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Differential-Difference Equation of the Glauber-Lachs and Perina-McGill Formula, QCD Branching Processes and Hadronization

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Abstract

By the use of the Poisson transformation or the method of difference, we obtain a differential-difference equation from the Fokker-Planck equation for the Glauber-Lachs and Peřina-McGill formula. It is found that a resulting equation is a same type equation of the QCD branching processes. From solutions in these equations, we can infer possible mechanisms for hadronization: The randomization in stochastic theory seems to be a plausible procedure. Some data(diffractive KNO scaling distribution and that of a single jet in e^+e^- annihilations) are analysed in our scheme.

Introduction. For recent several years, many authors[1)~3)] have stressed that the Planck-Ploya-Eggenberger distribution is a fundamental probability distribution, since it is the solution of the gluon distribution in the QCD barnching processes and, futhermore, it satisfies the KNO scaling observed at high energies.[4),5)] However recently an availability of the Peřina-McGill probability distribution (i.e., the generalized Glauber-Lachs formula: hereafter abbreviated as the GLPM formula) for hadronic reactions have been pointed out.[6),7)] Futhermore it is proved that the KNO scaling functions based on the geometrical aspect are connected with the stochastic processes of the GLPM formula.[8)] Therefore, in this paper, it is worth while to examine physical- and stochastic connections among the GLPM formula, Planck-Polya distribution and the QCD branching processes. For our purpose, we have to obtain the differential-difference(DD) equation of the GLPM formula, because tbe fundamental equation in the QCD branching processes is given by the DD equation.

DD Equation of the GLPM Formula. The Fokker-Planck equation for the probability density (i.e., the KNO scaling function) in the GLPM formula is given as follows(see also ref.7)),

$$\begin{cases} \dfrac{\partial}{\partial t}\Psi_k(z,t) = -\dfrac{\partial}{\partial z}\left[a(z) - \dfrac{1}{2}\dfrac{\partial}{\partial z}b(z)\right]\Psi_k(z,t)\,, & (1a) \\ a(z) = -s(z-1) \quad \text{and} \quad b(z) = (2s/k)\,z\,, & (1b) \end{cases}$$

$$\Psi_k(z,t) = \frac{k}{1-e^{-st}}\left(\frac{z}{e^{-st}}\right)^{(k-1)/2} \exp\left[-\frac{k(z+e^{-st})}{1-e^{-st}}\right] \times I_{(k-1)}\left[2k\,\frac{(z\,e^{-st})^{1/2}}{1-e^{-st}}\right], \qquad (1c)$$

In order to get the DD equation of eq.(1), i.e.,the master equation, it is useful to utilize the Poisson transformation[8)]

$$\int_c^{\infty} W^n/n! \cdot \exp(-W) \times [\Psi_k(t)] \big|_{t=n/\langle n\rangle} dW/\langle n\rangle . \qquad (2)$$

Otherwise, we can use the well-known replacement in difference calculas between the discrete- and continuous variables,

$$\partial \Psi_k/\partial t \longrightarrow [P_k(n) - P_k(n-1)], \qquad (3a)$$

$$\partial^2 \Psi_k/\partial t^2 \longrightarrow [P_k(n+1) - 2P_k(n) + P_k(n-1)] \qquad (3b)$$

By making use of eqs.(2) and/or (3), we obtain the following DD equation from eq.(1),

$$\partial P_k(n)/\partial t = s\,[(n+1)P_k(n+1) - nP_k(n)] - s\langle n\rangle [P_k(n) - P_k(n-1)]$$

$$+ s\langle n\rangle/k\,[(n+1)P_k(n+1) - 2nP_k(n) + (n-1)P_k(n-1)], \qquad (4a)$$

$$P_k(n) = \frac{(p\langle n\rangle/k)^n}{(1+p\langle n\rangle/k)^{n+k}} \exp\left[-\frac{\gamma p\langle n\rangle}{1+p\langle n\rangle/k}\right] L_n^{(k-1)}\left(-\frac{\gamma k}{1+p\langle n\rangle/k}\right), \qquad (4b)$$

where γ and p are, respectively, given as follows,

$$\gamma = \langle n_{c}(\text{ coherent })\rangle \,/\, \langle n_{in}(\text{ incoherent})\rangle , \qquad (4c)$$

$$p = 1/(1+\gamma) = (1 - \exp(-st)). \qquad (4d)$$

We can prove that no-singularity occurs in the Poisson transformation, by making use of the numerical computaion of both sides in eq.(4a) with eq.(4b) and/or the way of the generating function for them. In the next paraghraph, we resolve eq.(4a) with two different boundary conditions. Probably we can expect that the procedure might elucidate essential difference between the GLPM formula and results from the QCD branching processes.[1)~3)] The physical meaning of eq.(1c) elucidated in refs.6) and 7) is demonstrated in Fig.1.

/ Fig. 1 /

Generating Function's Method and Eq.(4). Here we solve eq.(4a) by using the generating function,

$$Q_k(w) = \sum_{n=0}^{\infty} P_k(n)\, w^n. \tag{5}$$

Equation (4) is rewritten as follows,

$$\partial Q_k(w)/\partial t = \lambda_0 (w-1) Q_k + [\lambda_1 - (\lambda_1+\lambda_2) w + \lambda_2 w^2]\, \partial Q_k/\partial w, \tag{6}$$

where the following substitutions are made,

$$\lambda_0 = s\langle n\rangle \;,\; \lambda_1 = s\langle n\rangle/k + s \quad \text{and} \quad \lambda_2 = s\langle n\rangle/k \quad .(7)$$

From eq.(6), we obtain the following equation

$$\begin{cases} (\lambda_2 w - \lambda_1)/(w-1) = A \exp[(\lambda_2-\lambda_1)t], & (8a) \\ Q_k(w,t) = B\left[\dfrac{-(\lambda_2-\lambda_1)A \exp[(\lambda_2-\lambda_1)t]}{\lambda_2 - A\exp[(\lambda_2-\lambda_1)t]}\right]^{-\lambda_0/\lambda_2}, & (8b) \end{cases}$$

where A and B are constants to be determined by the boundary condition.

Here we consider two cases of the boundary conditions.

I) The first case: At t = 0, the probability density is given by the Poisson distribution, which is obtained by the following calculation,

$$\int_0^{\infty} \frac{W^n}{n!} \exp(-W)\, \delta(z-1)\Big|_{z=W/\langle n\rangle} \frac{dW}{\langle n\rangle} = \frac{\langle n\rangle^n}{n!} \exp(-\langle n\rangle). \tag{9a}$$

Introducing a parameter β, we obtain the following boundary conditions for the generating function,

$$P_k(n,0) = \frac{(\beta\langle n\rangle)^n}{n!} \exp(-\beta\langle n\rangle), \tag{9b}$$

$$Q_k(w,0) = \sum_{n=0}^{\infty} P_k(n,0)\, w^n = \exp[\beta\langle n\rangle (w-1)] \tag{9c}$$

Combining eqs.(8) and (9), $Q_k(w,t)$ is expressed as follows,

$$Q_k(w,t) = \left\{1-(w-1)\frac{\lambda_2}{\lambda_2-\lambda_1}\left[\exp[(\lambda_2-\lambda_1)t]-1\right]\right\}^{-\lambda_0/\lambda_2} \cdot$$

$$\times \exp\left\{\beta\langle n\rangle (w-1) \exp[(\lambda_2-\lambda_1)t] \Big/ \left[1-(w-1)\frac{\lambda_2}{\lambda_2-\lambda_1}\right.\right. \tag{10}$$

$$\left.\left.\times(\exp[(\lambda_2-\lambda_1)t]-1)\right]\right\}$$

The GLPM formula is calculated by the well-known formula,

$$P_k(n) = 1/n!\ \partial^n/\partial w^n\ Q_k(w,t)\Big|_{w=0}$$

$$= \frac{(p\langle n\rangle/k)^n}{(1+p\langle n\rangle/k)^{n+k}} \exp\left[-\frac{\beta(1-p)\langle n\rangle}{1+p\langle n\rangle/k}\right] L_n^{(k-1)}\left(-\frac{\beta(1-p)}{p(1+p\langle n\rangle/k)}\right). \quad (11)$$

II) The second case. When we choose the following initial condition,

$$P_k(n,t=0) = \delta_{n,0} \quad (\text{ or } \ Q_k(w,0) = 1), \qquad (12)$$

we obtain the generating function $Q_k(w,t)$ from eq.(10), putting $\beta = 0$. The following Planck-Polya probability distribution is easily derived,

$$P_k(n) = 1/n!\ \partial^n/\partial w^n\ Q_k(w,t)\Big|_{w=0}$$

$$= k(k+1)\cdots(k+n-1)/n!\ \{p\langle n\rangle/k\}^n/\{1+p\langle n\rangle/k\}^{n+k}. \quad (13)$$

An average multiplicity is expressed as follows,

$$\langle n\rangle_{AV} = \partial/\partial w\ Q_k(w,t)\Big|_{w=1} = p\langle n\rangle . \qquad (14)$$

As p defined by eq.(4d) contains the time-dependent term, $\langle n\rangle_{Av}$ varies with time. Furthermore, it should be noticed that we also obtain the Planck-Polya distribution in the case of $\lambda_1 = 0$ in eq.(6).

<u>QCD Branching Processes.</u> According to ref.2), we give the equation of the QCD branching processes with the evolution function $\tilde{P}$ and the QCD proper parameter $Y = [\ln Q^2/\Lambda^2/\ln Q_0^2/\Lambda^2]/2\pi b$ where $12\pi b = (11N_c - 2N_f)$,

$$\partial\tilde{P}(n_g,n_f)/\partial Y = [-\tilde{A}n_f - Bn_g - An_g]\ \tilde{P}(n_g,n_f)$$
$$+ \tilde{A}n_f\tilde{P}(n_g-1,n_f) + A(n_g-1)\tilde{P}(n_g-1,n_f)$$
$$+ B(n_g+1)\tilde{P}(n_g+1,n_f-2). \qquad (15a)$$

The parameters A, $\widetilde{A}$ and B introduced in eq.(15a) denote the following processes: A;$(g \to g + g)$, $\widetilde{A}$; $(q \to g + q)$ and B; $(g \to q + \bar{q})$. Since to find the general solution is too difficult, we assume that n_i quarks are produced initially (mainly at Y = 0), and they can emit gluons through process ($\widetilde{A}$): $\widetilde{A}\, n_q(\text{quark}) = n_i \widetilde{\lambda}_0$. Quarks produced from gluons through process (B) cannot emit gluons futher. Without the loss of generality, this case study of the gluon distribution in the quark jet allows us to analyse the QCD branching processes.

$$\begin{aligned}
\partial \widetilde{P}(n_i, n_g, n_f)/\partial Y &= n_i \widetilde{\lambda}_0 \widetilde{P}(n_i, n_g-1, n_f) - n_i \widetilde{\lambda}_0 \widetilde{P}(n_i, n_g, n_f) \\
&+ (n_g+1) \widetilde{\lambda}_1 \widetilde{P}(n_i, n_g+1, n_f-2) - n_g \widetilde{\lambda}_1 \widetilde{P}(n_i, n_g, n_f) \\
&+ (n_g-1) \widetilde{\lambda}_2 \widetilde{P}(n_i, n_g-1, n_f) - n_g \widetilde{\lambda}_2 \widetilde{P}(n_i, n_g, n_f), \qquad (15b)
\end{aligned}$$

where $\widetilde{\lambda}_0 = (N_c^2-1)/(2N_c\varepsilon)$, $\widetilde{\lambda}_1 = N_f/3$ and $\widetilde{\lambda}_2 = A = N_c/\varepsilon$.

The parametars N_c,N_f and ε are the numbers of colors, flavors and cutoff, respectively. Comparing eqs.(4a) and (15b), the DD equation of the GLPM formula is the same type as the QCD branching process for the gluon distribution in the quark jet. Nobody noticed this similarity, in so far as the authors know. With an initial condition, $n_g = 0$ and j quarks are produced at Y = 0, we obtain the following probability distribution

$$\widetilde{P}_a(n, \delta) = \frac{\delta(\delta+1)\cdots(\delta+n-1)}{n!}\left[\frac{\langle n_g\rangle}{a}\right]^n \Bigg/ \left[1+\frac{\langle n_g\rangle}{a}\right]^{n+\delta} \Bigg|_{\delta=a} \qquad (16)$$

where $a = j\widetilde{\lambda}_0/\widetilde{\lambda}_1$ and $\langle n_g\rangle = \widetilde{\lambda}_0/(\widetilde{\lambda}_2 - \widetilde{\lambda}_1)(\exp((\widetilde{\lambda}_2 - \widetilde{\lambda}_1)Y) - 1)$.
It should be emphasized that eq.(15b) descibes the multiplicity of the soft gluons.

Secondly if we choose the other boundary condition, i.e., eq.(9a), we obtain eq.(11) with substituting time (t) by Y. However, since this initial condition seems to be unnatural for the multiplicity of the soft gluons, , we abandon this problem here. Now we have eq.(16) for the QCD-world and eq.(1c) (i.e., the GLPM formula) and/or the Planck-Polya distribution for the typical description of the meson-world. Thus we must consider physically smooth connections between them.(see Fig.2).

/ Fig.2 /

Hadronization Problem from Eq.(16). At least there are two possiblities. I) Suppose that the Planck-Polya distribution for the meson distribution and eq.(16) are correct, then the physical gap between the produced meson-world (mainly pions) and the QCD-world can easily be filled up by the following substitution,

$$\langle n_g \rangle / a \longrightarrow \langle n(\text{incoherent pion}) \rangle / k. \qquad (17)$$

It is well-known that eq.(16) with eq.(17) satisfies the KNO scaling,[2),5)] and the KNO scaling distributions with rapidity cutoffs (mainly central region) can be explained in this scheme.[6),7)]

II) If the physical contents in eqs.(4b) and eq.(16) are true, we have to find another possibility of the hadronization between them. In order to bridge eqs.(16) to (4b), a randomization method in the stochastic theory [9)] seems to be useful, which is defined as follows (see also the flow chart in Fig.2),

$$\langle n_g \rangle / a \longrightarrow p \langle n(\text{pion}) \rangle / k, \qquad (18a)$$

$$c \left(= a = j\tilde{\lambda}_0/\tilde{\lambda}_1\right) \longrightarrow i \,(\text{integer}) + k, \qquad (18b)$$

$$i = k\gamma = k\langle n_{ee}\rangle / \langle n_{in}\rangle \qquad (18c)$$

$$P_k(n) = \sum_{i=0}^{\infty} \tilde{P}_k(n, i+k)\, c^i \exp(-c)/i! = \text{Eq.}(4b) \qquad (19)$$

Contents of eq.(18) are shown in Fig.3. At $t = 0$, $n_g = n_{in} = 0$ and coherent dominance are certainly seen. Physical interpretations of eq.(19), an effect of hard gluon, is demonstrated in Fig.3.c). This physical picture is also found in ref.10).

Of course we know that eq.(19) satisfies the KNO scaling and it does excellently explain a wide range of experimental data:[6),7)] It should be emphasized that coherent component is necessary in those comparisons. Moreover, we have to add some analyses.

Comparisons of Data with Eq.(1c). Here we show new comparisons of data with eq.(1c). The diffractive KNO scaling distributions in hadron collisions,[11),12)] distribution in a single jet and full KNO scaling distributions in e^+e^- annihilations are analysed in Fig.4. It is very interesting that the former two data can be explained by eq.(1c) with the *same parameters* ($k = 2$ and $\gamma = 4.$). The analyses sugget us the coherent dominance in the fragmentation regions. In other words, Poisson-like component is necessary in analyses of the KNO scaling distributions. The full KNO scaling distribution in the two-jet ($e^+e^- \rightarrow q\,\bar{q}$) and an addition of three-jet ($e^+e^- \rightarrow q\,\bar{q}\,g$) by about 10 % can be explained by the parameters ($k = 4$) with smaller value, $\gamma = 3$. This comparison means that there is a weak correlation in two-jet, and in three-jet, respectively.

This fact also suggests us that the randomization is one of plausible and possible hadronization mechanisms. Futhermore eq.(18) might explain the integal number of meson charges from the fractional number of the quark charges.

/ Figs.3 and 4./

Conclusions and Discussions. By the use of the Poisson transformation, we have obtained the DD equation of the GLPM formula. This is essentially the same type of equation obtained in the QCD branching branching process. This fact seems to suggest us the similarity between QCD branching process [1),2)] and quantum optics.[14)] Comparing eq.(4) with eq.(16), we can infer possible mechanisms of hadronization: This is the randomization. To compare our results with those in ref. 15), we have to wait for various data.

Equation (4), i.e., DD equation and its solution are not known in the stochastic theory, as far as we know.[16)] However, the GLPM formula and its probability density have been used in electrical engineering,[17),18)] theoretical psychology,[19)] the critical phenomena[20)] as well as quantum optics for a comparably long time.

Thus we expect that our study sheds light on these fields.

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References

1) K.Konishi,A.Ukawa and G.Veneziano, Nucl.Phys. B157(1979),45.

2) A.Givannini, Nucl.Phys. B161(1979),429: See also M.Anselmino,A.Givannini and G.Vercellino, Nuo.Cim., 62A(1981),253.

3) A.Bassetto,M.Ciafaloni and G.Marchesini, Nucl.Phys. B163(1980),477: Somewhat different views about the KNO scaling function from refs. 1) and 2) are mentioned.

4) Z.Koba,H.B.Nielsen and P.Olesen, Nucl.Phys. B40(1972), 317.

5) P.Carruthers and C.C.Shih, Phys. Lett. 127B(1983),242: See also M.Blazek, contribution paper at the Hadron Structure '83 conference (Smolenice 1983).

6) M.Biyajima, Prog.Theor.Phys. 69(1983), 966: (a) 70(1983),1468.

7) M.Biyajima, Phys. Lett.B.in press : See also P.Carruthers and Shih, LA-UR-83-3544, Phys. Lett.B. in press

8) M.Biyajima, Phys. Lett.B. in press : See also M.Le Bellac,J.L.Meurier and G.Plaut, Nucl.Phys. B62(1973),350: They called eq.(2) a "compound Poisson distribution".

9) W.Feller, "An Introduction to Probability Theory and Its Applications" Vol. II, (Wiley, New York, 1966): See also ref.15).

10) J.D.Bjorken, in "Physics in Collision." ed.by P.Carlson and W.P.Trower, (Plenum Press, New York and London).p.343. P.Soeding, in the Proceedings of the International Europhysics Conference on High Energy Physics. (Rutherford Lab.,1983). p.567.

11) L.Cool et al., Phys.Rev.Lett., 48(1982),1451.

12) S.Barshay, Phys.Rev.Lett., 49(1982),1609.

13) M.Althoff et al., TASSO collaboration, DESY 83-130(1983): See also H.Graessler et al., MPI-PAE/EXP.El 113(1983).

14) J.Peřina, In " Progress of Optics", ed. by E.Wolf, Vol. 18 (North-Holland, Amsterdam, 1980),p.129.

15) F.Hayot and G.Sterman, Phys.Lett., 121B(1983),419.

16) K.Saitou, private communication: M.Tamaki, private communication.

17) J.J.Freeman, "Principles of Noise", (Wiley, New York,1958).

18) S.O.Rice, In "Noise and Stochastic Processes",(ed. by N.Wax,Dover, 1954)

19) W.J.McGill, J.Math.Psychol. 4(1967), 351.

20) M.Suzuki, K.Kaneko and S.Takesue, Prog.Theor.Phys. 67(1982),1756.

Figure Captions

Fig.1. Physical contents of eq.(1c). $\langle n_{co} \rangle$ and $\langle n_{in} \rangle$ denote the coherent and incoherent average multiplicities, respectively. It should be noticed that the Poisson distribution with high energy mesons at an initial stage is observed in refs. 6) and 7).

Fig.2. Flow chart of hadronization from the resulting Planck-Polya distribution in the QCD branching processes.

Fig.3. Contents of eq.(18). It should be noticed that at $t = 0$, $\langle n_g \rangle = \langle n_{in} \rangle = 0$ and coherence is dominant; this corresponds to contents in Fig.1.

Fig.4. Comparison of eq.(1c) with KNO scaling distributions.
a) The diffractive KNO scaling.[11] The dashed curve is given by Barshay.[12] b) Distribution in the single jet by TASSO collaboration.[13] c) Full KNO scaling distribution in e^+e^- annihilations.[13]

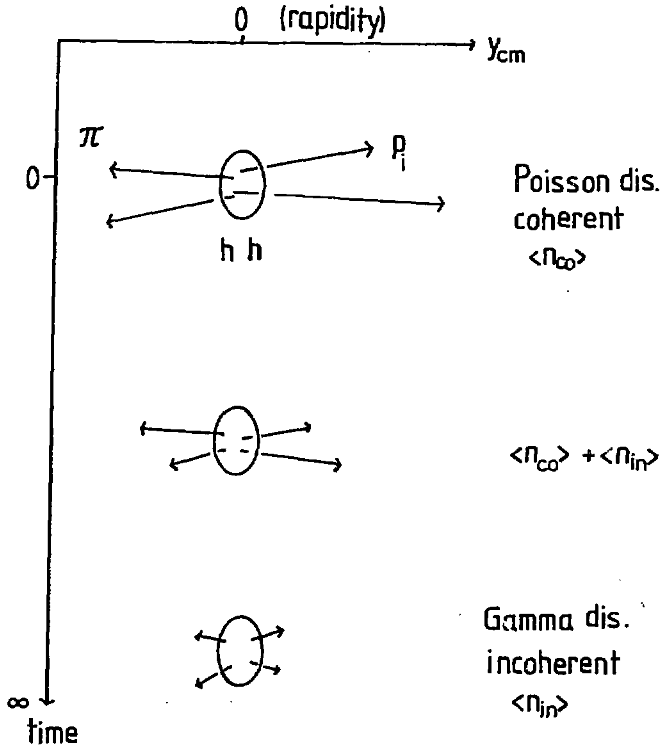

0 (rapidity)
ycm
π
pi
0
h h
Poisson dis.
coherent
<nco>
<nco> + <nin>
Gamma dis.
incoherent
<nin>
∞
time

Fig.1.

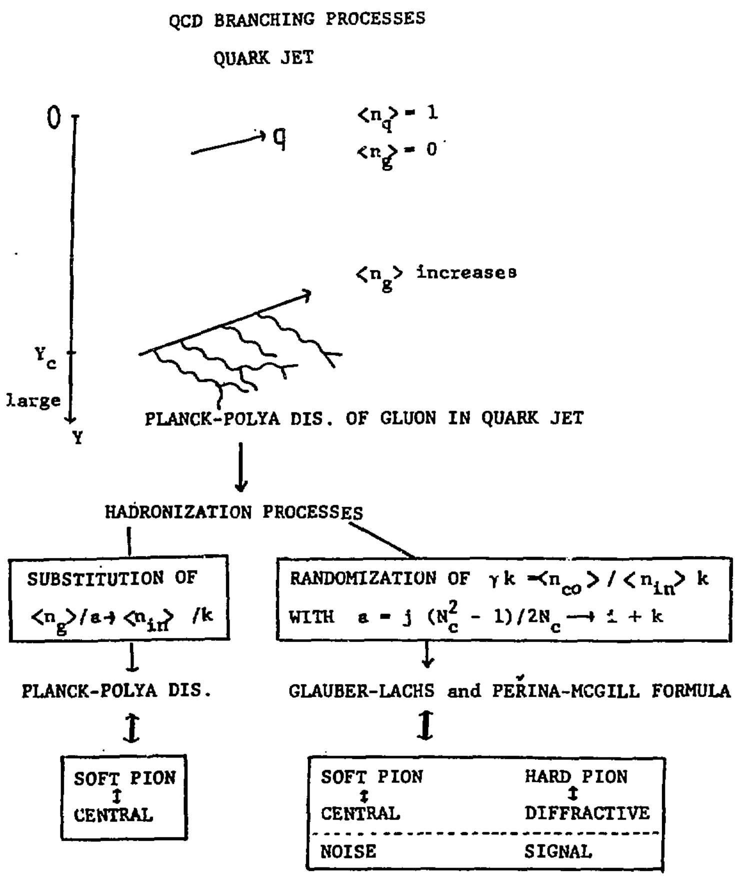
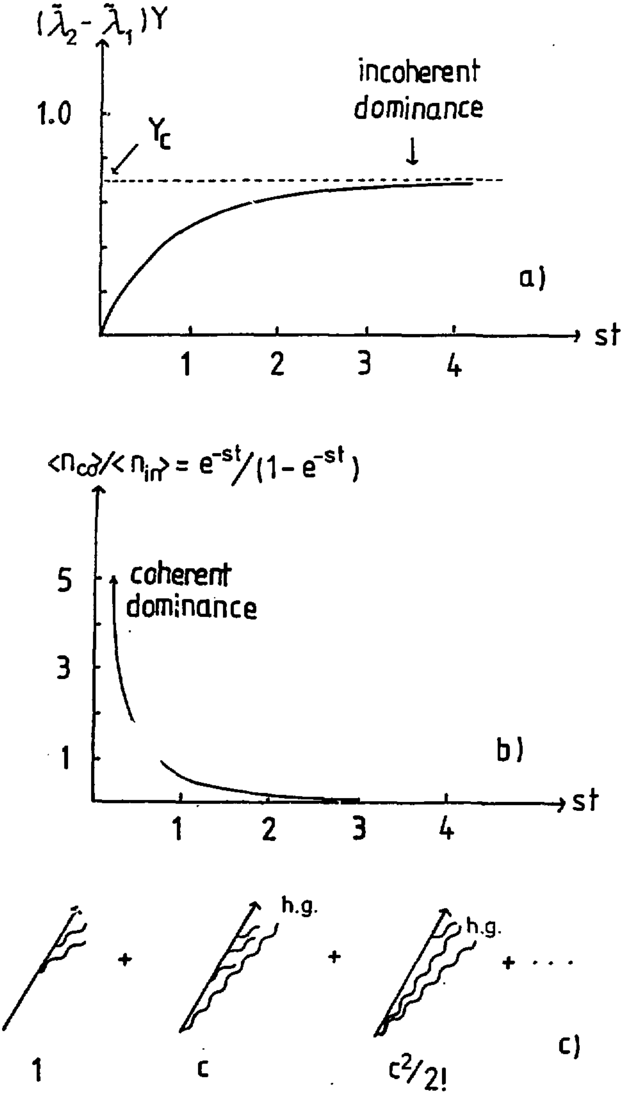
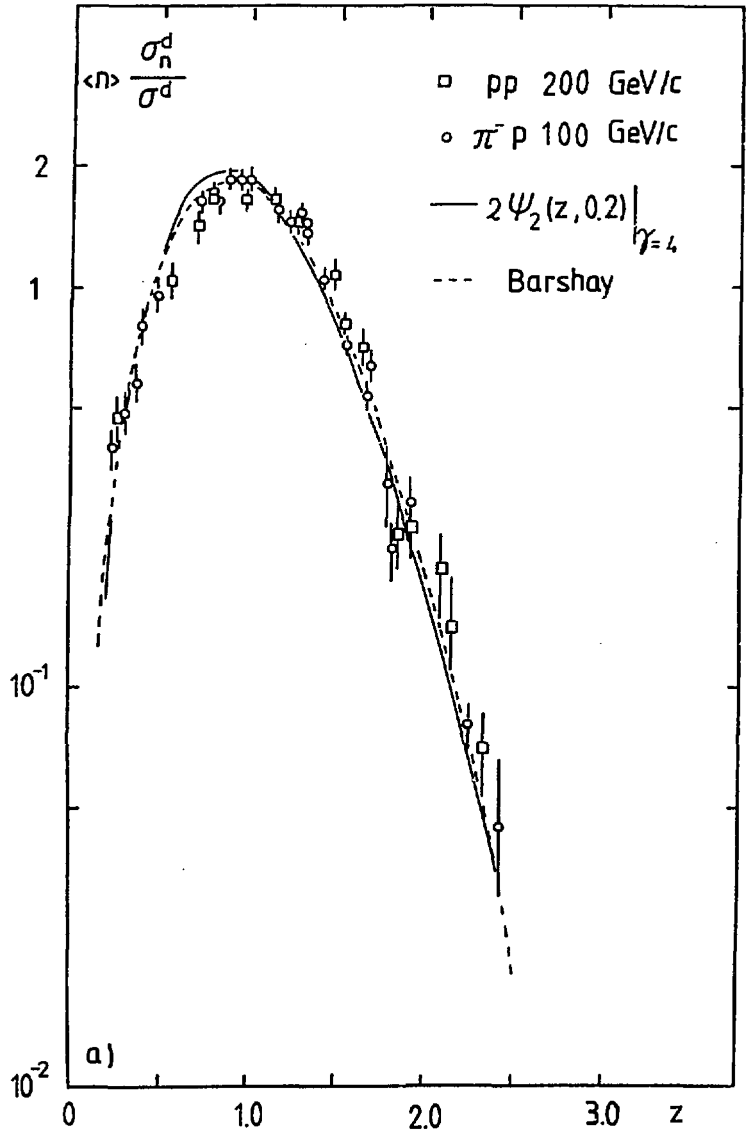

QCD BRANCHING PROCESSES
QUARK JET
0
q
$\langle n_q \rangle = 1$
$\langle n_g \rangle = 0$
$\langle n_g \rangle$ increases
Y_c
large
Y
PLANCK-POLYA DIS. OF GLUON IN QUARK JET
HADRONIZATION PROCESSES
SUBSTITUTION OF
$\langle n_g \rangle / a \to \langle n_{in} \rangle / k$
RANDOMIZATION OF $\gamma k = \langle n_{co} \rangle / \langle n_{in} \rangle k$
WITH a = j (N_c^2 - 1)/2N_c → 1 + k
PLANCK-POLYA DIS.
GLAUBER-LACHS and PEŘINA-MCGILL FORMULA
SOFT PION
CENTRAL
SOFT PION
CENTRAL
HARD PION
DIFFRACTIVE
NOISE
SIGNAL

FIG.2.



$(\tilde{\lambda}_2 - \tilde{\lambda}_1)Y$
1.0
Y_c
incoherent dominance
a)
st
1
2
3
4
$\langle n_{co}\rangle / \langle n_{in}\rangle = e^{-st}/(1-e^{-st})$
5
3
1
coherent dominance
b)
st
1
2
3
4
h.g.
h.g.
+
+
+ . . .
1
c
$c^2/2!$
c)

Fig. 3.



⟨n⟩ σ_n^d / σ^d
□ pp 200 GeV/c
○ π⁻ p 100 GeV/c
— 2Ψ₂(z, 0.2)|γ=4
--- Barshay
2
1
10⁻¹
10⁻²
a)
0
1.0
2.0
3.0
z

Fig. 4.

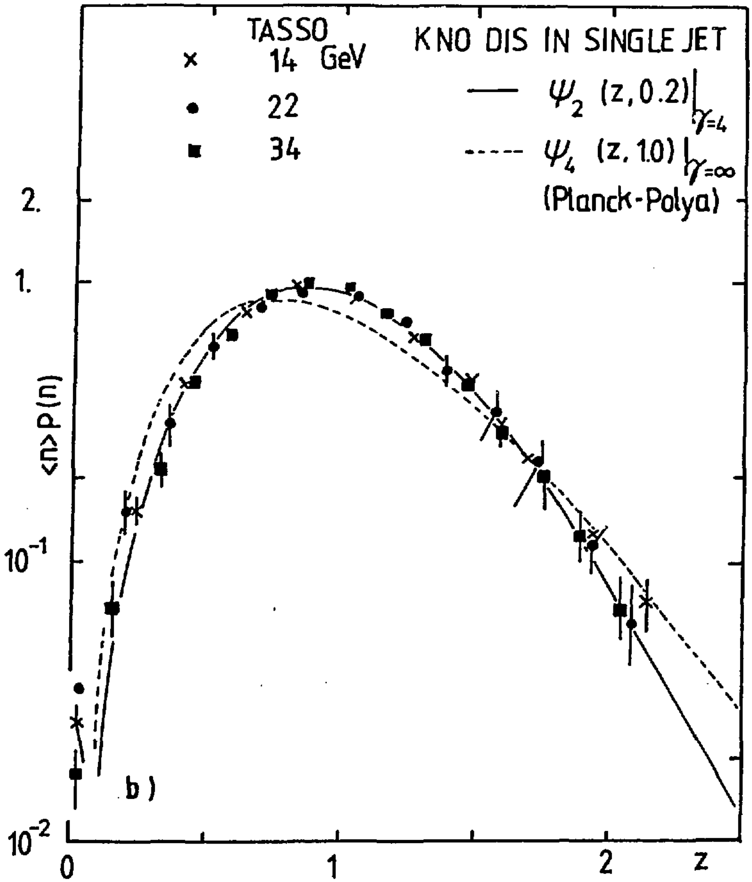

TASSO
× 14 GeV
● 22
■ 34
KNO DIS IN SINGLE JET
ψ2 (z, 0.2)|γ=4
ψ4 (z, 1.0)|γ=∞
(Planck-Polya)
2.
1.
10^-1
10^-2
<n>P(n)
b)
0
1
2
z

Fig. 4