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CRITICAL DIMENSIONS OF UNTAMPED CONICAL VESSELS

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Note: This extended version of the original report was prepared for three reasons (a) the only available copies were illegible, (b) revived interest in bucklings of unusual shapes in connection with TMI-2 recovery, (c) to provide improved accuracy.

Raymond L. Murray September 10, 1984

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CRITICAL DIMENSIONS OF UNTAMPED CONICAL VESSELS

Abstract

The need often arises for determining the critical chemical concentration of uranium solution in the conical bottom of a plant reactor or storage vessel, or the dimension if the concentration is known. This report describes the mathematical analysis of Poisson's equation for a spherical sector, which approximates a right circular cone. The ratio of the critical dimension of an equivalent sphere to the height of the sector for various sector angles is derived from a comparison of first eigenvalues. No description of further relations between composition and dimensions is discussed in the report.

Analysis

The "pile equation" of thermal neutron diffusion theory is, for steady state,

$$\nabla^2 n + k^2 n = 0$$

a form of Poisson's equation appearing frequently in heat flow problems. Here k is a geometry parameter that must be equated to a function of the composition of the fissionable solution. The value of k is to be found in terms of the boundary conditions for neutron density in an untamped spherical sector, i.e.,

$$a = 0 \text{ at } r = 0,\ r = a,\ \theta = \theta_1.$$

Since neutron density is independent of the polar angle, only the radial and azimuthal portion of the laplacian is needed:

$$\nabla^2 = \frac{1}{r^2 \sin\theta} \left\{ \sin\theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} \right\}$$

Substitute in Poisson's equation and rearrange to obtain

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial n}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 n}{\partial \theta^2} + \frac{\cot\theta}{r^2} \frac{\partial n}{\partial \theta} + k^2 n = 0$$

The usual method of separation of variables is applied, by assuming that

$$n = R\Theta$$

thus

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2 = \frac{-\frac{d^2\Theta}{d\theta^2} \quad \cot\theta\frac{d\Theta}{d\theta}}{\Theta^2}$$

Set each side equal to a constant, $\ell(\ell + 1)$, to obtain two linear differential equations of second order, each involving one variable only.

$$\frac{R''}{R} + \frac{2}{r}\frac{k'}{R} + k^2 = \frac{\ell(\ell+1)}{r^2} \qquad (1)$$

$$\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} = - \ell(\ell+1) \qquad (2)$$

where primes refer to differentiation with respect to the appropriate variable.

Equation (1) may be put in the form of the familiar Bessel equation by the substitutions $x = dr$ and $R = kf/\sqrt{x}$ *, thus

$$\frac{d}{dr} = k\frac{d}{dx} \quad \text{and} \quad \frac{d^2}{dr^2} = k^2\frac{d^2}{dx^2}$$

Differentiation leads immediately to

$$f'' + \frac{f'}{x} + \left[1 - \frac{1}{4x^2} - \frac{\ell(\ell+1)}{x^2}\right] f = 0$$

or, since $\ell(\ell+1) + \frac{1}{4} = (\ell + \frac{1}{2})^2$

$$f'' + \frac{f'}{x} + \left[1 - \frac{(\ell + \frac{1}{2})^2}{x^2}\right] f = 0$$

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* For analogous derivations, see Margenau and Murphy, The Mathematics of Physics and Chemistry .

Its solution is the half-odd order Bessel function

$$f = J_{\ell+\frac{1}{2}}(x)$$

thus

$$R \sim \frac{J_{\ell+\frac{1}{2}}(x)}{\sqrt{x}} = \frac{J_{\ell+\frac{1}{2}}(kr)}{\sqrt{kr}}$$

Equation (2) may be put in the form of the familiar Legendre equation by the substitutions $x = \cos\theta$ and $\Theta = y$. Thus

$$\frac{d}{d\theta} = \frac{d}{d(\cos\theta)} \frac{d(\cos\theta)}{d\theta} = -\sin\theta \frac{d}{d(\cos\theta)} = -\sqrt{1-x^2} \frac{d}{dx}$$

and

$$\frac{d^2}{d\theta^2} = -\sqrt{1-x^2} \frac{d}{dx}\left(-\sqrt{1-x^2} \frac{d}{dx}\right) = (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx}$$

Differentiation leads to

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)y = 0$$

Its solution is the Legendre function

$$y = P_\ell(x) = P_\ell(\cos\theta) = \Theta$$

The neutron density, except for an arbitrary multiplying constant, is then

$$n = R\Theta = \frac{J_{\ell+\frac{1}{2}}(kr)}{\sqrt{kr}} P_\ell(\cos\theta)$$

Application

The physical significance of this solution is that the variation of density in the radial direction is independent of that in the angular sense. To satisfy the boundary condition $n = 0$ at $\theta = \theta_1$, for a given sector, $P_\ell(\cos\theta_1) = 0$. In general then, ℓ is non-integral and its determination would be difficult. A simpler device is to choose a set of values of θ_1 such that P_1, P_2, - - - - vanish. Once the ℓ's are chosen, the heights of the sectors for criticality, a, are found from the roots of the half-odd Bessel function, using the boundary condition $n = 0$ at $r = a$. Thus,

$J_{\ell+\frac{1}{2}}(ka) = 0.$

As an example, choose a sector of about 54° half angle, i.e., $\theta_1 = \cos^{-1} \frac{1}{\sqrt{3}}$. This satisfies the equation

$$P_2(\cos \theta_1) = \frac{1}{2}(3 \cos^2\theta_1 - 1) = 0.$$

The value of ℓ is thus 2. Solve the equation

$$J_{5/2}(x) = 0,$$

or

$$(3/x^2 - 1) \sin x - \frac{3}{x} \cos x = 0.$$

This root is x = 5.76, so that ka = 5.76. If k is given by the composition of the pile, a may be found, or the composition may be found for a given sector height.

Since the critical relation for a sphere of radius A is given by

$$k = \pi/A$$

it is convenient to determine the ratio of this "equivalent" sphere radius to the height of the spherical sector, a, as a function of the half angle. For the example above, if

$$k = \pi/A = \pi/5.76a$$

then A = 0.545a for $\theta_1 = 54°44'$.

Data on the roots of Legendre and Bessel functions are listed in Table 1 and the correlation of critical angle θ and ratio A/a is given in Table 2. Pairs of data for integers ℓ are augmented by data deduced by Lagrange interpolation. A graph through the calculated points is provided in Figure 1.

TABLE 1   Roots of Functions

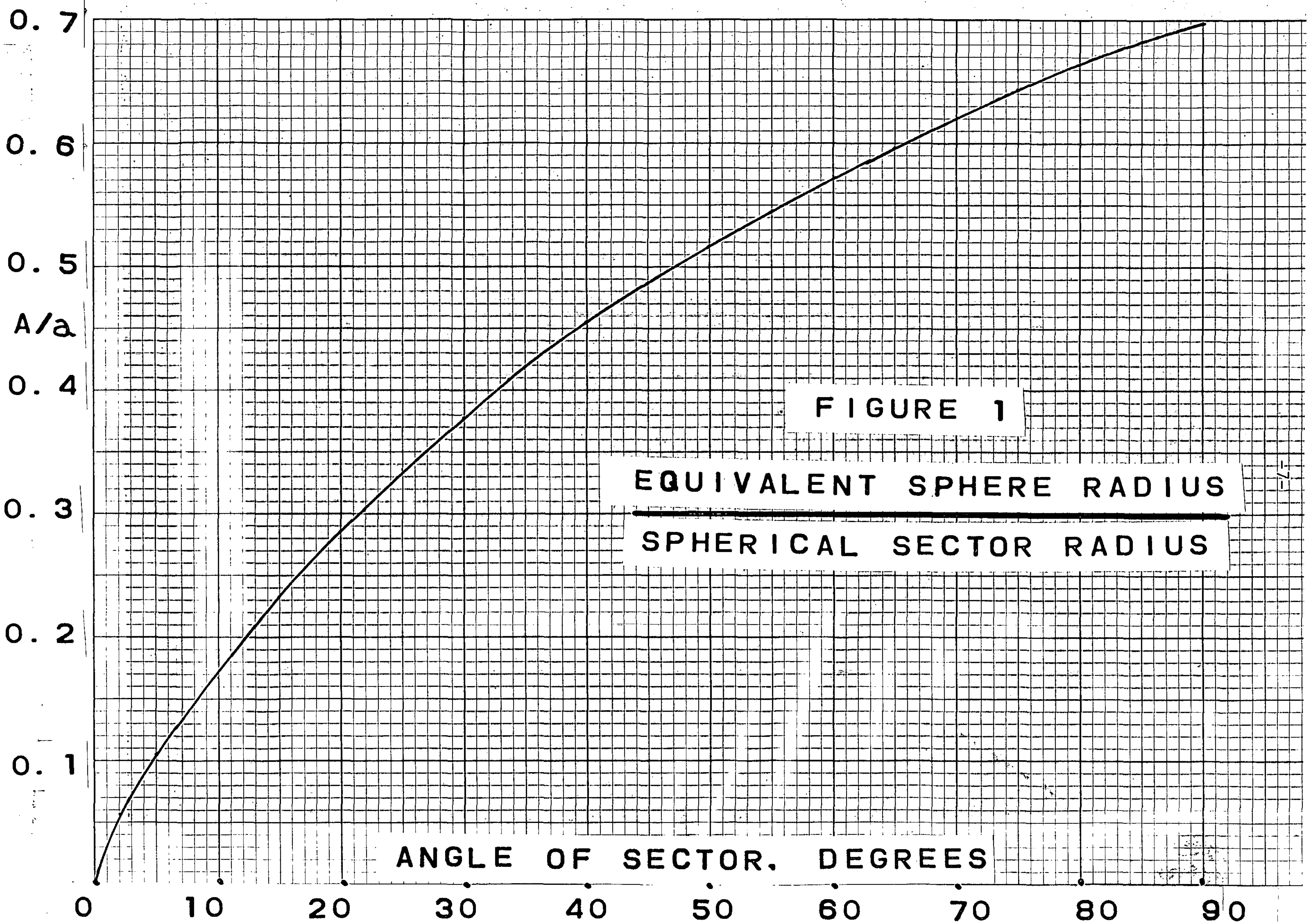
Order ℓ	Legendre* $P_\ell(x) = 0$	Half-order Bessel** $J_{\ell+\frac{1}{2}}(x) = 0$	
0	-	3.141593	sphere
1	0	4.493409	hemisphere
2	0.5773503	5.763459	
3	0.7745967	6.987932	
4	0.8611363	8.182561	
5	0.9061798	9.355812	
6	0.9324695	10.512835	
7	0.9491079	11.657032	
8	0.9602899	12.790782	
9	0.9681602	13.915823	
10	0.9739065	15.033469	
11	0.9782287	16.144743	
12	0.9815606	17.250455	
13	0.9841831	18.351261	
14	0.9862838	19.447703	
15	0.9879925	20.540230	

* Arnold N. Lowan, Norman Davids and Arthur Levenson, Bulletin of the American Mathematical Society, Vol. 48, No. 10, pp. 739-743. Oct. 1942.

** Editors, Milton Abramowitz and Irene A. Stegun, Handbook of Mathematical Functions, New York, Dover Publications, 1964.

TABLE 2 Critical Spherical Sectors

Order, ℓ	Angle, degrees	Ratio A/a
	0	0
	5	0.106
16	8.3494	0.145248
15	8.8879	0.152948
14	9.5006	0.161541
	10	0.1684
13	10.2040	0.171192
12	11.0202	0.182117
11	11.9775	0.194589
10	13.1175	0.208973
9	14.4971	0.225757
	15	0.2317
8	16.2008	0.245614
7	18.3579	0.269502
	20	0.2868
6	21.1769	0.298834
	25	0.336
5	25.0173	0.335790
	30	0.379
4	30.5557	0.383938
	35	0.419
3	39.2315	0.449574
	40	0.455
	45	0.488
	50	0.518
2	54.7356	0.545088
	55	0.5465
	60	0.573
	65	0.598
	70	0.621
	75	0.643
	80	0.663
	85	0.682
1	90	0.699156



0.7
0.6
0.5
A/a
0.4
0.3
0.2
0.1
FIGURE 1
EQUIVALENT SPHERE RADIUS
SPHERICAL SECTOR RADIUS
ANGLE OF SECTOR, DEGREES
0
10
20
30
40
50
60
70
80
90

Since a right circular cone is closely approximated by a spherical sector, at least for narrow cones, Figure 1 will give (conservatively) the critical relations for many plant reactors having conical bottoms.

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