}\left(\frac{\pi^2}{a}\right)^{1/4} E^{*\,-5/4}\,\exp\,2\sqrt{aE^*} \qquad (5)$$

where the level density parameter a is $A\pi^2/4\varepsilon_F$ with $A = N + Z$. This formula has to be improved near the origin to avoid the divergences it would produce in equation 4 (17). Also the value of a needs to be refined in order to include surface effects which are not taken into account in the Fermi gas model (26). At large excitation equation 5 is expected to overestimate the level density since it assumes an infinite number of equidistant single particle levels while there are only a few resonances in the continuum of the shell model potential. However the resulting error has been found to be still small up to temperatures of 10 MeV, both in semiclassical (18) and Hartree-Fock calculations (19).

3. INGREDIENTS OF MICROSCOPIC METHODS

When density is higher than about 10^{13} g/cm^3 it becomes necessary to treat in the same way the nucleons inside nuclei and nucleons in the vapor, i.e. to use a microscopic description based on nucleon-nucleon interactions. In spite of their great variety

microscopic methods almost always use the same two ingredients namely Skyrme forces and the Wigner Seitz approximation. Skyrme-type density-dependent effective forces (20) have the great advantage to lead to nuclear Hartree-Fock equations which have a simple structure and which give an excellent description of many nuclear properties such as radii, binding energies, ground state deformations, fission barriers, and giant resonances (21). The most recent such forces are the modified SKM force of Bartel et al (21) and the forces manufactured by the Brussel group (22).

The Wigner-Seitz approximation (23) is quite appropriate to calculate the free energy in the high density region because in this case the nuclei are expected to be arranged into a lattice. In this approximation one divides the lattice up into unit Wigner-Seitz cells with one nucleus at the center of the cell. One neglects interactions between cells and one uses Wigner-Seitz boundary conditions (23) within a cell. In Hartree-Fock calculations this means that single nucleon wave function should vanish or have zero derivatives at the cell edge. A further simplification is to assume the cells to be spherical, which is legitimate as long as the density at the cell edge in sufficiently low or uniform. Notice that when using the Wigner-Seitz approximation one assumes implicitely that there is only one kind of nucleus present in hot dense matter, while in Boltzmann gas approaches it was possible to have mixtures of different nuclei. This limitation is however not very important and has been shown to lead to negligible errors (6, 9). In order to be accurate, the Wigner-Seitz approximation requires that nuclei should form a lattice. This is the case if the plasma paramater $\Gamma = Z^2e^2/(RkT)$ is larger than 155 (24), where Z is the nucleus charge, R the cell radius, and T the temperature. This condition is usually well satisfied at densities greater than 3×10^{13} g/cm^3 (9). Below this value it is necessary to include correction terms such as those derived by Hansen (24). Up to now the only calculations that go beyond the Wigner-Seitz approximation are those of the Munich group (10), who use the augmented plane wave method. This method requires the use of single-nucleon wave functions without spherical symmetry.

4. THE BULK MATTER APPROXIMATION (3, 4)

The simplest microscopic way to calculate the equation of state is the bulk matter approximation, which describes the nucleus as a piece of homogeneous nuclear matter, in thermal equilibrium with an external uniform nucleon gas. If we denote by u the fraction of the cell volume V occupied by the nucleus, and by $f(\rho_n, \rho_p, T)$ the free energy density of nuclear matter, calculated e.g. from a Skyrme force, the free energy of the cell is given by

$$\frac{F}{V} = u \quad f(\rho_n, \rho_p, T) + (1 - u) \quad f(\bar{\rho}_n, \bar{\rho}_p, T) \qquad (6)$$

where ρ_n, ρ_p and $\bar{\rho}_n$, $\bar{\rho}_p$ are the neutron and proton densities in the nucleus and the vapor respectively. Note that F/V in equation 6 is independent of V because surface and Coulomb effects have not been included. Minimizing F/V with respect to the five parameters ρ_n, ρ_p, $\bar{\rho}_n$, $\bar{\rho}_p$, u, with the two constraints that the total density should be equal to ρ and the proton fraction Z/A to Y_p, yields three equilibrium conditions. These conditions require that the pressures P and $\bar{P}$, and the neutron and proton chemical potentials μ_n, $\bar{\mu}_n$ and μ_p, $\bar{\mu}_p$ should be equal in the nucleus and the vapor. These conditions are sufficiently simple to allow calculations of the equation of state over wide ranges of density, temperature, and proton fraction. The bulk matter approximation also has the advantage to provide a transparent picture which is useful to explore the coexistence of nuclei and vapor, and to determine the transition temperature, corresponding to u = 1, beyond which nuclei become uniform nuclear matter. However this approximation neglects important effects such as surface and Coulomb contributions. For this reason Lamb, Lattimer, Pethick and Ravenhall (LLPR), have developed a finite temperature liquid drop model, which includes a proper treatment of surface and Coulomb effects, while maintaining a tractable amount of numerical work.

5. THE COMPRESSIBLE LIQUID DROP MODEL (5, 6)

This model is a generalization of the liquid drop model of Baym, Bethe and Pethick (25) to finite temperatures. As in the previous section the Wigner-Seitz cell contains in this model a piece of homogeneous nuclear matter located at the center of the cell, which describes the nucleus, and a low density external gas between the nuclear radius R and the cell radius R_c. However the cell free energy F now includes surface and Coulomb terms. Explicitely

$$F = uV \quad f(\rho_n, \rho_p, T) + (1 - u)V \quad f(\bar{\rho}_n, \bar{\rho}_p, T)$$
$$+ 4\pi R^2 \sigma (\rho_n/\rho_p, T) + F_{Coul} \qquad (7)$$

where the notations u, ρ_n, ρ_p, $\bar{\rho}_n$, $\bar{\rho}_p$ are those of section 4. The Coulomb term F_{Coul} can be taken from the work of Baym, Bethe, Pethick (25), while the surface energy coefficient is most easily evaluated from Thomas-Fermi calculations of semiinfinite nuclear

matter with various proton fractions at finite temperature (26). In reference (26) it was found that the temperature dependence of the surface free energy is well described at low T by the approximate formula

$$\sigma\,(\rho_n/\rho_p,\ T) = \sigma\,(\rho_n/\rho_p,\ 0)\ (1 - T^2/T_s^2) \qquad (8)$$

where T_s is of the order of 12.5 MeV.

In actual calculations the free energy given by equation 7 is corrected to include contributions from alpha particles and from the translational motion of the nucleus. However these contributions are rather small and can be neglected to a first approximation. Since the specific free energy F/V given by equation 7 depends on the cell size due to surface and Coulomb effects, there are now six variational parameters ρ_n, ρ_p, $\bar{\rho}_n$, $\bar{\rho}_p$, u and R and two constraints ρ and Y_p i.e. four equilibrium conditions. These conditions, given in reference (6), are still simple enough to allow systematic calculations of the equation of state. It is in fact the main advantage of the compressible liquid drop model to provide a good compromise between simplicity and reliability. It indeed contains all the ingredients necessary for a reasonable equation of state and still leads to simple numerical calculations. The only point which may still need to be improved is the temperature dependence of the surface energy since it is determined from Thomas-Fermi calculations. Indeed the Thomas-Fermi method does not give a very accurate description of the nuclear surface. To conclude this section we would like to mention that a simplified version of the model, using an incompressible liquid drop, has been proposed recently by Bethe, Brown, Cooperstein and Wilson (27).

6. THOMAS-FERMI CALCULATIONS (7, 8)

In this approach the free energy of a Wigner-Seitz cell is a functional of the neutron and proton density distributions $\rho_n(r)$ and $\rho_p(r)$. It is given by

$$F[\rho_n(r),\ \rho_p(r)] = \int \{\mathcal{H}(\rho_\alpha,\ \tau_\alpha,\ \vec{\nabla}\rho_\alpha) - T\sum_\alpha S(\rho_\alpha)\}\, d\vec{r} \qquad (9)$$

where the index α labels neutron and proton densities. The energy density $\mathcal{H}$ is usually taken to be the Skyrme energy functional, which is a polynomial in the densities ρ_α, kinetic energy densities τ_α, and in $(\nabla\rho_\alpha)^2$. For a given value of the density ρ_α the kinetic energy density τ_α is given, in lowest order Thomas-Fermi approximation by eliminating y_α in the relations (6)

$$\rho_\alpha = \frac{1}{2\pi^2}\left(\frac{\hbar^2}{2m^*_\alpha}\beta\right)^{-3/2} F_{1/2}(y_\alpha)$$

$$\tau_\alpha = \frac{1}{2\pi^2}\left(\frac{\hbar^2}{2m^*_\alpha}\beta\right)^{-5/2} F_{3/2}(y_\alpha) \qquad (10)$$

where F is the Fermi function. The entropy density is given by

$$kT\, S_\alpha = \frac{5}{3}\frac{\hbar^2}{2m^*_\alpha}\tau_\alpha - kT\,\rho_\alpha\, y_\alpha \qquad (11)$$

where y_α is a function of ρ_α via equation 10. To determine the density profiles $\rho_n(r)$ and $\rho_p(r)$ one has to minimize the free energy F with respect to $\rho_n(r)$ and $\rho_p(r)$, with the constraint that the total baryon density is equal to ρ and the proton fraction to Y_p. Up to now Thomas-Fermi calculations were restricted to Fermi-type profiles for the distributions $\rho_n(r)$ and $\rho_p(r)$, because of numerical instabilities. However, in has been possible recently to perform fully variational calculations by using the so-called imaginary time step method (28). The amount of numerical work in Thomas-Fermi calculations is large since the number of variational parameters is the number of mesh points necessary to describe the distributions $\rho_n(r)$ and $\rho_p(r)$. Fortunately the imaginary time method provides a very efficient way to carry out minimizations. In practice the Thomas-Fermi method is very useful when there are large changes in the densities, namely when nuclei begin to overlap, leading to configurations, called bubble configurations (5), where the lattice is built up by holes or bubbles rather than nuclei.

7. HARTREE-FOCK CALCULATIONS (9, 10)

The most detailed treatment of nucleon-nucleon interactions is provided by the Hartree-Fock (or mean field) approximation at finite temperature. In this approximation the nucleons in the Wigner-Seitz cell are described by an independent particle density matrix of the form

$$D = \frac{1}{Z}\exp\left(\sum_i \alpha_i\, a_i^+\, a_i\right) \qquad (12)$$

where Z is defined by Trace D = I and where the α_i'S and the single particle states $|\varphi_i> = a_i^+|0>$ are variational parameters. The matrix D describes both nucleus and vapor. The free energy of the Wigner-Seitz cell reads

$$F = \text{Trace } (HD) + kT \text{ Trace } (D \text{ Log } D) \qquad (13)$$

where H is the nuclear hamiltonian, which is usually chosen to be a Skyrme type hamiltonian. An explicit expression for F in terms of the variational parameters is easily obtained by means of standard many body techniques. Minimizing the free energy F with respect to α_i leads to the condition that the occupation number of the i-th orbit $n_i = \text{Trace } (D a_i^+ a_i) = 1/(1 + \exp(\alpha_i))$ should be given by a Fermi distribution. The minimization with respect to the single particle orbits φ_i leads to Hartree-Fock type equations. After solving these equations one still has to minimize the free energy with respect to the cell radius R_c which determines the optimal size of the clusters. The nucleon contribution to the pressure is then calculated as the opposite of the grand potential $F - \mu_n N - \mu_p Z$ divided by the cell volume V. Such mean field calculations are difficult numerically because they correspond to a very large number of variational parameters, equal to the number of mesh points times the number of orbits. They are however very useful as a reference. In particular it would be worthwhile to use mean field results to extract the temperature dependence of surface free energy coefficients.

8. DISCUSSION

In figure 1 we compare the equations of state obtained in the bulk matter approximation by Pi, Marcos and Barranco (PMB) (29), in the incompressible liquid drop model by Bethe, Brown, Cooperstein and Wilson (BBCW) (27), in Thomas-Fermi calculations by Marcos, Barranco and Buchler (MBB) (7), and in the Hartree-Fock approximation by Bonche and Vautherin (BV) (9). All these curves correspond to a lepton fraction $Y_\ell = 0.35$ and an entropy equal to unity per baryon. The corresponding density domain is from 0.02 to 0.07 or 0.08 nucleons per fm^3. Nuclear matter density is 0.17 nucleons/fm^3 i.e. 2.45×10^{14} g/cm^3. From figure 1 it can be noted that even though the bulk matter approximation neglects surface effects it gives an equation of state similar to those obtained from more sophisticated approaches. The only important difference is that it gives a slightly higher value of the adiabatic index $\gamma = \partial \text{Log} P / \partial \text{Log} \rho$. All other approaches give very similar values of γ, near 1.295, i.e. somewhat smaller than the value of 4/3 which would be obtained from electrons only. This general agreement seems to indicate that, contrary to the

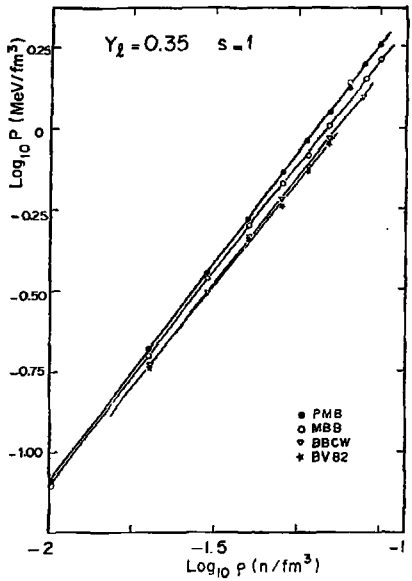

Figure 1.- Pressure as a function of density, plotted on a doubly logarithmic scale. The results of the bulk matter approximation (PMB) (29), are compared with those of Thomas-Fermi (MBB) (7), Hartree-Fock (BV) (5) and incompressible liquid drop model calculations (BBCW) (27).

situation a few years ago only, there is now no major uncertainty remaining in the determination of the equation of state at subnuclear density. It should be remembered however that all calculations use Skyrme forces which give a rather low density of single particle states near the Fermi levels. It would be worthwhile to perform calculations with forces giving better level densities, such as those built by Campi and Stringari (30). Also all calculations assume temperature independent effective nucleon-nucleon forces. However since this assumption has been checked over a limited range of temperatures only, up to about 3 MeV (31), it would definitely be worthwhile to perform further investigations. At densities higher than nuclear density ρ_0 one has very useful informations from the observed energies of monopole vibrations. These modes indeed involve the equation of state up to about twice nuclear density (32), as can be checked by looking at collective wave functions calculated for monopole states (33). Most recent Skyrme forces do reproduce these data quite accurately and, eventhough some ambiguities may remain, it thus seems reasonable to trust them up to about twice nuclear density. More information about the high energy region should become available in a nearby future, with the advent of new heavy ion facilities such as GANIL (32).

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