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ATOMIC ENERGY OF CANADA LIMITED

L'ÉNERGIE ATOMIQUE DU CANADA LIMITÉE

INTRODUCTION TO FIRST ORDER OPTICS
TWO LECTURES GIVEN TO SCANDITRONIX STAFF
MAY 1982 by W.G. DAVIES

Introduction à l'optique de premier ordre
Deux conférences données au personnel de Scanditronix
en mai 1982, par W.G. Davies

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Chalk River Nuclear Laboratories

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Chalk River, Ontario

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Résumé

On présente les données fondamentales de l'optique ionique, en commençant par les équations de base non relativistes de mouvement et par les conditions dans lesquelles les équations para-axiales sont possibles. Les équations relativistement correctes sont ensuite présentées.

Les propriétés de focalisation de base des dipôles et des quadripôles sont décrites ainsi que le fonctionnement du doublet et du triplet du quadripôle. On développe la formulation de l'optique par matrice et l'on s'en sert pour déduire un certain nombre de résultats de base comme les relations de simulé, les plans principaux, la dispersion et l'achromatisme.

Les concepts de l'émittance, de l'acceptance, de la brillance et de l'espacement des phases sont présentés et illustrés au moyen d'exemples. Un étranglement de l'espacement des phases est défini et l'on commente la relation qu'il a avec un foyer. On présente les formulations "Sigma matrix" et "Twiss" pour les transformations de l'espacement des phases et l'on indique leurs rapports mutuels. On commente également le théorème de Liouville ainsi que les conditions de sa validité.

Des exemples de l'action d'une lentille et d'un espace de glissement sur l'espacement des phases des faisceaux sont donnés pour illustrer la puissance de ces concepts et pour développer chez le lecteur une certaine familiarité avec eux.

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Mai 1983

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ABSTRACT

The fundamentals of ion optics are presented, beginning with the basic non-relativistic equations of motion, and the conditions under which the paraxial equations hold. The relativistically correct equations are then presented.

The basic focussing properties of dipoles and quadrupoles are described as well as the operation of the quadrupole doublet and triplet. The matrix formulation of optics is developed and used to derive a number of basic results such as imaging relations, principal planes, dispersion and achromatism.

The concepts of emittance, acceptance, brightness and phase space are introduced and illustrated by examples. A waist in phase space is defined, and its relationship to a focus is discussed. Both the "Sigma matrix and Twiss formulations" for phase-space transformations are presented and their inter-relationship shown. Liouville's Theorem is also discussed along with conditions for its validity.

Examples of the action of a lens and a drift space on the beam phase space are given to illustrate the power of these concepts and develop some familiarity with them on the part of the reader.

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Introduction to First Order Ion Optics

TWO LECTURES GIVEN TO SCANDITRONIX STAFF, MAY 1982 BY W.G. DAVIES

1. Why do we need ion optical systems?

Ion optical systems are used for many purposes amongst which are:

i) To transport an ion beam from an accelerator to a target. This is the most common application.

ii) To produce special beam properties at this target. (i.e. at a microtron or neutron therapy target.)

iii) To prepare the beam for injection into another accelerator such as a synchrotron or cyclotron. (In many respects this is a similar problem to ii above.)

2. Basic design requirements

When designing and operating a beam transport system, it is necessary to know two important things:

i) What does the beam we are starting with look like?

ii) What must it end up looking like?

If we do not know the answers to one or both of the above questions, then we must design a system that has sufficient flexibility to accommodate our ignorance. The extent of our ignorance will be reflected in the complexity and cost of the system. Here knowledge saves money and time.

If we do know what we have and what we want, then the problem remaining is how to get from the beginning to the end. As with any other design problem, the simplest and best design is usually the result of a thorough understanding of the problem and the possible methods of solution. Except for very simple problems this is an evolutionary process.

3. Basic physics of beam optics

In these lectures I will restrict myself to a discussion of magnetic systems, and for the moment to systems where the energy of the beam is constant - i.e. non-accelerated systems. In many instances electrostatic lenses look the same as magnetic lenses, but the beam is always accelerated or decelerated inside an electrostatic lens so in general they can have quite different properties.

The force on a particle travelling in a magnetic field is

$$\overline{F} = q\overline{v} \times \overline{B} \tag{1}$$

Here F = force, q = charge of the particle (+ for protons, - for electrons) v = velocity and B = magnetic field. The "x" means vector cross product and the result is that the force is perpendicular to the plane formed by v and B.

The force due to the magnetic interaction is countered by the "Centrifugal force" experienced by the deflected particle.

$$F = mv^2/\rho \tag{2}$$

where m is the mass and ρ is the radius of curvature. These two forces must be equal for us to have a stable trajectory. If the arrangement is as shown in Fig. 1, we can equate the two expressions with the result

$$qv_z\ B_y = mv_z^2/\rho \tag{3}$$

which becomes

$$qB_y\rho = mv_z = P_z \tag{4}$$

From (4) we obtain

$$B_y = P_z/q\rho \tag{5}$$

and

$$\rho = P_z/qB_y \tag{6}$$

For particles with relativistic velocities these equations still hold if we replace P = mv by

$$P = \frac{1}{c}\sqrt{1863mE + E^2} \qquad (7)$$

where m is now in AMU (atomic mass units) and E is the kinetic energy in MeV. P is the momentum of the particle in units of MeV/c.

Equation 7 is easily derived from the relation connecting the total energy T and the momentum (see for example Goldstein p.204).

$$T^2 = p^2c^2 + m^2c^4 \qquad (8)$$

so that

$$P = \frac{1}{c}\sqrt{(mc^2 + E)^2 - m^2c^4} \qquad (9)$$

where we have substituted

$$T = mc^2 + E \qquad (10)$$

the rest energy plus kinetic energy.

Finally we have

$$P = \frac{1}{c}\sqrt{2mc^2E + E^2} \qquad (11)$$

The equation

$$B\rho = P/q \qquad (12)$$

defines what is referred to as the B-rho or the magnetic rigidity of the particle beam; $B\rho$ is in Tesla-meters (T-m) if P is in (MeV/c) and q is in multiples of the electron charge. Equations (5) and (6) above can be written in an "obvious" manner as

$$\rho = (B\rho)/B \tag{13}$$

and

$$B = (B\rho)/\rho \tag{14}$$

4. Elements of First Order Systems

There are only two focusing elements available to the designer of a first order ion-optical system. They are the dipole and quadrupole. A dipole is shown in Fig. 1 and the quadrupole in Fig. 2. The magnetic field as a

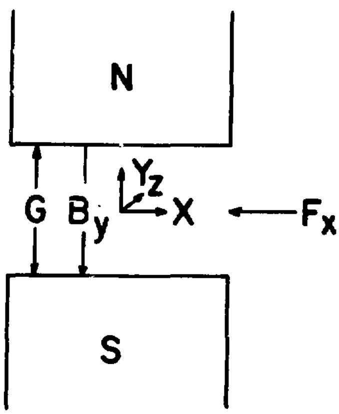

Fig. 1 Dipole magnet showing the definitions of the coordinate system, the direction of the field and the direction of the force for a positively charged particle going into the paper. $Fx = -v_z By$.

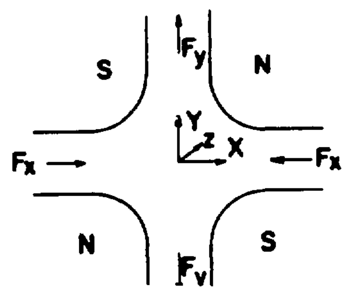

Fig. 2 Quadrupole magnet showing definitions of the coordinates, forces and fields as in Fig. 1. Once again, a positively charged particle is assumed to be entering the paper.

function of the coordinate X are shown in Fig. 3. We see that a dipole has a a perfectly uniform field, so that except for end effects the particle will travel along the arc of a circle with radius given by equations (6) or (13). A simple uniform field dipole, with perpendicular entrance and exit boundaries has the properties of a thick lens in the bending plane, the X-plane, and the properties a drift space in the axial or Y-plane. The distributed, or "thick" lens properties of a dipole can easily be demonstrated graphically, as illustrated in Fig. 4. The trajectory of a particle travelling in a quadrupole field is more complicated. Here the deflection of the particle increases linearly as the particle goes farther away from the center of the quadrupole. In the example of Fig. 2, we see that a positively charged particle is always deflected towards the axis if it lies near the X-plane. Application of eq. 1 shows that the particle will always be deflected away from the axis if it lies on or near the Y-plane. This is a general property of all quadrupole singlets. They can be oriented so that they are focusing "F" or defocusing "D" in either the X or Y planes, but never in both at the same time. By convention the symbols "F" and "D" always refer to the X-plane, i.e. F means focusing in the X-plane **AND** defocusing in the Y-plane etc.

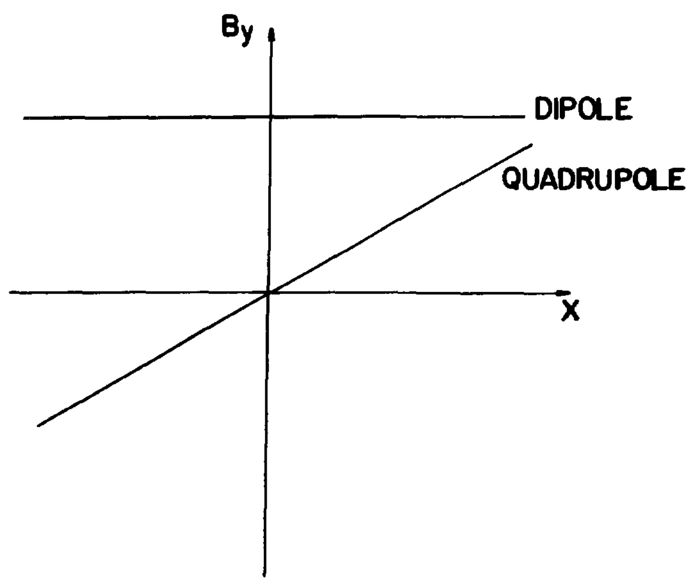

Fig. 3 Shows the magnetic field as a function of the X coordinate for an ideal dipole and an ideal quadrupole magnet.

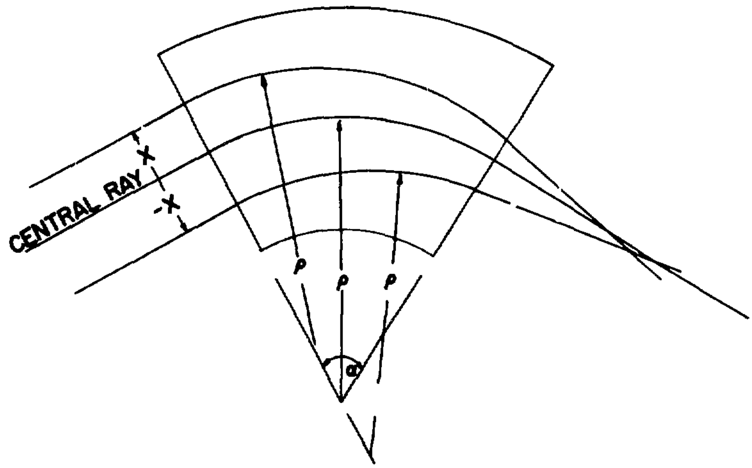

Fig. 4 The thick lens or distributed focussing property of a magnetic dipole is illustrated by a simple geometrical construction.

5. Doublet and Triplet Lenses

Fig. 5 and 6 show schematically how a doublet and triplet lens are formed. It is, of course, possible to interchange the X and Y planes. Let us assume that all singlets have the same strength. Referring to Fig. 5a, we see that a ray leaving the optic axis at the object position is bent back towards the axis by the first element. After travelling a distance ℓ, it encounters the defocusing lens, but closer to the axis where its strength is less (see Fig. 3). Hence it is not defocused as much as the first lens focused it. The net result is focusing. However, the focusing power of the doublet is about half of that of the singlet. In the other plane (Fig. 5b), the ray first encounters the defocusing lens where it is deflected outwards. After traversing the distance ℓ, it is deflected back towards the axis by the focusing lens because, by having been bent outwards initially, it passes through the focusing lens in a region of higher field strength (see Fig. 3).

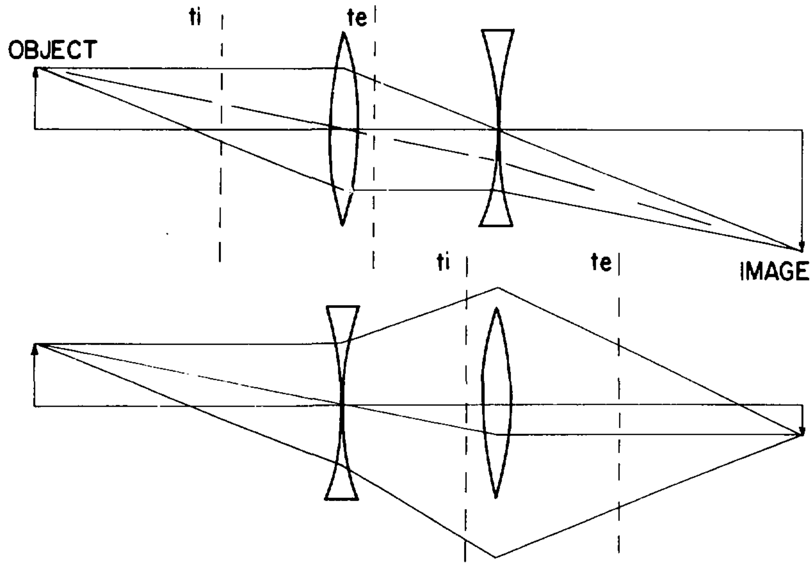

Fig. 5 Schematic representation of a quadrupole doublet in each of the two planes. The choice of FD in X and DF in Y is arbitrary and X and Y can clearly be interchanged. The "imaging" properties of a doublet and the "astigmatism" associated with doublet lenses is also indicated. The lines denoted ti and te are the so-called principal planes of the lens and they illustrate why the doublet has unequal magnifications in the X and Y planes.

Thus once again, the effect is net focusing. So, we see that two singlet lenses of approximately equal strength are net focusing in both planes whether or not they are connected FD or DF even though each singlet is focusing in one plane and defocusing in the other. However, their imaging properties are not at all alike. As indicated in the figure, the FD combination always has a magnification greater than 1, and the DF combination always has a magnification less than 1. This is, in some sense, the same as having a lens with a lot of astigmatism. This intrinsic "astigmatic" property of doublet lenses can often be a problem. On the other hand, if one starts with a beam that is astigmatic (that is, the virtual objects in the X and Y planes are at different positions) a doublet lens is very useful in correcting this astigmatism.

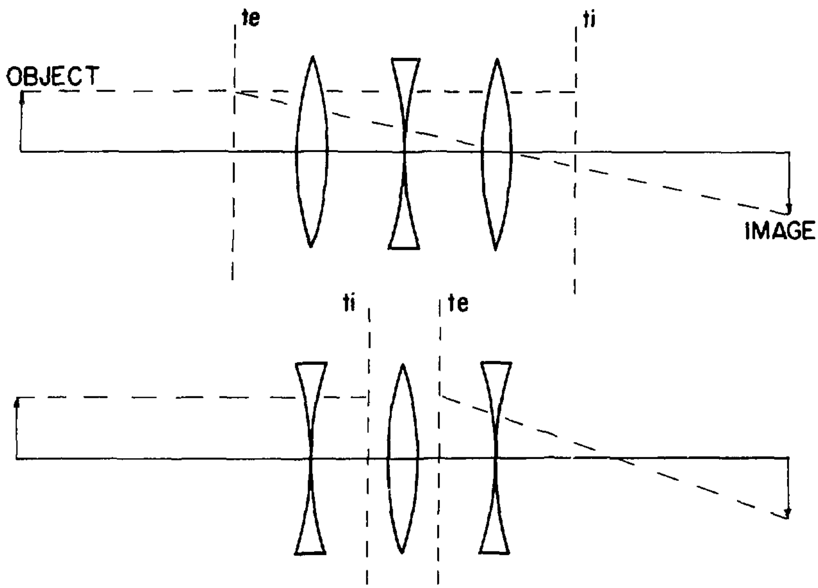

Fig. 6 Schematic representation of a triplet quadrupole lens. Once again, the choice of FDF in the X plane is arbitrary, and the two planes can be interchanged. The figure illustrates that a symmetric triplet lens forms a stigmatic image with equal magnifications in the X and Y planes. The principal planes are not coincident, though, so even though the image is stigmatic, the focal lengths are not the same in the X and Y planes.

We can avoid the problems associated with the astigmatism of a doublet lens by using a triplet lens as shown in Fig. 6. Arguments identical to those given above for a doublet lens can be used to prove that a triplet lens is also net focusing whether or not it is connected as FDF or DFD. Furthermore, if the lens is operated symmetrically, that is the first and last elements have equal strengths, we see immediately from the symmetry of the system that the imaging properties must be the same, i.e. the magnifications will be equal in both the X and Y planes. One penalty we must pay for this symmetry is that the focal length of the triplet lens is even weaker than that of a doublet when compared with the strengths of the singlets. Obviously, a triplet lens that is operated asymmetrically will have unequal magnifications in the two planes. The range of magnification will vary from that of a symmetric triplet to that of a doublet in the extreme case where we turn off one of the end elements.

6. Paraxial Optics

An optical system is said to be paraxial or first order when the rays transform according to a system of linear equations. That is:

$$x_1 = ax_0 + b\theta_0 \tag{15a}$$

$$\theta_1 = cx_0 + d\theta_0 \tag{15b}$$

In general, the rays do not satisfy the above equations, but contain a very large number of "higher order terms" such as x^2 or θ^2 or $x\theta$ etc. In fact what we have done is to expand the exact ray equations in a power series about the optic axis or central trajectory. The optic axis is the trajectory followed by a particle with momentum P_0 and $x = \theta = 0$. For many systems all rays lie very close to the optic axis and the paraxial approximation is very good. The higher order terms are called aberrations. They can come either from approximations in expanding the geometry of the system (i.e. $\tan\theta = \theta$) or from higher order terms in the expansion of the magnetic field, for example the field in a real quadrupole is never perfectly linear. I will say no more in these lectures about aberrations. However, their existence should not be forgotten!

More precise definitions of the requirements for paraxial optics and some elegant consequences of this formalism are given in the following sections.

6.1 The Paraxial Condition

The first order approximations are valid when all rays remain close to the "optic axis" or central trajectory.

In general the approximation is valid if $\tan\theta \approx \theta$, and $x/\rho \ll 1$ where ρ is the radius of curvature of the central ray.

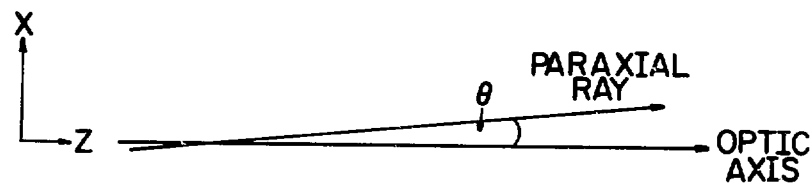

Fig. 7 Diagram showing the relation between the paraxial ray and the Optic axis.

6.2 Coordinate System

A general Curvilinear Coordinate System that is locally orthogonal is used because the origin of the local system always lies on the central trajectory as shown in Fig. 8.

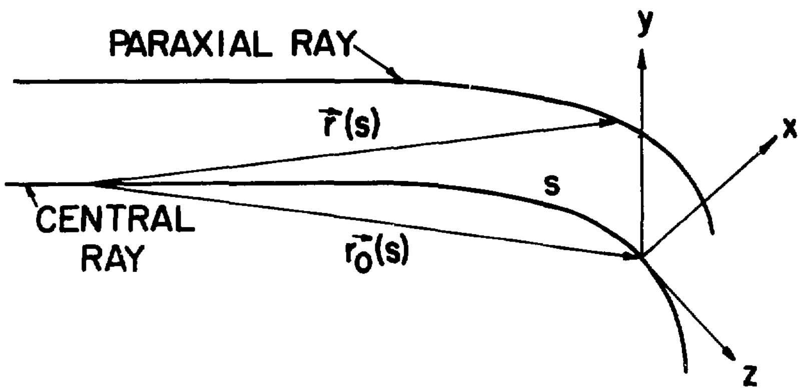

Fig. 8 Curvilinear coordinate system used in computing ray trajectories. The local orthogonal coordinate system (x,y,z) is oriented such that (i) x is in the bending plane and directed along the local radius vector ρ, (ii) z is tangent to the central trajectory, and (iii) y is the upward pointing vector perpendicular to the (x,z) plane. (Note the system can be either right handed or left handed depending on the direction of the curvature.) The coordinate of the ray is given as a function of s, the distance along the central trajectory from the origin to the position of local coordinate system.

In a field free region the central trajectory is a straight line and the Curvilinear system reduces to a simple Cartesian system. In a uniform field bending magnet S is an arc of a circle, that is the radius of curvature ρ is constant. Most beam optics systems consist of a series of straight lines and arcs of circles, so the generalized coordinate system is really not so complicated to use in practice.

6.3 Equations of Motion

The paraxial equations of motion are obtained most elegantly by expanding the relativistic Hamiltonian in the curvilinear system.

$$H = eV + c\left\{m^2c^2 + (\overline{p} - \overline{eA})^2\right\}^{1/2} \tag{16}$$

where V and A are the scaler and vector potentials of the electromagnetic field. The Hamiltonian (eq. 16) is transformed such that the independent variable is s, the distance along the central ray rather than the time t. Hence Hamilton's equations become, after a few pages of algebra. (See for example Courant and Snyder.)

$$x' = -\frac{\partial P_s}{\partial Px} \qquad P'_x = \frac{\partial P_s}{\partial x} \qquad x' = \frac{\partial x}{\partial s} \quad \text{etc.}$$

$$z' = -\frac{\partial P_s}{\partial Pz} \qquad P'_z = \frac{\partial P_s}{\partial z} \qquad ' = \frac{\partial}{\partial s} \tag{17}$$

$$H' = -\frac{\partial P_s}{\partial t} \qquad t' = \frac{\partial P_s}{\partial H}$$

In the paraxial approx. $x' = \theta$ $y' = \phi$. The expanded Hamiltonian when substituted into Hamilton's equations leads to two second order differential equations. For a dipole magnet they are

$$x'' + (1 - n)h^2x = h\delta \tag{18a}$$

$$y'' + nh^2y = 0 \tag{18b}$$

$$h = 1/\rho \tag{18c}$$

where $\delta = \Delta p/P_o$ and P_o is the central momentum.

$$n = - \left(\frac{\rho_o}{B_y} \right) \left(\frac{By}{\partial x} \right) \qquad B_y = \frac{hP_o}{e} = \frac{P_o}{\rho_o e} \tag{19}$$

We assume that the magnetic field B is in the vertical direction so that $\vec{v} \times \vec{B}$ bends the particles in the horizontal direction.

For a field free region eq. 16 and 17 reduce to

$$x'' = 0 \tag{20a}$$

$$y'' = 0 \tag{20b}$$

These equations are easily integrated leading to the solutions

$$x = as + b \tag{21a}$$

$$y = cs + d \tag{21b}$$

Now, $b = x_o$, the initial value of x and

$$a = x' = \frac{P_x}{P_o} = \tan\theta = \theta \tag{22}$$

The relationship between P, P_o and P_x is shown in Fig. 9.

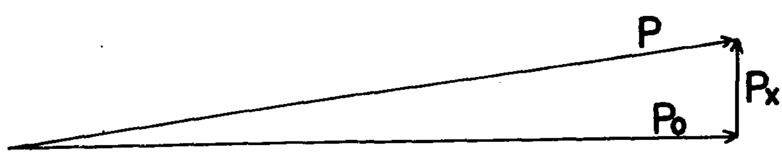

Fig. 9 Relationship between the total particle momentum P, the momentum along the optic axis P_o and the transverse momentum P_x.

We make the approximation that $P = P_o$ (22a)

Where $$P^2 = P_o^2 + P_x^2 \qquad (22b)$$

so that $$P_x^2 \ll P_o^2. \qquad (22c)$$

Hence

$$x' = k_x = \theta_o \qquad (23a)$$
$$y' = k_y = \phi_o \qquad (23b)$$

where here kx and ky are constants of the integration. Thus we see that for a drift space the transformation is linear. This linear transformation property is also approximately valid for all electrostatic lenses, magnetic dipoles and quadrupoles.

6.4 Matrix Formalism

The drift space (eq. 21,23) can be written as a matrix transformation:

L

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & (S-S_o) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ \theta_o \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ \theta_o \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_o \\ \phi_o \end{bmatrix} \qquad (24)$$

If the beam is bunched and we wish to know what happens to the bunch, we can write a similar matrix for the differential bunch length ℓ and the momentum spread δ as a function of S. Thus

$$\begin{bmatrix} t_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} 1 & L/v_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_o \\ \delta_o \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \ell_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ell_o \\ \delta_o \end{bmatrix} \qquad (25)$$

Thus we see that in this case the 3 pairs of canonically conjugate coordinates of Hamilton's equations (x,θ), (y,ϕ), (ℓ,δ) transform independently of each other but with the same transform.

This property is only valid under special circumstances. It is true for drifts and quadrupoles but not for dipole bending magnets. In a dipole the term $h\delta$ in eq. 18a produces coupling of the (x,θ) and (ℓ,δ) coordinates and leads to the phenomenon of dispersion. Then the transformation between S_0 and S_1 cannot be written as 3 independent 2 x 2 matrices.

In general the transformation through the region from S_0 to S_1 can be written as a 6 x 6 matrix, $R(S_1 - S_0) = R_0$. The theory of linear transformations tells us that the results for a set of successive piecewise constant regions can be obtained by matrix multiplication.

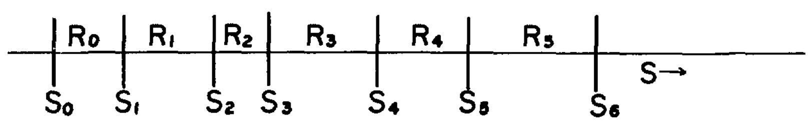

Fig. 10 Schematic illustration of how 6 elements comprising an optical system can be related by multiplying their respective transforms matrices.

$$R(S_n - S_o) = R(S_n - S_{n-1}).......R(S_2 - S_1)R(S_1 - S_0) \qquad (26)$$

Hence $X_6 = R(S_6 - S_0)X_0$

$$\text{where } X = \begin{bmatrix} x \\ \theta \\ y \\ \phi \\ \ell \\ \delta \end{bmatrix} \qquad (27)$$

If we have midplane symmetry then the (x,Θ) and (y,ϕ) coordinates are always orthogonal and the general 6 x 6 matrix has the form

$$\begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ y \\ \phi \\ \ell \\ \delta \end{bmatrix} \qquad (28)$$

What do the matrix elements mean?
This is most easily seen by inspecting their dimensions.

$R_{11} = (x_n/x_o)$
$R_{33} = (y_n/y_o)$ = transverse linear magnification

$R_{22} = (\theta_n/\theta_o)$
$R_{44} = (\phi_n/\phi_o)$ = angular magnification

$R_{21} = (\theta_n/x_o)$
$R_{43} = (\phi_n/y_o)$ has dimensions of 1/L and is the inverse of the focal length

$R_{12} = (x_n/\theta_o)$
$R_{34} = (y_n/\phi_o)$ *has dimensions of L and represents the effective drift lengths respectively through the system.

$R_{16} = (x_n/\delta_o)$ = d_x, the linear momentum dispersion

$R_{26} = (\theta_n/\delta_o)$ = d_θ angular momentum dispersion

$R_{51} = (\ell/x_o)$ terms which mix $(x,\theta)(\ell,\delta)$ coordinates pairs in bending magnets

$R_{52} = (\ell/\theta_o)$

$R_{56} = (\ell/\delta_o)$ *has dimensions of L and represents the effective length of the system for the longitudinal coordinates

*Note! The effective lengths are often different for each of the 3 pairs of canonically conjugate variables. Their values depend upon the details of the optical system and to some extent can be adjusted independently of each other.

6.5 Simple Optical System

For simplicity let us look at a simple non dispersive system which is the same in x and y. Thus we need only consider the 2 x 2 transformation matrix.

$$RX_o = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_o \end{bmatrix} \tag{29}$$

Let us check a few simple properties. The focal point of a system is defined as the point where a principal ray crosses the axis, as shown in Fig. 11.

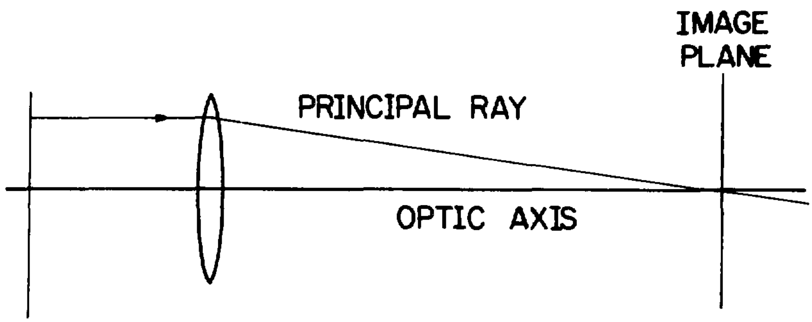

Fig. 11 Parallel to point focus. When the focal length $f = 1/R_{21} = L$ the distance from the lense to the image plane, a parallel to point focus condition exists.

Thus

$$\begin{bmatrix} 0 \\ \theta \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x_o \\ 0 \end{bmatrix} \tag{30}$$

For this to be true $R_{11} = 0$.

Thus $R_{11} = 0$ is the condition for a parallel to point focus;
$\theta_1 = R_{21}x_o = x_o/f$ where f is called the focal length.

A point to point focus is said to exist when all rays emanating from a point are refocused to a point in the image plane. This is shown in Fig. 12.

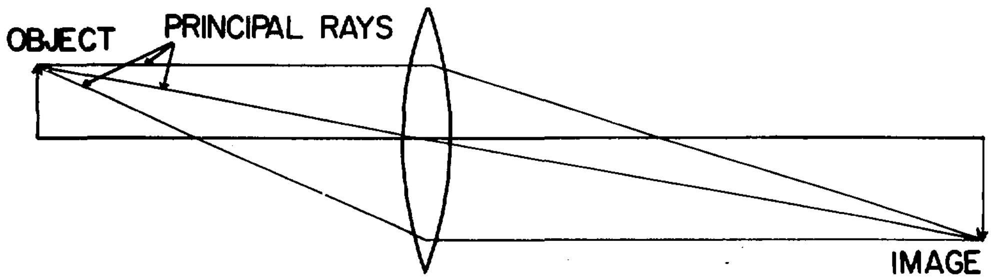

Fig. 12 Point to point focus. A point to point focus exists when all rays emanating from the object point pass through the image point. (See eq. 31, 39 and 40.)

Thus

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x_o \\ \theta_o \end{pmatrix} \qquad (31)$$

If all rays emanating from x_o pass through x_1 then $R_{12} = 0$. This is a necessary condition for a point to point focus.

6.6 Principal Planes

The principal planes of an optical system are defined as the pair of planes that produce an image-object relationship with a magnification of +1. These planes are found by transforming the optical system R into a new system R_1 which satisfies the above conditions. For a 2-dimensional case

$$R_1 = \begin{bmatrix} 1 & t_e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i & t_i \\ 0 & 1 \end{bmatrix} \quad (32)$$

where t_i and t_e are the distances from the entrance and exit boundaries of the system under consideration to the principal planes.

Hence

$$R_1 = \begin{bmatrix} R_{11} + t_e R_{21}, & R_{12} + R_{11} t_i + (R_{22} + R_{21} t_i) t_e \\ R_{21} & R_{22} + t_i R_{21} \end{bmatrix} \quad (33)$$

The condition that

$$\begin{aligned} R_{11} + t_e R_{21} &= 1 \\ R_{22} + t_i R_{21} &= 1 \end{aligned} \quad (34)$$

leads to

$$\begin{aligned} t_i &= f(1 - R_{22}) \\ t_e &= f(1 - R_{11}) \end{aligned} \quad (35)$$

where $R_{21} = 1/f$.

Since the matrix elements R_{11} and R_{22} are both unity in R_1 (eq. 33 and 34) and since in general $R_{21} = 1/f \neq 0$ and since the determinant, det $|R_1| = 1$, then $R_{12} = 0$. Thus the matrix R_1 has the form

$$R_1 = \begin{bmatrix} 1 & 0 \\ 1/f & 1 \end{bmatrix} \tag{36}$$

where if r is positive R_1 is divergent and if f is negative R_1 is convergent.

The matrix R_1 (eq. 36) is identical to the transformation matrix of a "thin" lens.

Thus we see that any arbitrary region can be transformed into the matrix of a thin lens with focal length $f = 1/R_{21}$ and two principal planes a distance $t_i = f(1 - R_{22})$ and $t_e = f(1 - R_{11})$ from the boundaries of "R". (See Fig. 13 and eq. 32.)

The sign convention for the distance to the principal planes (see eq. 35 and Fig. 13) is as follows: t_i and t_e are measured outward (inward) from the entrance and exit boundaries of R when their signs are positive (negative). Thus both t_i and t_e are negative in Fig. 13.

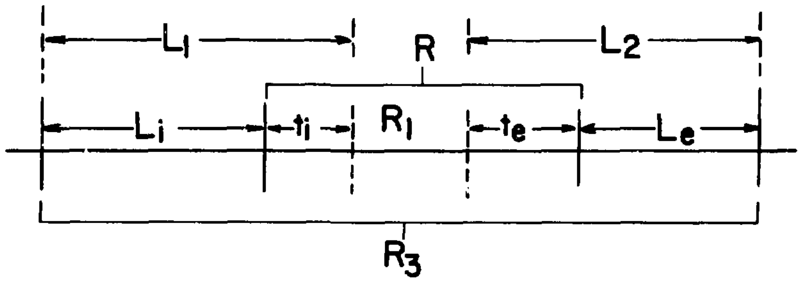

Fig. 13 An example of how the principal planes of the optical system represented by the matrix R exhibit the optical properties of an arbitrary system in a simple way.

Let us consider the example shown in Fig. 13. Let us also assume that the optical system represented by "R" is convergent. Then as indicated in the figure, eq. (36) becomes

$$R_1 = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

and the principal planes are located a distance t_i and t_e from the entrance and exit boundaries of the system R.

Thus we can define two new lengths, L_1 and L_2 as

$$L_1 = L_i - (-t_i)$$
$$L_2 = L_e - (-t_e) \tag{38}$$

The overall transformation, R_3, is:

$$R_3 = \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - L_2/f; & L_1 + L_2 - L_1L_2/f \\ -1/f & 1 - L_1/f \end{pmatrix} \qquad (39)$$

This matrix exhibits most of the elementary properties of optics. For example, if we want a focus-to-focus condition then $R_{12} = 0$ so that

$$L_1 + L_2 - \frac{L_1L_2}{f} = 0 \qquad (40)$$

which reduces to the well known imaging formula

$$\frac{1}{L_1} + \frac{1}{L_2} = \frac{1}{f} \qquad \text{(Note convergence is assumed.)} \qquad (41)$$

If we want a parallel to point focus then $R_{11} = 0$ in eq. 39, so $1 - L_2/f = 0$ or $L_2 = f$ which is what we mean by focal length. The linear magnfication is $R_{11} = 1 - L_2/f$ which for a point to point focus becomes $R_{11} = -L_2/L_1$ where we substitute for f from the condition that $R_{12} = 0$ into the expression for R_{11} in eq. 39. This is the well known formula for the magnification of a thin lens imaging system.

6.7 Achromatic Bending Systems

In beam transport systems in which the energy spread of the beam, $\Delta E/E$ is not extremely small (i.e. << 1%), the dispersive properties of dipole magnets must be taken explicitly into account. Very often the dispersion is useful and is why spectrometers and analysing magnets are useful. At other times it becomes a problem such as in the microtron and neutron therapy systems, or in the transport of bunched beams where preserving the time structure of the beam is important.

The dispersion of a dipole magnet is analogous to the separation of white light into different colors by a glass prism. To first order, the theory is the same, although the derivation of the matrix elements is quite different. Also, the methods of correcting dispersive effects in light optics and ion optics are different.

As was noted earlier (see section 6.4), dispersion couples together the (x,θ) plane and the (ℓ,δ) plane. Thus, one must have a 4 x 4 matrix instead of a 2 x 2 matrix. For convenience, we will write it as follows:

$$R = \begin{bmatrix} R_{11} & R_{12} & 0 & d_x \\ R_{21} & R_{22} & 0 & d_\theta \\ \ell_x & \ell_\theta & 1 & L_{eff} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (42)$$

Here we write the dispersive terms R_{16} and R_{26} as d_x and d_θ to make their meaning more transparent.

Two important terms will now be defined:

NON-DISPERSIVE: A system is said to be non-dispersive when the matrix element $d_x = 0$. (When $d_x = 0$, $\ell_\theta = 0$ also.)

ACHROMATIC: A system is said to be achromatic when both $d_x = 0$ and $d_\theta = 0$. (Both $\ell_x = 0$ and $\ell_\theta = 0$ also.) Thus we see from the matrix, eq. 42, that a necessary condition for an optical system to be decoupled in each of the three pairs of canonically conjugate variables is for the system to be achromatic. Another necessary condition is that the system have midplane symmetry. This separates the (x,θ) plane from the (y,ϕ) plane as discussed earlier.

Misalignment (in particular a rotation about the optic axis) of dipoles and especially quadrupoles, leads to coupling of the (x,θ) and (y,ϕ) planes. If the system is also dispersive such misalignments lead to coupling of all three pairs of coordinates. As we will see later, such couplings can make it impossible to obtain the required beam properties.

Achromatic systems have another notable property. Not only is (x,θ) independent of (ℓ,δ) where $\delta = \Delta P/Po$, but they are also insensitive to small errors in the dipole field settings, because $\Delta\rho/\rho = \Delta P/Po = -\Delta B/Bo$.

The most commonly used achromatic bend is composed of two identical bending magnets (but in reflection) and a quadrupole singlet, as shown in Fig. 14.

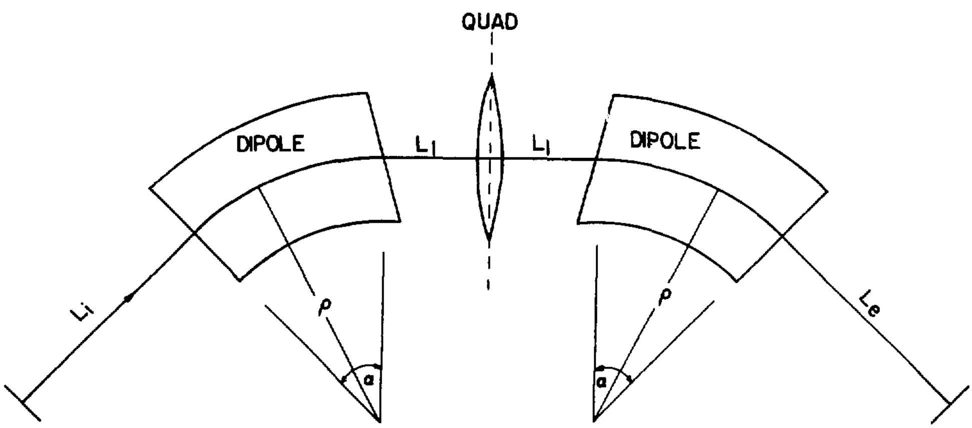

Fig. 14 Schematic representation of a simple achromatic bending system. The dashed line represents the symmetry plane of the system, i.e. the half to the right is a reflection of the half to the left. This system is made achromatic by adjusting the strength of the quadrupole. It appears that we can set two independent conditions ($d_x = d_\theta = 0$) with only one free parameter. This is not true because the imposition of exact reflection symmetry is equivalent to setting one parameter.

7. Emittance, Acceptance, Brightness and Phase-Space

Up until now we have dealt with the problem of how to "trace" the path of a single ray or particle through an optical system with the help of first order matrix transformations. In reality, we cannot deal with a single ray, but must deal with bundles of rays or particles. Often there are millions of particles present in these bundles. We can deal with this problem very elegantly by making use of the concept of phase-space. There exists a number of powerful mathematical theorems which allow us to find some "simple" solutions to what otherwise would be a very time-consuming and costly problem if solved by "brute-force". (i.e. perhaps by tracing hundreds of rays through the system.)

The total phase-space of the beam is the minimum "volume" occupied by all of the particles in the n-dimensional space required to describe the system. We will find out what this means in a few moments. Consider a typical bundle, such as the one represented in Fig. 15.

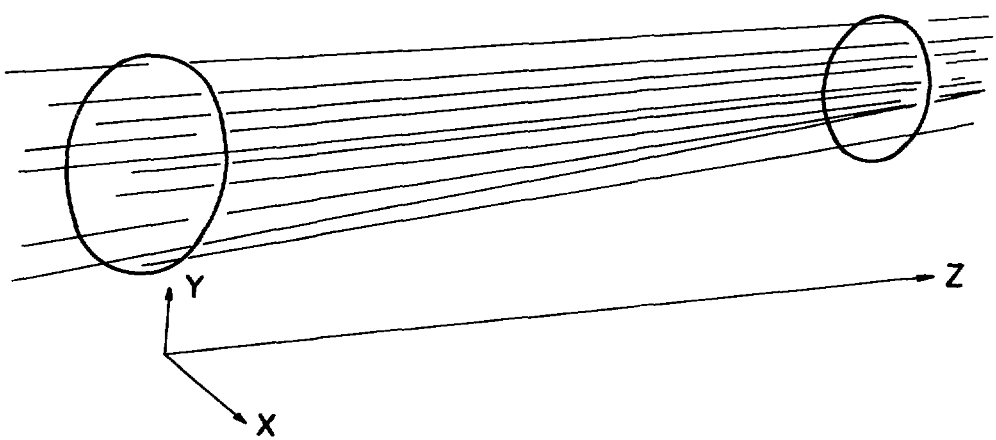

Fig. 15 Bundle of rays. All of the rays in this bundle will fit into a certain "volume" in phase-space. Once this volume has been determined it remains constant in non-accelerated systems, if no beam is lost through scattering or by running into something.

The "volume" in phase space can be defined as the product of all the canonical variables.

$$\varepsilon = x\theta y\phi \ell\delta \qquad (43)$$

Each particle in the bundle occupies a certain phase space volume given by eq. 43.

Somewhere there will be three particles that occupy the largest volume in the 6 dimensional space. This volume defines the maximum phase-space of the beam. All other "volumes" are smaller. A nice property of these volumes is that they can lie within each other! (There are other ways of defining phase-space, but I don't think they are easier to understand.) If the optical system has mid-plane symmetry and does not contain any dispersive elements, then the phase-space defined by eq. 43 reduces to 3 independent 2-dimensional phase-space "volumes" or areas.

$$\begin{aligned} \varepsilon_x &= x\theta \\ \varepsilon_y &= y\phi \qquad (44) \\ \varepsilon_\ell &= \ell\delta \end{aligned}$$

For simplicity, let us deal with only the phase-space in the (x,θ) plane. The particle with the maximum $x\theta$ defines the maximum phase-space area in the (x,θ) plane. It is said to have a phase-space or "emittance" of $\varepsilon_x = x\theta$ (cm-mrad) or whatever units one is using. (44a)
We note here that the phase-space or emittance is a property associated with canonical pairs of variables. It is not meaningful to define a phase-space from the xy variables, for example.

The concept of emittance is most useful when we know the percentage of the total intensity contained in a certain phase-space volume (or area). For example, perhaps 68% of all particles are contained in, say, a phase-space area of 1 cm-mrad (in the x-plane), 95% in 2 cm-mrad, 99% in 3 and 100% in 5. The emittance of this beam is 5 cm-mrad. However, if we are willing to throw away only 5% of the beam, the remaining 95% is contained in an area of only 2 cm-mrad! Perhaps we will decide that it's not economic to have a system with big enough apertures to accommodate the full beam. BE VERY VERY CAREFUL HERE, BECAUSE THE PHASE-SPACE AREA OF THE BEAM IS NOT THE SAME AS THE BEAM AREA IN THE X-Y PLANE! Also, although the maximum aperture requirements of a beam transport system are a function of the emittance of the beam, the relationship is not necessarily simple.

Now is a good time to clearly define some of the terms that have been used or are going to be used.

EMITTANCE: The emittance is a property of the BEAM. It is the PHASE-SPACE VOLUME OR AREA occupied by a specified intensity fraction of the beam (e.g. 95%).

BRIGHTNESS: The brightness of the beam is the INTENSITY of the beam within a specified PHASE-SPACE volume. The more current we have in a given phase-space volume, the brighter is the beam.

ACCEPTANCE: Acceptance is a property of a DEVICE The acceptance of an accelerator or beam transport system is the MAXIMUM phase-space that the device can ACCEPT . Often we must match the EMITTANCE of the beam to the ACCEPTANCE of the device. This includes not only matching the phase-space volumes or areas, but also matching the orientation and shape of the PHASE ELLIPSES.

Now that we have defined what the phase-space, or emittance, of the beam is, we must find out how it transforms through the system. The phase-space area or volume is not so useful if we only use it as defined in eq. 4_ and 44. We will define the so-called "beam matrix" as follows: If the motion of all of the particles is random (i.e. their trajectories are uncorrelated or independent of each other) as is more or less the case at the plasma boundary of an ion source or at an electron gun filament, then we can define a matrix by taking the outer product of the ray vectors defining the phase-space contour of interest - for example the maximum phase-space of the beam.

Define:

$$[\sigma] = [X][X]^T = \begin{bmatrix} x_o^2 & & & & & 0 \\ & \theta_o^2 & & & & \\ & & y_o^2 & & & \\ & & & \phi_o^2 & & \\ & & & & p_o^2 & \\ & 0 & & & & \delta_o^2 \end{bmatrix} \qquad (45)$$

Where T means the transpose of the matrix. (A matrix is transposed by interchanging its rows and columns.)

This matrix is a "quadratic form" in matrix algebra and under the right conditions it is an "invariant" which means its determinant remains constant as the matrix is "transformed" -- as we shall see shortly. If we can define the beam matrix as in eq. 45 at some place then we can find out what it looks like at any other place in the optical system as follows: We already know that

$$[X_1] = [R][X_o]$$

By definition,

$$[I] = [X_o]^T [\sigma_o]^{-1} [X_o] \quad (46)$$

Where $[I]$ is the unit matrix.

Also,

$$[X_o] = [R]^{-1} [X_1] \quad (47)$$

Thus, using the rules of matrix algebra, eq. 46 can be written as:

$$[I] = [R^{-1}X_1]^T [\sigma_0][R^{-1}X_1] = [X_1]^T [R^{-1}]^T [\sigma^{-1}][R^{-1}][X_1] \quad (48)$$

So that we can obtain the result

$$[\sigma_1] = \Big[[R^{-1}]^T [\sigma^{-1}] [R^{-1}] \Big]^{-1} = [R][\sigma_o][R]^T \quad (49)$$

We see from eq. 43 and 45 that the phase-space volume is

$$\varepsilon = \det|\sigma|^{1/2} \quad (50)$$

Going back to our 2 x 2 example, the above can be summarized as

$$[\sigma] = \begin{bmatrix} x \\ \theta \end{bmatrix} [x \quad \theta] = \begin{bmatrix} x^2 & 0 \\ 0 & \theta^2 \end{bmatrix} \quad (51)$$

and

$$\det|\sigma|^{1/2} = \sqrt{x^2\theta^2} = x\theta = \varepsilon_x. \tag{52}$$

As we have seen before, the optical transformation matricies have $\det|R| = 1$ for systems with constant energy. Matrix algebra tells us that if $\det|R| = 1$, then $\det|\sigma|$ is unchanged by the transformation given by eq. 49. We can invert this statement too. That is if we know that σ is invariant then $\det|R| = 1$. We can do this because of a very powerful theorem in classical mechanics called "Liouville's Theorem'.

LIOUVILLE'S THEOREM

For systems that satisfy Hamilton's equations (as given in section 6.3), the density of particles in the appropriate 2, 4, or 6-dimensional phase-space is invariant as we move (transform σ) through the system.

So, to summarize, phase-space is conserved, that is $\det|\sigma|$ = constant, or "invariant" if:

i) the beam is not accelerated or decelerated

ii) there are no beam losses. (Part of the beam does not run into something.)

iii) there is no scattering of the beam by gas or foils

iv) space-charge effects are negligible.

In general, it is the 6-dimensional phase-space that is conserved. If we also have midplane symmetry and an achromatic system (or a system without dipoles), then clearly the three 2 x 2 phase-space areas are conserved independently of each other and in general will have different areas.

If the system contains dispersive elements (dipoles) then the 2-dimensional phase-space in the dispersive, or X-plane is not conserved. However, the 4-dimensional phase-space

$$\varepsilon = x\theta\ell\delta \tag{53}$$

is conserved. The reason that ε_x, the (x,θ) phase-space area is not conserved by itself is because the dispersion mixes the (x,θ) and (ℓ,δ) coordinates as we saw earlier. Thus one must be very careful when measuring or using the 2-dimensional phase ellipses, because dispersive effects can make it appear that the emittance has increased even though this is not really true. Nevertheless, the beam width and hence aperture requirements as well as the maximum beam angle are increased in the (x,θ) plane by the dispersion.

Equations 45 and 51 can be shown to be identical with the equations of an ellipse, in 6 and 2 dimensions respectively. In order to see how this works, let us consider once again a 2-dimensional example. The 2-dimensional phase space is given by eq. 51 for an "uncorrelated" beam. This matrix is equivalent to the equation for an ellipse that is oriented along the X-θ axes. (i.e. The major and minor axes of the ellipse lie along the axes of the canonical coordinates.) This is shown in Fig. 16. It is easy to show that in this case the equation has the form.

$$\gamma x^2 + \beta\theta^2 = \varepsilon_x \qquad (54)$$

Here $\beta = x_0^2/\varepsilon_x$ and $\gamma = \theta_0^2/\varepsilon_x$

If this ellipse represents the phase-space for 100% of the beam then the contours for 95% and 68% of the beam intensity, for example, will be concentric ellipses lying inside the 100% contour. One of the most useful and most powerful properties of phase-space is that if a particle lies on the ellipse shown in Fig. 16, it always lies on the ellipse no matter how the ellipse is rotated or stretched. Perhaps a more important fact is that a particle that lies somewhere inside a given phase ellipse can never move outside that ellipse (unless it suffers some interaction that violates Liouville's Theorem). Thus, particles that lie between the 95% and 100% contours at some point in the system will always lie between these two contours.

Let us see how our phase ellipse, equations 51 and 54, transforms through a drift space of length L. Making use of eq. 49, we can write

$$[\sigma_1] = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_o^2 & 0 \\ 0 & \theta_o^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} = \begin{bmatrix} x_o^2 + L^2\theta_o^2, & L\theta_o \\ L\theta_o & \theta_o^2 \end{bmatrix} \quad (55)$$

Equation 55 can be written as

$$[\sigma_1] = \begin{bmatrix} x_1^2 & rx_1\theta_1 \\ rx_1\theta_1 & \theta_1^2 \end{bmatrix} \quad (56)$$

Where "r" is the statistical correlation coefficient which is a measure of the degree to which x and θ are correlated or dependent upon each other. Formally we can define the correlation coefficient as follows:

$$r = \frac{\langle x\theta\rangle}{\langle x\rangle\langle\theta\rangle} \quad (57)$$

where $\langle\rangle$ means average value or expectation value of the quantity within the symbols $\langle\rangle$. The equation of the ellipse representing eq. 55 and 56 has the form:

$$\gamma x^2 + 2\alpha x\theta + \beta\theta^2 = \varepsilon_x \quad (58)$$

This ellipse is shown in Fig. 16b. We see that the original ellipse, Fig. 16a, has been rotated to the right and stretched. Nevertheless, the areas of the two ellipses are identical and are given by

$$A = \pi \det |\sigma| = \pi \varepsilon_x \tag{59}$$

This is easily demonstrated by direct calculation of the determinant, which for a 2 x 2 matrix is

$$\det |\sigma| = \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} \tag{60}$$

The coefficients of the equation of the rotated ellipse, eq. 58, can be found from the following matrix equation:

$$[\sigma_1] = \begin{bmatrix} x_1^2 & rx_1\theta_1 \\ rx_1\theta_1 & \theta_1^2 \end{bmatrix} = \varepsilon \begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix} \tag{61}$$

β, γ, and α are the so-called "Twiss" parameters.

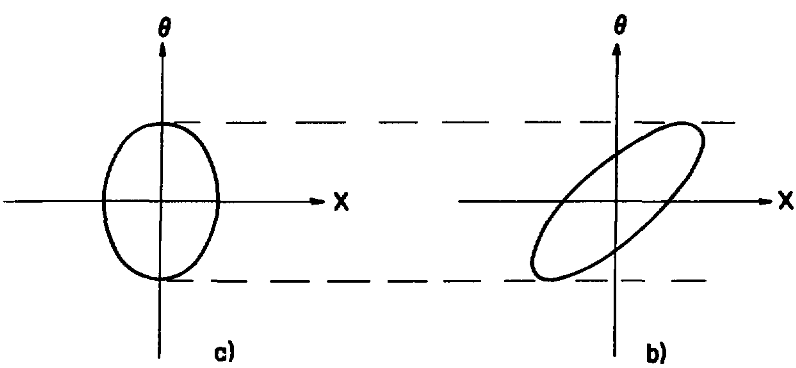

Fig. 16a Erect phase ellipse for an uncorrelated beam as given by eq. 50 and 54.

Fig. 16b Rotated phase ellipse after drifting a distance L. See eq. 55 and 58. Both θ and ε_x remain constant even though x increases.

NOTE: The shape of the phase ellipse is NOT the same as the shape of the beam. One of the coordinates of the phase ellipse is the angle of the beam envelope with respect to the optic axis. This one cannot "see".

We can understand physically why the phase ellipse is rotated and stretched in Fig. 16 by studying Fig. 17.

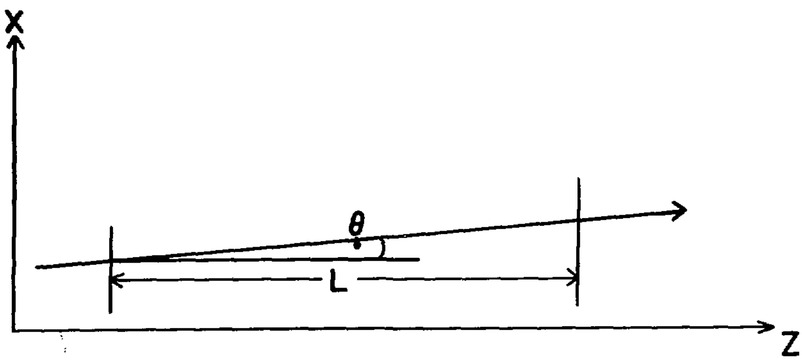

Fig. 17 Transformation of the X-coordinate through a drift distance L. The angular coordinate is a constant and the X-coordinate increases linearly with L.

We also see from equations 55 and 56 that if the drift distance L is very large, the correlation coefficient, r, approaches unity. In this case the phase ellipse approaches a straight line. The area of the ellipse remains constant, however, even for this extreme case.

In an analogous way, we can see how the phase ellipse transforms under the action of a thin lens.

In analogy to eq. 55 we have:

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_o^2 & 0 \\ 0 & \theta_o^2 \end{bmatrix} \begin{bmatrix} 1 & -1/f \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_o^2 & -x_o^2/f \\ -x_o^2/f & \left(\frac{x_o}{f}\right)^2 + \theta_o^2 \end{bmatrix} \qquad (62)$$

where we assume convergence. (See eq. 36.)

Equation (62) can obviously be written in the form of eq. 56 and eq. 58. The initial and transformed phase ellipses are shown in Figs. 18a and b respectively. In this case the emittance ellipse is rotated to the left, instead of to right as for a drift (see eq. 55 and Fig. 16b). Note that in this case the beam width x is constant and the angle increases as the focal length decreases.

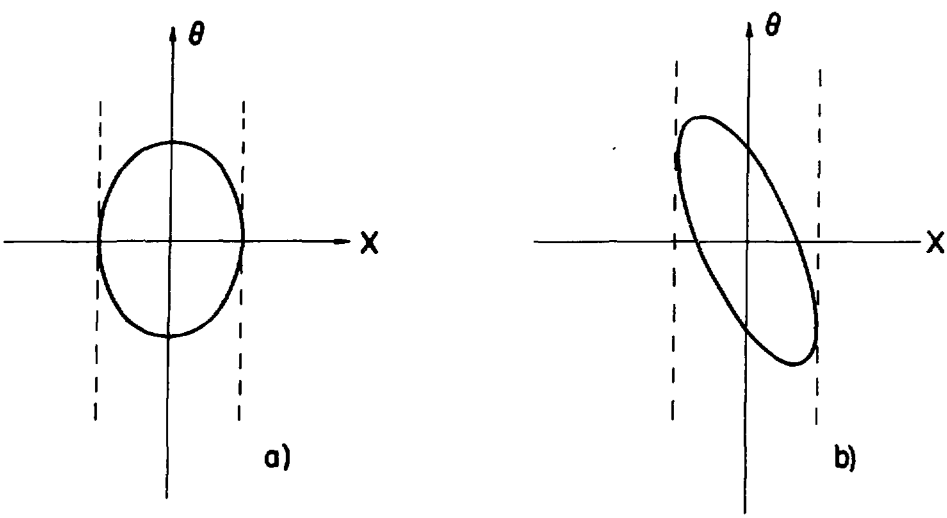

Fig. 18 a) Initial phase ellipse as in Fig. 16a.
b) Rotated phase ellipse after the action of a focussing lens.

We should note here also that phase-space is ELLIPTICAL only if FIRST ORDER optics is valid. When aberrations are included the phase-space is no longer elliptical but its area is still conserved. In extreme cases, such as when passing through the fringe field of a cyclotron, for example, the phase-space can look more like a hockey stick than an ellipse. If we have a situation that is describable to first order, we can obtain some very important information about the beam properties from the beam matrix. For instance, the maximum size of the beam is given by

$$x_{max} = \sqrt{\sigma_{11}} = \sqrt{x^2} \tag{63}$$

The maximum angle of the beam is given by

$$\theta_{max} = \sqrt{\sigma_{22}} = \sqrt{\theta^2} \tag{64}$$

Thus, it is easy to trace the maximum beam size in the X-plane by calculating the beam matrix at various points using eq. 61. A very useful thing to do is to plot x along the system and obtain "a beam envelope plot". The beam envelope plot for the 100% phase-space contour tells us what the aperture of the system must be to pass all of the beam. A representative example is shown in Fig. 19.

Let us now look at some other information that we can obtain from the beam matrix. The points where the phase ellipse intersects the coordinate axes are as follows:

$$\text{X-intercept} \quad x_{int} = \sqrt{\sigma_{11}(1 - r^2)} = \sqrt{x^2(1 - r^2)} \tag{65a}$$

$$\theta\text{-intercept} \quad \theta_{int} = \sqrt{\sigma_{22}(1 - r^2)} = \sqrt{\theta^2(1 - r^2)} \tag{65b}$$

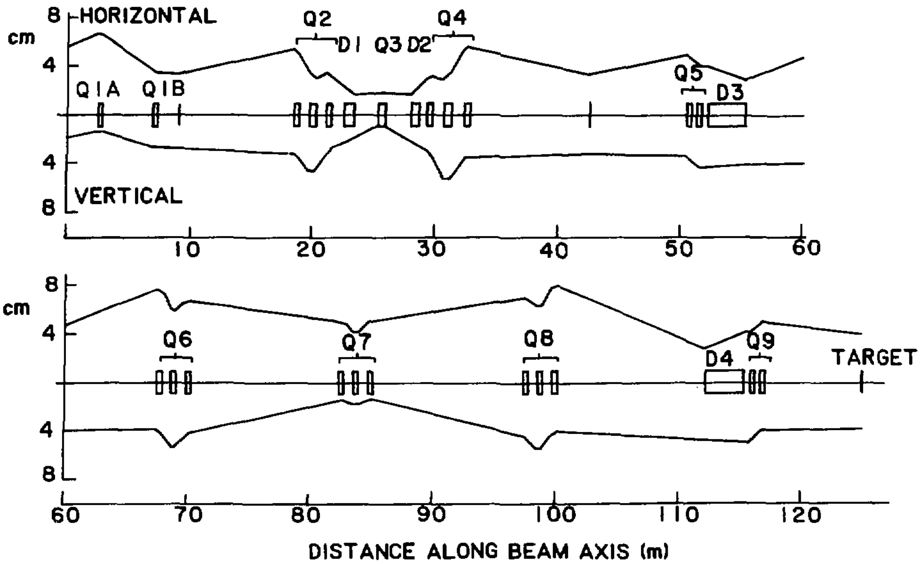

Fig. 19 Representative beam envelope for the X and Y planes of a beam transport system.

A **WAIST** occurs when the correlation coefficient r = 0. That is when the major and minor axes of the phase ellipse are oriented along the coordinate axes. A waist is not at all the same as a focus as we will see from the following example. Fig. 20 shows a simple optical system that forms an "image" of a waist with unity magnification. For convenience, the focal length is assumed to be f = -1. The focus or image always occurs at the same place as noted in the figure, but the position and size of the waist depend upon the phase-space of the beam as can be seen in Table 1.

The phase ellipse at the image plane or focal plane of the example in Fig. 20 can be found from eq. 49 as in our previous example. Thus:

$$\begin{bmatrix} -1 & 0 \\ -1/f & -1 \end{bmatrix} \begin{bmatrix} x_o^2 & 0 \\ 0 & \theta_o^2 \end{bmatrix} \begin{bmatrix} -1 & -1/f \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} x_o^2 & x_o^2/f \\ x_o^2/f & (x_o/f)^2 + \theta_o^2 \end{bmatrix} \qquad (66)$$

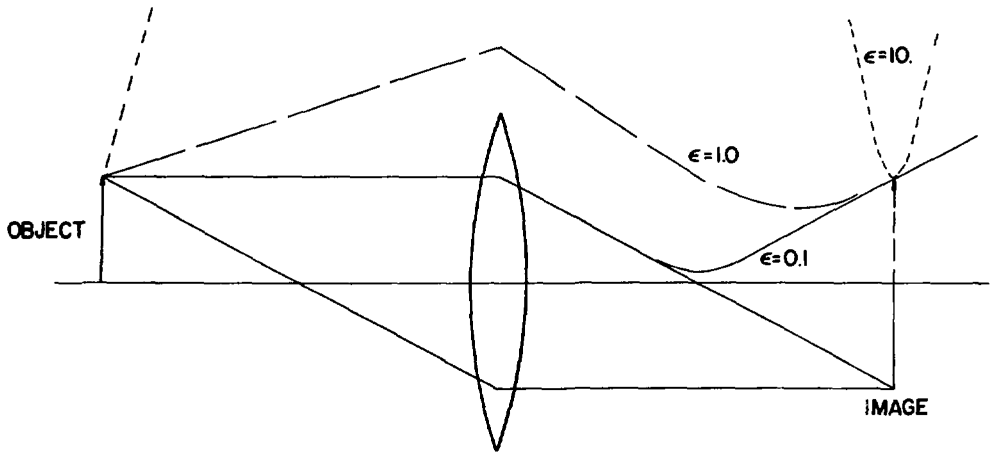

Fig. 20 A simple optical system which forms an image of an initial object which is also a waist. The image is independent of the emittance of the beam, but the position and size of the waist is very dependent on the emittance of the beam. See text for clarification.

So we see that at the focal point,

$$x_1 = x_o, \tag{67a}$$

$$\theta_1 = \sqrt{\theta_o^2 + (x_o/f)^2} \tag{67b}$$

and

$$r = \frac{+x_o/f}{\left[\theta_o^2 + (x_o/f)^2\right]^{1/2}} \tag{66c}$$

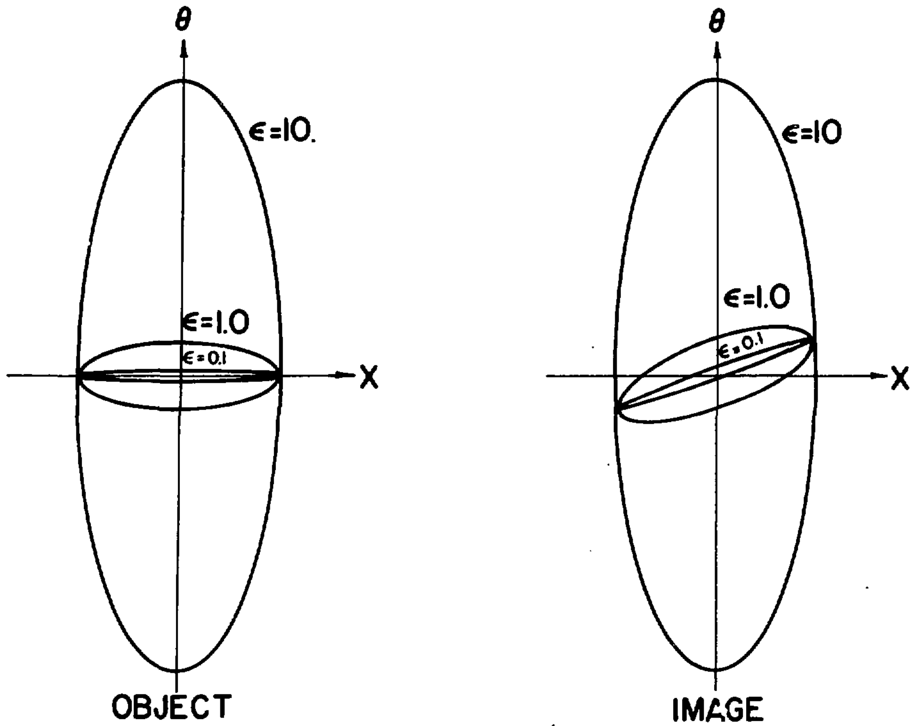

Fig. 21 Object and Image phase-space or emittance ellipses for the example shown in Fig. 20.

Table 1 shows the beam parameters at the object and image planes, as well as the distance from the image plane to the waist for 3 different values of the initial phase-space. The example illustrates some of the problems and differences one encounters when one deals with images and waists. Images are easier to calculate, but are hard to establish in practice. Waists are easier to find, because if we adjust a lens such that we minimize the beam width, then we are very near a waist. The problem is that the lens strength required to produce a waist depends upon the magnitude of the beam emittance -- if the emittance changes because we scrape off beam somewhere, then the position of the waist will also change.

The distance from the focus to the waist can be found by solving for the distance Lw that is needed to transform eq. 66 into a diagonal matrix. This distance is

$$Lw = -rx_1/\theta_1 \tag{68a}$$

The size of the waist at Lw is

$$Xw = x_1\sqrt{1 - r^2} \tag{68b}$$

These two equations are very useful and should be remembered!

Table 1

Xo = 1 in all cases.

θo	ε_x	X_1	θ_1	r	Lw*	Xw
.1	.1	1	1.005	+.99501	-.9901	.09978
1.	1.	1	1.414	+.70711	-.5000	.70711
10.	10.	1	10.05	+.09950	-.0099	.99504

*Note: Lw is the distance from the focal plane to the waist in units of the focal length.

There is one other difference between a waist and a focus that is worth noting; a focus is stable with respect to changes in the average beam direction at the object position, a waist is not.

Before finishing, perhaps it is worth noting how the phase space of a beam transforms in an accelerated system. Here I will not go into details, but only point out that the AREA of the phase ellipse for each of the 3 pairs of canonically conjugate variables, (x,θ), (y,ϕ), and (ℓ,δ) is inversely proportional to the beam momentum. That is:

$$\varepsilon_x(P_1) = \varepsilon_x(P_o)P_o/P_1 \text{ etc.} \tag{69}$$

Thus as the beam momentum (and energy) increases, the phase space decreases if "noise" is not added to the system at the same time.

These lectures have presented a brief, perhaps too brief, introduction to the basic concepts of beam optics. In some cases there are many pages of algebra connecting the results presented here. Some of the concepts, such as phase-space, are not easy to understand, however one is not required to understand everything in order to make use of some of the results. The only problem then is being sure that the formulas or concepts fit the situation under consideration.

APPENDIX

The first order transformations for a drift, a quadrupole singlet and a uniform field bending magnet are given for the convenience of the readers and agree in format to those given in ref. 4.

(a) Drift Space:

$$\begin{bmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma^{-2}L \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ \theta \\ y \\ \phi \\ \ell \\ \delta \end{bmatrix}$$

where $L = s_1 - s_0$ the length of the central ray of the drift, $\gamma = (1 - \beta^2)^{-1}$ and $\beta = v/c$; c is the velocity of light.

(b) Quadrupole singlet:

$$\begin{bmatrix} \cos kL & k^{-1}\sin kL & 0 & 0 & 0 & 0 \\ -k \sin kL & \cos kL & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh kL & k^{-1} \sinh kL & 0 & 0 \\ 0 & 0 & k \sinh kL & \cosh kL & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma^{-2}L \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $k^2 = (1/B\rho)(\partial B_y/\partial x)$.

(c) Uniform field bending magnet

$$\begin{bmatrix} \cos \alpha & \rho \sin \alpha & 0 & 0 & 0 & \rho(1-\cos \alpha) \\ -\rho^{-1}\sin \alpha & \cos \alpha & 0 & 0 & 0 & \sin \alpha \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin \alpha & -\rho(1-\cos a) & 0 & 0 & 1 & \gamma^{-2}L-\rho(\alpha-\sin \alpha) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where α is the bending angle of the central ray of the magnet, and ρ is the radius of curvature.

The matrix elements of a rotated pole face are given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \tan\beta/\rho & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\tan\beta/\rho & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Bibliography

There are many books and articles covering the topics of ion optics, classical mechanics (including relativistic mechanics) and classical electricity and magnetism. The ones listed here are chosen on the basis of personal prejudice but are in any event not only readable but give thorough accounts of the subject matter.

1) Classical Mechanics by Herbert Goldstein, Addison-Wesley, Reading, MASS. (1959)

2) Classical Electricity and Magnetism, by W.K.H. Panofsky and Melba Phillips, Addison-Wesley, Reading, MASS. (1955)

3) Classical Electrodynamics, by J.D. Jackson, John Wiley and Sons Inc., New York (1962)

4) A First and Second-Order Matrix Theory for the Design of Beam Transport Systems and Charged Particle Spectrometers, K.L. Brown SLAC Report #75, (1967).

5) Calculations of Properties of Magnetic Deflection Systems, S. Penner, Rev. Sci. Instr. 32 (1961) 150.

6) Transport, A Computer Program for Designing Charged Particle Beam Transport Systems, K.L. Brown and S.K. Howry SLAC Report #91, (1970)

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