ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Napoli

INFN/TC-81/16
9 Settembre 1981

S. De Nicola, P. Luchini, G. Prisco and S. Solimeno:

ELECTRON MOTION IN AN ELLIPTICALLY POLARIZED FREE ELECTRON LASER AMPLIFIER.

Servizio Documentazione
dei Laboratori Nazionali di Frascati

The authors wish to express their gratitude to the late Professor Ettore PANCINI for his inspiring teaching and youthful enthusiasm shown by holding a grip on the challenges posed by the opening up of new scientific frontiers.

The legacy of his long teaching career will be a precicus source of values for those willing to accept in whole the risk of serving science bravely and honestly.

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ABSTRACT.

The equations of motion for an electron in a free electron laser amplifier are written in hamiltonian form, and exactly solved for the circular polarization case. Successively, an approximate solution for a general polarization (elliptical or linear) and weak laser field is obtained by a relativistic generalization of the two-timing technique.

Finally, the theoretical perturbative results are confirmed by comparison with numerical solutions.

1. - INTRODUCTION.

The free electron laser amplifier is a device in which a relativistic electron beam interacts with a static period magnetic field (wiggler field) and a travelling optical wave (laser field).

The mean initial conditions of the beam may be chosen to yield a net energy transfer from the electrons to the laser field, thus producing an amplification of the latter.

If the beam density is not too high, the Coulomb interaction between the electrons, as well as quantum effects, may be neglected.

In these conditions, a theoretical investigation of the behaviour of a FEL starts with the calculation of the classical one-electron trajectories in the combined fields.

Among the various designs which have been proposed for the FEL, specially interesting are the circular case (both fields circularly polarized) and the linear case (both fields linearly polarized in the same direction).

Though the circular case has been deeply investigated, there is a lack of theoretical (and experimental) results for the linear case, which is the main object of our investigation.

This work is organized in five sections. In Section 2 the principal results of previous analysis are briefly considered, and a Hamiltonian approach to the equations of motion is presented. In Section 3 some interesting cases are discussed, and an exact solution for the circular case is obtained. In Section 4, we describe a scheme of perturbative analysis, the expansion parameter being the laser field strength, which generalizes the well-known nonrelativistic two-timing technique. In Section 5, some plots of computer solutions of the equations of motion are presented and discussed.

2. - EQUATIONS OF MOTION.

The classical one-electron trajectories for the circular case are calculated in ref. (1). In this analysis, which assumes the electron longitudinally injected, the longitudinal motion (once obtained this, writing down the transverse motion is straightforward, as will be shown) is described by

$$\ddot{\xi}(t) = -\Omega^2 \sin(\xi(t)) \tag{1}$$

where Ω^2 is a function of the field strenght and electron energy, and

$$\xi(t) = \xi_o + \Delta\omega t + K_1\, \delta z(t)\ ,$$

$$\delta z(t) = z(t) - \langle \dot{z} \rangle t\ ,$$

$$\Delta\omega = \frac{\langle \dot{z} \rangle}{c}\, \omega_w - \omega_1 \left(1 - \frac{\langle \dot{z} \rangle}{c}\right),$$

$\langle \dot{z} \rangle$ being the mean electron longitudinal velocity, and

$$\omega_w = \frac{2\pi C}{\lambda_w}\ , \qquad \omega_1 = \frac{2\pi C}{\lambda_1}$$

where λ_w and λ_1 are the period of the wiggler field and the wavelength of the laser field, respectively.

As shown in refs. (1, 2), maximum gain is achieved near the resonance condition

$$\Delta\omega = 0\ .$$

which, in turn, may be written as

$$\langle \dot{z} \rangle = C\left(\frac{\omega_1}{\omega_1 + \omega_w}\right). \tag{2}$$

The right side of (2) is easily recognized as the velocity, measured in the laboratory frame, of a frame in which the two fields have the same frequency (W frame). Thus, the resonance condition amounts to saying that the velocity of the W frame is the same as that of a frame in which the electron mean velocity is zero. Equivalently, when $\Delta\omega = 0$ the electron motion is bounded in the W frame.

All of the following calculations will be carried out in the W frame, where the two fields can be represented, with fairly good approximation, by two plane monochromatic waves, travelling in opposite directions along the z-axis.

In units where the speed of light, as well as the charge-mass ratio of the electron are equal to 1, the equations of motion for an electron in an electromagnetic field are[4]

$$\frac{d^2X^i}{ds^2} = \left(\frac{\partial A^k}{\partial X_i} - \frac{\partial A^i}{\partial X_k}\right)\frac{dX_k}{ds} \tag{3}$$

where s is the proper time, A^i is the electromagnetic four-potential, and summation over repeated indexes is understood.

For electromagnetic waves it is always possible to choosė a gauge in which the 4-potential is purely transverse. Supposing, then, the potential independent of the transverse coordinates, (3) implies the conservation of the transverse components of the generalized 4-momentum

$$\frac{dX^i}{ds} + A^i = P^i = \text{const} \qquad i = 1,2 . \tag{4}$$

Using (4) in (3), we get the equation of longitudinal motion

$$\dot{z}\frac{\partial M}{\partial t} + \frac{M\ddot{z}}{1-\dot{z}^2} = -\frac{\partial M}{\partial z} \tag{5}$$

where

$$M^2(z,t) = 1 + \left|\underline{P} - \underline{A}(z,t)\right|^2$$

the bars denoting Euclidean modulus of the two-dimensional vector. We may attain a more useful formulation of (5) by passing to the Hamilton-Jacobi equation, which in its relativistic formulation reads[3,4]

$$\left(\frac{\partial S}{\partial X^i} - A_i\right)\left(\frac{\partial S}{\partial X_i} - A^i\right) = 1 . \tag{6}$$

The action is defined as the line integral of the electron Lagrangian along the effective path, which obeys the variational principle

$$\delta S = 0$$

the variation being taken for given initial and final positions. Equation (4) permits us to look for solutions of (6) of the form

$$S = P_i X^i + S'(z,t) . \tag{7}$$

Equation (6), with S given by (7), becomes

$$(\frac{\partial S'}{\partial t})^2 - (\frac{\partial S'}{\partial z})^2 = M^2 \tag{8}$$

from which we see that the scalar M plays the role of a space-time dependent effective mass of the electron in its longitudinal motion. Regarding (8) as a Hamilton-Jacobi equation for the electron motion in the z-t subspace, we also see that it obeys the variational principle

$$\delta\int M\, d\sigma = 0 \tag{9}$$

with

$$d\sigma^2 = dt^2 - dz^2 .$$

Equation (9) may be written as

$$\delta\int L_t(z, \dot{z}, t)\, dt = 0$$

being

$$L_t(z, \dot{z}, t) = M\frac{d\sigma}{dt} = - M\sqrt{1 - \dot{z}^2} . \tag{10}$$

Hence, the Lagrange equation for the electron longitudinal motion reads

$$\frac{d}{dt}(\frac{M\dot{z}}{\sqrt{1-\dot{z}^2}}) = -\sqrt{1-\dot{z}^2}\,\frac{\partial M}{\partial z} \tag{11}$$

which is equivalent to (5).

The canonical momentum conjugated to z is given by

$$P_z = \frac{\partial L_t}{\partial \dot{z}} = \frac{M\dot{z}}{\sqrt{1-\dot{z}^2}} \tag{12}$$

and the Hamiltonian corresponding to (10) is

$$H = \dot{z}P_z - L_t = \frac{M}{\sqrt{1-\dot{z}^2}} . \tag{13}$$

Note that, though relative only to the longitudinal motion, H effectively represents the electron energy. In fact, it is readily verified that

$$(\frac{dt}{ds})^2 - (\frac{dz}{ds})^2 = M^2$$

which in turn implies

$$\frac{dt}{ds} = \frac{M}{\sqrt{1-\dot{z}^2}} = H .$$

The Hamiltonian may be written in terms of z, P_z and t as

$$H = \sqrt{P_z^2 + M^2} \tag{14}$$

from which

$$\dot{z} = \frac{P_z}{\sqrt{P_z^2 + M^2}} , \qquad \dot{P}_z = - \frac{M}{\sqrt{P_z^2 + M^2}} \frac{\partial M}{\partial z} . \tag{15}$$

The variational principle (9) permits us to write the equations of motion by using any couple of variables. For example, choosing the phases

$$\theta = t + z , \qquad \tau = t - z$$

as independent variables, and noting that

$$d\sigma^2 = d\theta \, d\tau$$

the Lagrangian reads

$$L_\tau = M \frac{d\sigma}{d\tau} = M \sqrt{\dot{\theta}} \tag{16}$$

(whenever the phase variables will be used the dot will indicate differentiation with respect to τ, and the brackets will indicate τ-average). Consequently

$$\frac{d}{d\tau} \left(\frac{M}{2\sqrt{\dot{\theta}}} \right) = \sqrt{\dot{\theta}} \frac{\partial M}{\partial \theta} . \tag{17}$$

Whenever M depends on either one of the couple of independent variables, a new constant of motion is found, and the solution of the equations of motion is obtained by quadratures. We discuss now some exact solutions of the equations of motion.

3. - EXACT SOLUTIONS.

Suppose $\underline{A}$ depending only on τ. A potential of this kind represents the more general superposition of waves, travelling in the positive z-direction. Being M independent of θ, equation (17) implies the conservation of the θ-component of the canonical momentum

$$\frac{M}{2\sqrt{\dot{\theta}}} = P_\theta = \text{const} . \tag{18}$$

As may be readily verified

$$\frac{d\theta}{ds} \frac{d\tau}{ds} = M^2$$

which gives

$$\frac{d\tau}{ds} = \frac{M}{\sqrt{\delta}} . \tag{19}$$

So equation (18) implies that, for the motion in a superposition of waves travelling in one direction, the phase is a linear function of the proper time.

The law of longitudinal motion is

$$\theta = \frac{1}{4P_\theta^2} \int M^2 d\tau \tag{20}$$

while the transverse motion is obtained by (4). For a wave of unity angular frequency

$$\underline{A} = \mathrm{Re}\,(\underline{A}_o \exp i\tau) .$$

Supposing $\underline{P} = 0$, which means absence of constant transverse drift, and using the vector identity

$$\left[\mathrm{Re}\,(\underline{x})\right]^2 = \frac{1}{2}\,|\underline{x}|^2 + \frac{1}{2}\,\mathrm{Re}\,(\underline{x}^2) \tag{21}$$

yield

$$M^2 = 1 + \frac{1}{2}\,|\underline{A}_o|^2 + \frac{1}{2}\,\mathrm{Re}\,(\underline{A}_o^2 \exp 2i\tau) . \tag{22}$$

Being $z = \frac{1}{2}(\theta - \tau)$, $\frac{dz}{dt} = \frac{1}{2}\frac{d\tau}{dt}\left(\frac{d\theta}{d\tau} - 1\right)$ the condition for bounded motion, viz.

$$\left\langle \frac{dz}{dt} \right\rangle = 0$$

is satisfied if

$$\left\langle \frac{d\theta}{d\tau} \right\rangle = 1 \tag{23}$$

which means that the coefficient of the linear part in $\theta(\tau)$ must be equal to 1. Using (20) and (22), the bounded motion condition may be written as

$$1 + \frac{1}{2}\,|\underline{A}_o|^2 = 4P_\theta^2 . \tag{24}$$

For a circularly polarized wave, $\underline{A}_o^2 = 0$ and the longitudinal motion is uniform. When the initial conditions satisfy (24), the trajectory is a circumference in the transverse plane.

For a linearly polarized wave, assuming the polarization directed along the x-axis, the trajectory is an eight-figure in the z-x plane. For weak fields, it reduces to its nonrelativistic limit, a segment parallel to the polarization axis.

Consider now two waves of the same frequency, travelling in opposite directions along the z-axis. For unity frequency, what may always be imposed by an appropriate choice of the length unit, the potential is given by

$$\underline{A}(z,t) = \mathrm{Re}\,(\underline{A}_1^{o} \exp i\,(t-z) + \underline{A}_2^{o} \exp i\,(t+z))\ . \tag{25}$$

Supposing again $\underline{P} = 0$ and using (21) lead to

$$M^2 = 1 + \frac{1}{2}\,|\underline{A}_1^{o}|^2 + \frac{1}{2}\,|\underline{A}_2^{o}|^2 + \mathrm{Re}\Big[\frac{1}{2}\,\underline{A}_1^{o\,2} \exp 2i\,(t-z) + \frac{1}{2}\,\underline{A}_2^{o\,2} \exp 2i\,(t+z) + \underline{A}_1^{o*}\cdot\underline{A}_2^{o} \exp 2iz + \underline{A}_1^{o}\cdot\underline{A}_2^{o} \exp 2it\Big]\ . \tag{26}$$

If both waves are circularly polarized, $\underline{A}_1^{o\,2} = \underline{A}_2^{o\,2} = 0$.

Furthermore, depending on the concordance of polarizations, either $\underline{A}_1^{o*}\cdot\underline{A}_2^{o}$ or $\underline{A}_1^{o}\cdot\underline{A}_2^{o}$ vanish. In both cases M depends on only one variable, what permits us to solve exactly the equation of motion.

If the polarization are concordant, $\underline{A}_1^{o*}\cdot\underline{A}_2^{o} = 0$ and M^2 is given by

$$M^2 = 1 + \frac{1}{2}\,|\underline{A}_1^{o}|^2 + \frac{1}{2}\,|\underline{A}_2^{o}|^2 + \mathrm{Re}\,(\underline{A}_1^{o}\cdot\underline{A}_2^{o} \exp 2it)$$

and is independent of z. The z-component of the canonical momentum is then conserved

$$\frac{M\dot{z}}{\sqrt{1-\dot{z}^2}} = P_z = \text{const.} \tag{27}$$

the law of longitudinal motion being given by

$$z = \int \frac{P_z\, dt}{\sqrt{P_z^2 + M^2(t)}}\ . \tag{28}$$

Note that the electron never changes its direction, what makes bounded motion impossible (save of course, when the electron does not move at all). If the polarization are discordant,

$$M^2 = 1 + \frac{1}{2}\,|\underline{A}_1^{o}|^2 + \frac{1}{2}\,|\underline{A}_2^{o}|^2 + \mathrm{Re}\Big[\underline{A}_1^{o*}\cdot\underline{A}_2^{o} \exp 2iz\Big] \tag{29}$$

does not depend on t, and energy is conserved,

$$\frac{M}{\sqrt{1-\dot{z}^2}} = E = \text{const.} \tag{30}$$

which yields

$$\frac{dt}{dz} = \pm \frac{E}{\sqrt{E^2 - M^2(z)}}$$

where the sign must be chosen in such a way as to mantain dt positive. The law of motion is

$$t = \pm \int \frac{E\, dz}{\sqrt{E^2 - M^2(z)}} \tag{31}$$

and motion can be bounded or unbounded, depending on the value of E. For the transverse motion, equation (4) has to be used.

4. - PERTURBATIVE ANALYSIS FOR ELLIPTICALLY POLARIZED WAVES.

In nonrelativistic mechanics there exists a method, due to Kapitza, known as "two-timing approximation", which permits to separate a slow component of motion from a rapidly oscillating one(3, 5). What follows may be viewed as a relativistic generalization of the Kapitza method.

The motion in the one-wave field is given by (20). Using (22), it may be written as

$$\vartheta = \theta + \frac{\alpha\dot{\theta}}{2}\,\mathrm{Re}\left(\frac{1}{2i}\,\underline{A}_1^{o\,2}\exp 2i\tau\right) \tag{32}$$

where

$$\alpha = \left[1+\frac{1}{2}\,|\underline{A}_1^o|^2\right]^{-1}, \qquad \dot{\theta} = \text{const.}, \qquad \theta = \dot{\theta}\tau + \text{const.}$$

In case of bounded motion, $\dot{\theta} = 1$.

Suppose, now, that a second wave, weak respect to the first, is added. We can look for a solution of the form (32), where $\dot{\theta}$ is now a slowly varying function of τ.

Being $\ddot{\theta} \ll 1$

$$\dot{\vartheta} = \dot{\theta}\left(1+\alpha\,\mathrm{Re}\left(\frac{1}{2}\,\underline{A}_1^{o\,2}\exp 2i\tau\right)\right) \tag{33}$$

Neglecting all terms containing $\underline{A}_2^{o\,2}$ and supposing $\underline{P} = 0$, the effective mass $M^2(\vartheta,\tau)$ reduces to

$$M^2 = 1+\frac{1}{2}\,|\underline{A}_1^o|^2 + \mathrm{Re}\left(\frac{1}{2}\,\underline{A}_1^{o2}\exp 2i\tau\right) + \mathrm{Re}\left(\underline{A}_1^o\cdot\underline{A}_2^o\exp i(\tau+\vartheta) + \underline{A}_1^o\cdot\underline{A}_2^{o*}\exp i(\tau-\vartheta)\right). \tag{34}$$

The lagrangian $L_\tau = M\sqrt{\dot{\vartheta}}$ may then be written as

$$L_\tau = \sqrt{\frac{\dot{\theta}}{\alpha}}\Big[\left(1+\alpha\mathrm{Re}\left(\frac{1}{2}\,\underline{A}_1^{o2}\exp 2i\tau\right)\right)\left(1+\alpha\mathrm{Re}\left(\frac{1}{2}\underline{A}_1^{o\,2}\exp 2i\tau\right)\right. +$$

$$\left. + \alpha\mathrm{Re}\left(\underline{A}_1^o\cdot\underline{A}_2^o\exp i(\tau+\vartheta) + \underline{A}_1^o\cdot\underline{A}_2^{o*}\exp i(\tau-\vartheta)\right)\right)\Big]^{1/2}$$

and, up to the first order in $\underline{A}_2^o$,

$$L_\tau \simeq \sqrt{\alpha\dot{\theta}}\Big[1+\frac{1}{2}\,|\underline{A}_1^o|^2 + \mathrm{Re}\left(\frac{1}{2}\,\underline{A}_1^{o\,2}\exp 2i\tau\right) +$$

$$+ \frac{1}{2}\,\mathrm{Re}\left(\underline{A}_1^o\cdot\underline{A}_2^o\exp i(\tau+\vartheta) + \underline{A}_1^o\cdot\underline{A}_2^{o*}\exp i(\tau-\vartheta)\right)\Big].$$

As in the Kapitza method, being $\ddot{\theta} \ll 1$, we may replace the effective Lagrangian by its τ-average, obtained keeping θ constant.

$$L_\tau \simeq \sqrt{\alpha\dot\theta}\left[1+\frac{1}{2}\ \underline{A}_1^o\ ^2+\frac{1}{2}\mathrm{Re}(\underline{A}_1^o\cdot\underline{A}_2^o\ \langle\exp i(\tau+\theta)\rangle+\underline{A}_1^o\cdot\underline{A}_2^{o*}\langle\exp i(\tau-\theta)\rangle)\right]. \quad (35)$$

Suppose $\dot\theta \simeq 1$, which corresponds to an almost bounded motion in absence of the second wave. We may then consider $\theta-\tau$ constant in one period, thus getting

$$\langle\exp i(\tau-\theta)\rangle = \exp - i(\theta-\tau)\langle\exp - i\frac{\alpha\dot\theta}{2}\mathrm{Re}(\frac{1}{2i}\underline{A}_1^{o\,2}\exp 2i\tau)\rangle ,$$

$$\langle\exp i(\tau+\theta)\rangle = \exp i(\theta-\tau)\langle\exp i(2\ +\frac{\alpha\dot\theta}{2}\mathrm{Re}(\frac{1}{2i}\underline{A}_1^{o\,2}\exp 2i\tau)\rangle .$$

Supposing now $\underline{A}_1^{o\,2}$ real, what may always be imposed by an appropriate choice of coordinates, and using the integral representation of Bessel functions of integral order

$$J_N(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin t - Nt)\,dt$$

it is readily verified that

$$\langle\exp(-i\frac{\alpha\dot\theta}{4}\underline{A}_1^{o\,2}\sin 2\tau)\rangle = J_o(\frac{\alpha\dot\theta\underline{A}_1^{o\,2}}{4}),$$

$$\langle\exp i(2\tau+\frac{\alpha\dot\theta}{4}\underline{A}_1^{o\,2}\sin 2\tau)\rangle = -J_1(\frac{\alpha\dot\theta\underline{A}_1^{o\,2}}{4}),$$

the imaginary parts of complex exponents giving no contribution to the integrals. Then

$$L_\tau \simeq \sqrt{\alpha\dot\theta}\left[1+\frac{1}{2}\left|\underline{A}_1^o\right|^2+\mathrm{Re}(B\exp i(\theta-\tau)\right],$$

where

$$B = \underline{A}_2^o(\frac{1}{2}J_o(\frac{\alpha\underline{A}_1^{o\,2}}{4})\underline{A}_1^{o*}-\frac{1}{2}J_1(\frac{\alpha\underline{A}_1^{o\,2}}{4})\underline{A}_1^o), \quad (36)$$

and we have put $\dot\theta = 1$ in the arguments of Bessel functions, which are multiplied by $\underline{A}_2^o$. The action S is given by

$$S = \int\sqrt{\alpha}\left[\frac{1}{\alpha}+\mathrm{Re}(B\exp i(\theta-\tau)\right]\sqrt{\dot\theta}\,d\tau = \int\frac{1}{\sqrt{\alpha}}\left[1+\alpha\mathrm{Re}(B\exp i(\theta-\tau)\right]\sqrt{\dot\theta}\,d\tau =$$

$$= \int\frac{1}{\sqrt{\alpha}}\left[1+\alpha\mathrm{Re}(2B\exp i(\theta-\tau)\right]^{1/2}\sqrt{\dot\theta}\,d\tau .$$

$$S = \int\left[1+\frac{1}{2}\left|\underline{A}_1^o\right|^2+\mathrm{Re}(2B\exp i(\theta-\tau)\right]^{1/2}\sqrt{\dot\theta}\,d\tau .$$

Using the variables: $T=\frac{1}{2}(\theta+\tau)$, $Z=\frac{1}{2}(\theta-\tau)$ we see that S may be written in terms of T and Z as

$$S = \int M(Z)\, d\Sigma$$

where $d\Sigma^2 = dT^2 - dZ^2$ and

$$M^2(Z) = 1 + \frac{1}{2}\, \underline{A}_1^{o\,2} + \mathrm{Re}\,(2B \exp 2i\, Z)\ . \tag{37}$$

Noting that, $\frac{\alpha \underline{A}_1^{o\,2}}{4}$ never becoming greater than $\frac{1}{2}$, B never differs too much from $\frac{1}{2}\, \underline{A}_1^{o\,*} \cdot \underline{A}_2^{o}$, we see that, as long as $\underline{A}_2^{o} \ll 1$, Equation (37) is formally equivalent to (29).

The function T(Z) obeys then the same equation as t(z) for the circular cases. Moreover, suppose

$$t\,(z = z_o) = 0\ , \qquad \left.\frac{dt}{dz}\right|_{z = z_o} = \frac{1}{\dot{z}_o}\ .$$

Opportunely shifting the T-origin, what does not affect the previous considerations,and choosing $\cos 2z_o = 0$, we have

$$T(Z = z_o) = 0\ , \qquad \left.\frac{dT}{dZ}\right|_{Z = z_o} = \frac{1}{\dot{z}_o}\ .$$

The law of longitudinal motion then reads

$$T = \pm \int_{z_o}^{z} \frac{E\, dz}{\sqrt{E^2 - M^2(z)}} \tag{38}$$

where M^2 is given by (37), and E still represents the initial electron energy. The electron position is implicitely given by

$$z = Z + \frac{\alpha}{4}\, \mathrm{Re}\, (\frac{1}{2i}\, \underline{A}_1^{o\,2} \exp 2i\tau)\ . \tag{39}$$

For a qualitative understanding of motion, consider first the slow oscillation Z. Its amplitude is implied from the condition

$$E^2 - M^2(Z) \geq 0$$

and is the same as for the circular case. Denoting its T-period by T^* and the t-period by t^*, gives

$$t^* \quad T^* + \frac{\alpha}{4}\, \mathrm{Re}\, (\frac{1}{2i}\, \underline{A}_1^{o\,2} \left[\exp 2i\,(t_o + t^* - z(t_o + t^*) - \exp 2i\,(t_o - z(t_o)\right]\,)$$

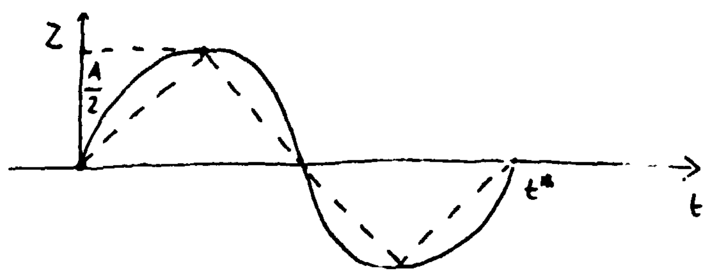
from which, since $T^* \gg 1$

$$t^* \simeq T^*\ .$$

Thus also the period of Z is the same as for the circular case. Denoting next the fast oscillation as z_1, (39) gives

$$z_1 = \frac{a}{4} \mathrm{Re}\left(\frac{1}{2i} \underline{A}_1^{o\,2} \exp 2i(t - z_1 - Z)\right) . \qquad (40)$$

In a first approximation we may consider $Z = 0$, which gives Z_1 coinciding with the motion in the one-wave field. For a better approximation, consider Z given by the broken line in the figure below.



A indicating the amplitude of the Z-oscillation. As may be seen from (40), the frequency of Z_1 is

$$\omega = 2\left(1 - \frac{A\omega^*}{\pi}\right) \qquad \text{for Z increasing, and}$$

$$\omega = 2\left(1 + \frac{A\omega^*}{\pi}\right) \qquad \text{for Z decreasing,}$$

where $\omega^* = 2\pi/t^*$ is the frequency of the Z-oscillation.

We conclude that, in the bounded situation considered, the electron longitudinal motion is formed by the superposition of the circular case solution corresponding to the same initial conditions, and a frequency modulated oscillation, whose amplitude is the same of the one-wave solution, and whose frequency oscillates between $\omega = 2(1 \mp \frac{A\omega^*}{\pi})$.

Note that, being $\omega^* \ll 1$ the fast oscillation may be identified with the one-wave solution, the approximation becaming better as the laser field strenght decreases.

5. - NUMERICAL SOLUTIONS.

The Hamilton equations (15), together with the transverse motion equations (4), have been numerically solved by Runge-Kutta methods. The expressions assumed for the fields are

$$A_x = A_{1x} \cos(t - z + \phi_{1x}) + A_{2x} \cos(t + z + \phi_{2x}) ,$$

$$A_y = A_{1y} \cos(t - z + \phi_{2y}) + A_{2y} \cos(t + z + \phi_{2y}) .$$

The data required are the initial electron position, the constant vector $\underline{P}$ and the initial z-momentum P_{zo}. A change in the initial transverse position does not affect the shape of the solution, while this is strongly dependent, as should be clear, on the initial longitudinal position, a change in which corresponds .o a phase shift.

Figures are plots of numerical solutions obtained for the linear case ($O = A_{1y} = A_{2y}$). The initial values have been chosen to yield, for one wave only, a bounded solution (what happens if the resonance condition is verified). Figs. 1-4 represent the electron z-position as a function of time, for different values of the laser field strenght A_{2x}. In Fig. 1, this is mantained small with respect to the wiggler field strenght. This is the case considered in the perturbative analysis, and the reader may convince himself that the results of Section 4 are confirmed. Particularly, the slow oscillation always corresponding to the circular case solution.

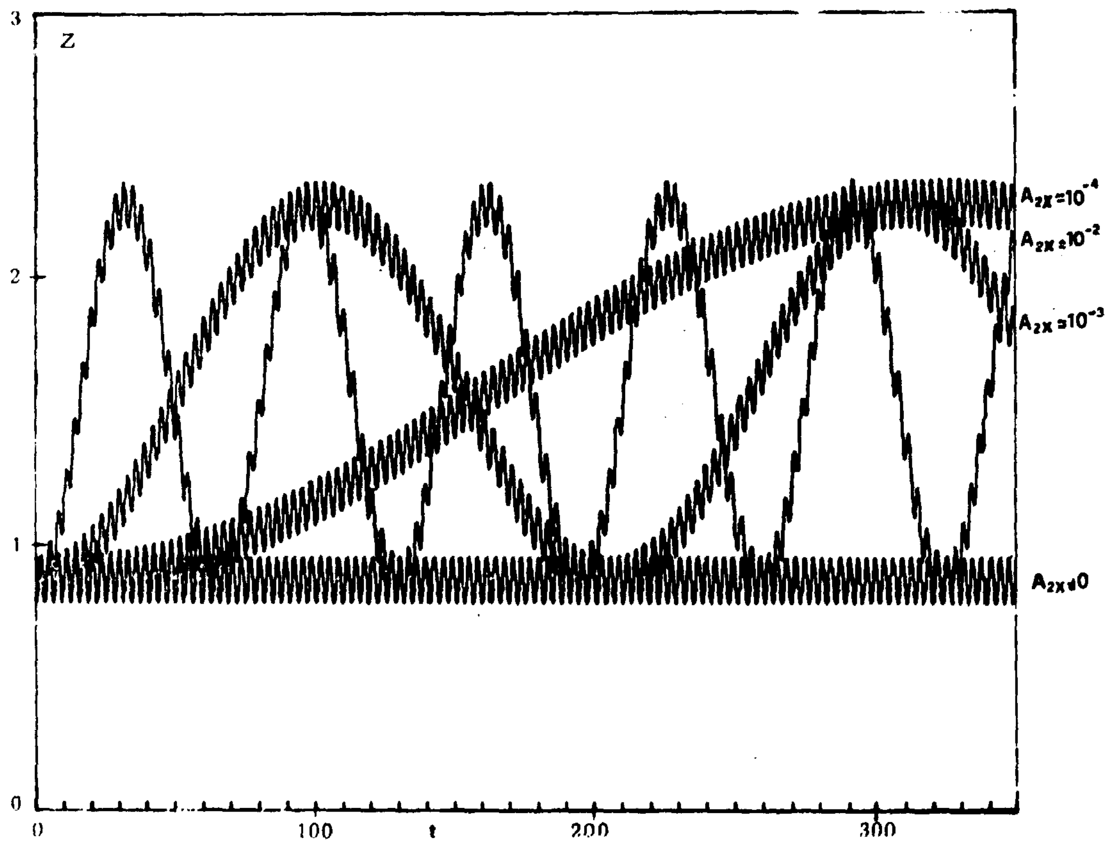
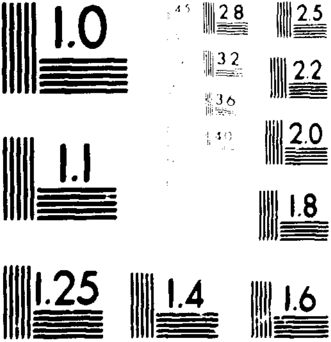


FIG. 1

In Figs. 2 and 3 the two fields have a comparable strenght. Note that the decomposition of motion in two oscillations of different frequencies may no longer be made. Furthermore, motion is now unbounded. In Fig. 4 the two fields have the same strenght. We see that motion is again bounded.

Moreover, it appears as a single oscillation whose period is exactly double than the one-wave motion frequency.



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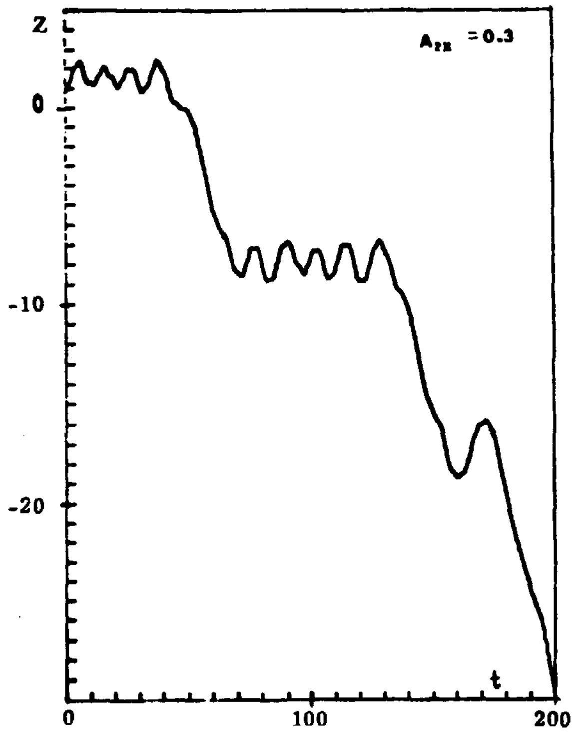

FIG. 2

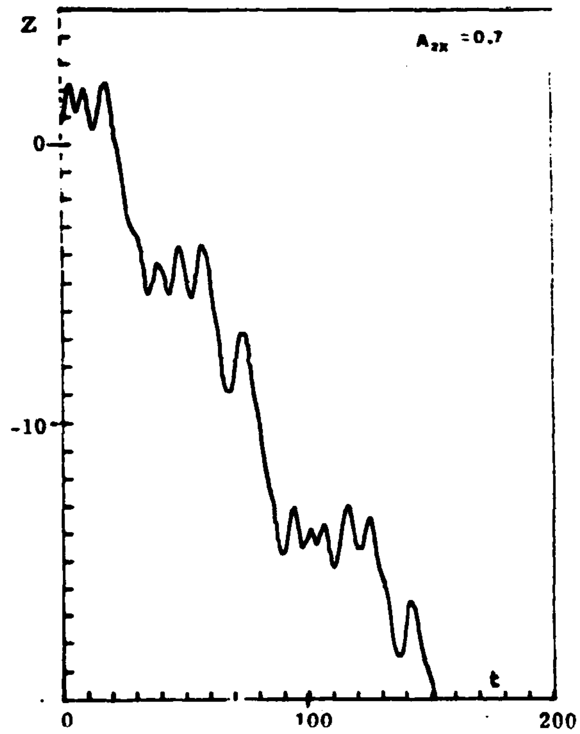

FIG. 3

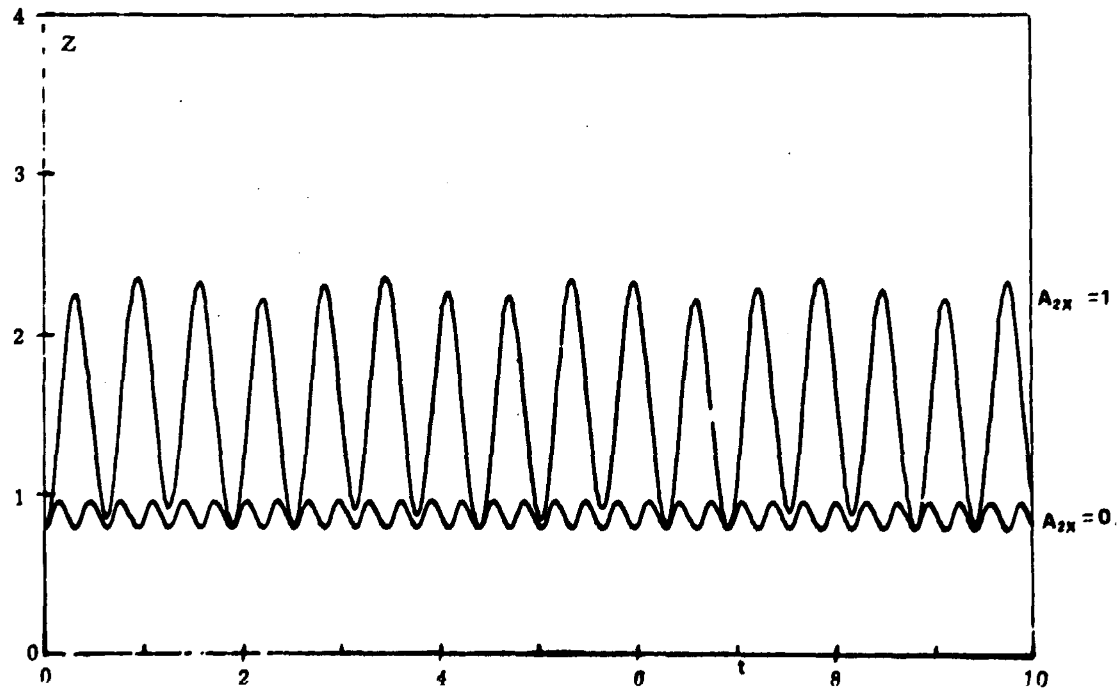

FIG. 4

Figs. 5-10 represents the electron trajectories corresponding to the same values of the parameters. When the laser field strenght is small (Figs. 5, 6, 7) the trajectory is an eight-figure oscillating back and forth between the bounds of the slow motion. In Figs. 8, 9 (comparable strenghts) the trajectory appears completely disordered. In Fig. 10 (equal strenghts) we note again some sort of order.

We may observe that the electron motion undergoes two order-disorder "phase transitions" at some critical values of the laser field strenght, the order parameter being the mean longitudinal velocity.

Figs. 11, 12, 13 show the electron phase space paths. For the circular case, the phase space paths have proved to be a powerful tool for an intuitive understanding of the gain mechanism. In Figs. 1-13 the following parameters are held fixed: $z_0 = \pi/4$, $P_{z0} = 0$, $\underline{P} = 0$, $A_{1x} = 1$, $\phi_{1x} = 0$, $\phi_{2x} = 0$, the varying parameter being A_{2x}.

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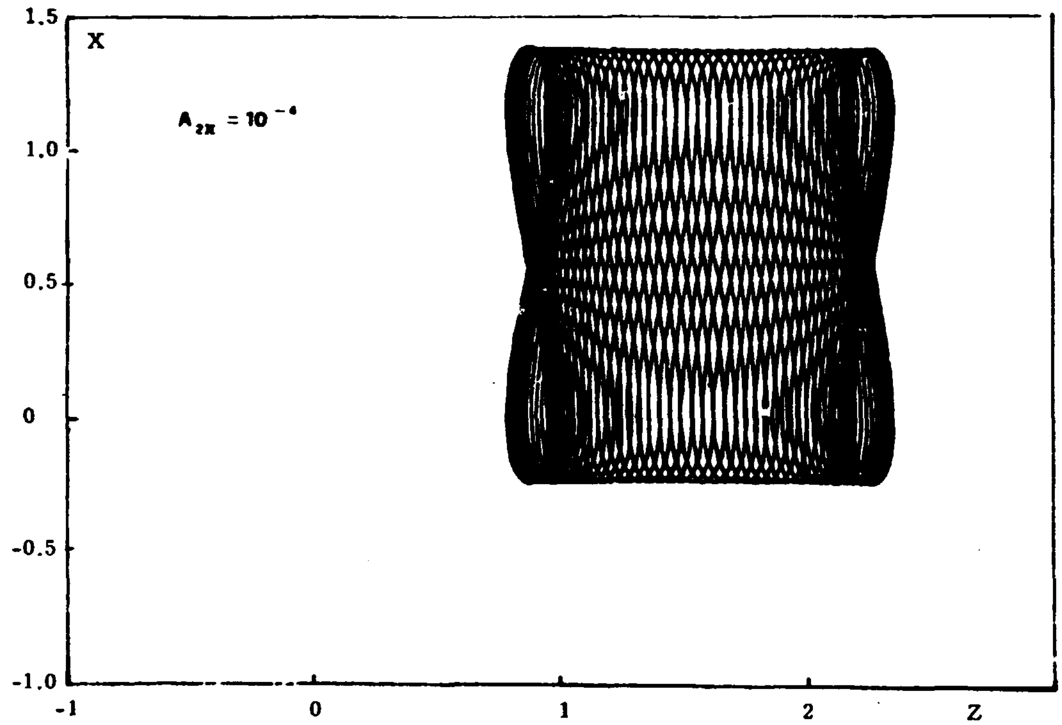

FIG. 5

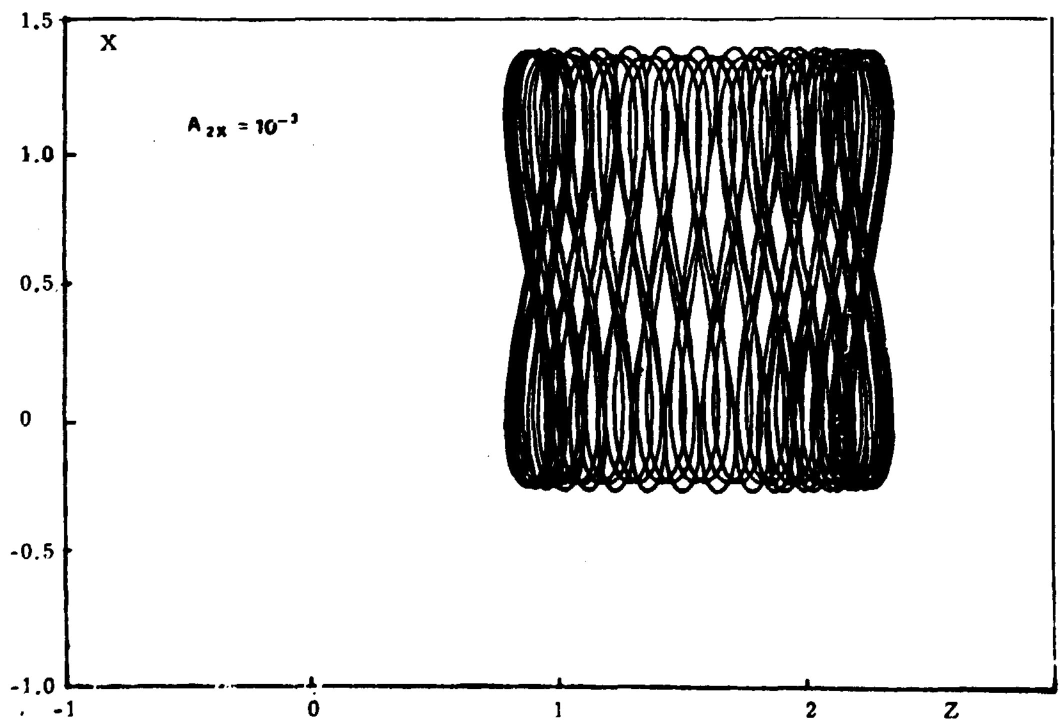

FIG. 6

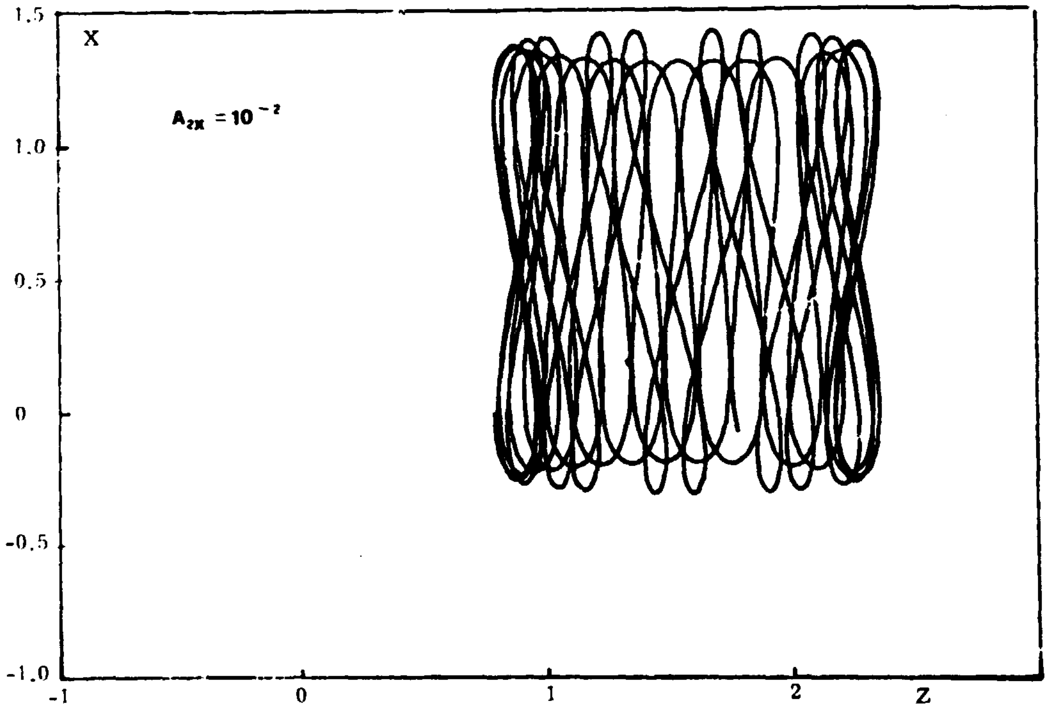

FIG. 7

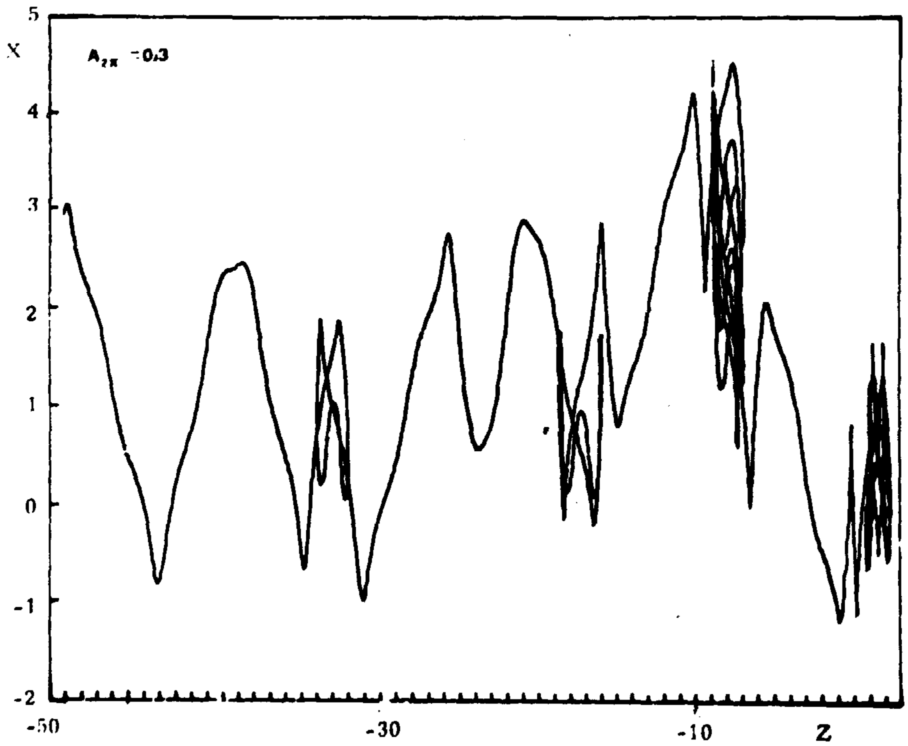

FIG. 8

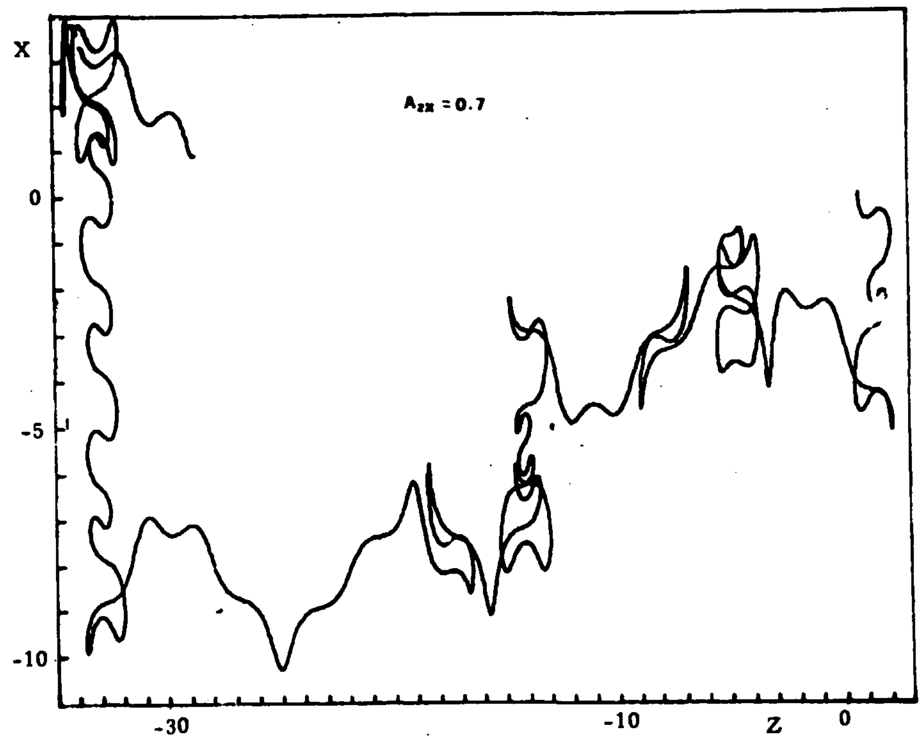

FIG. 9

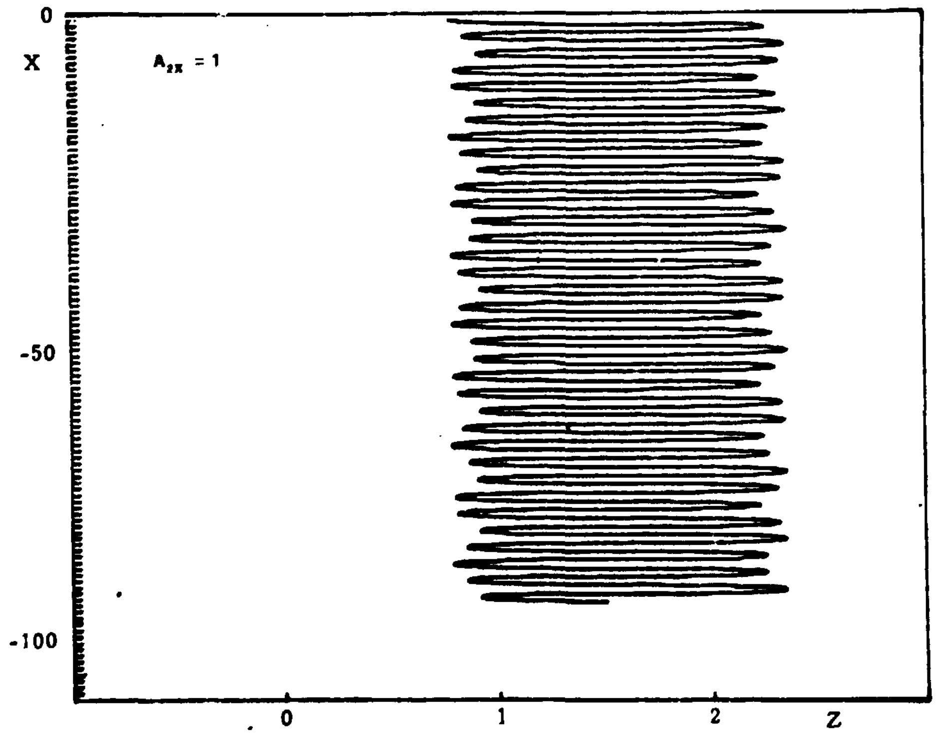

FIG. 10

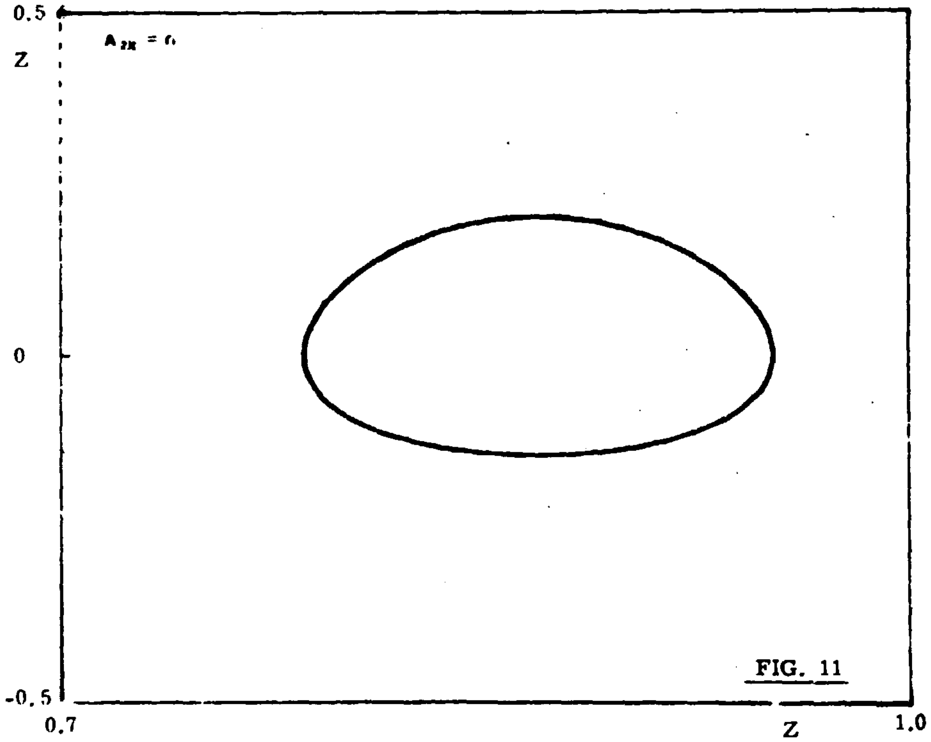

FIG. 11

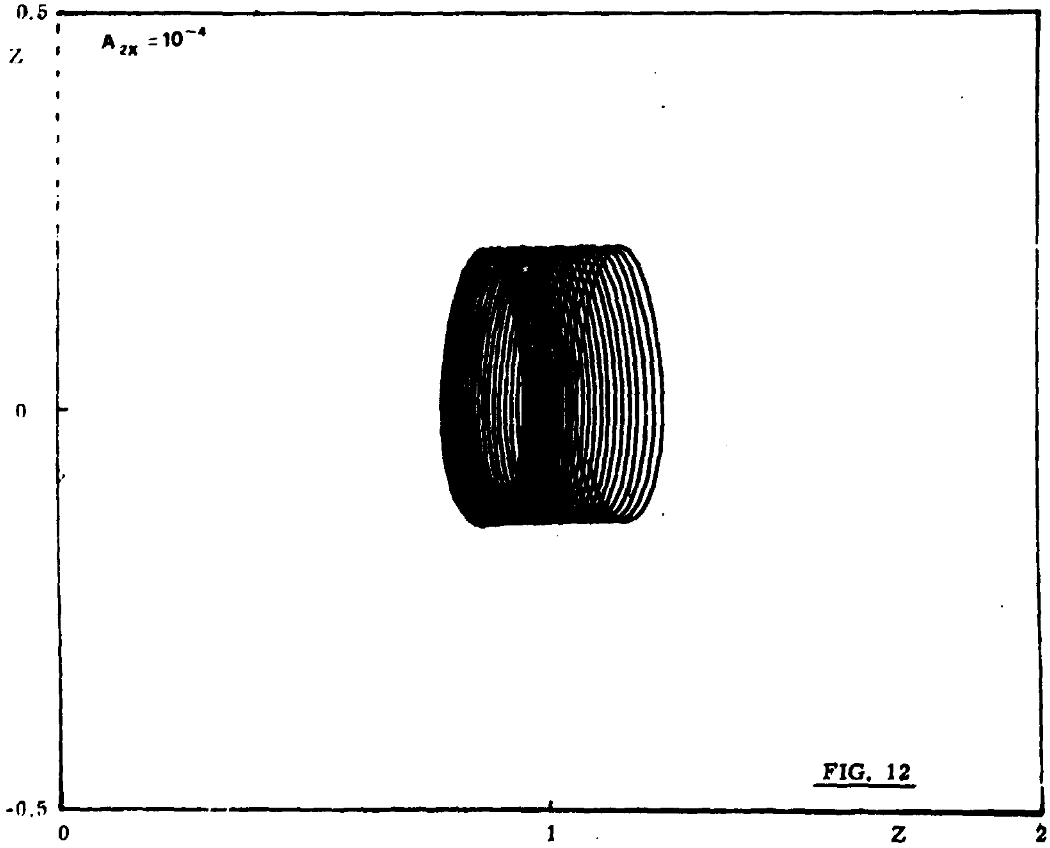

FIG. 12

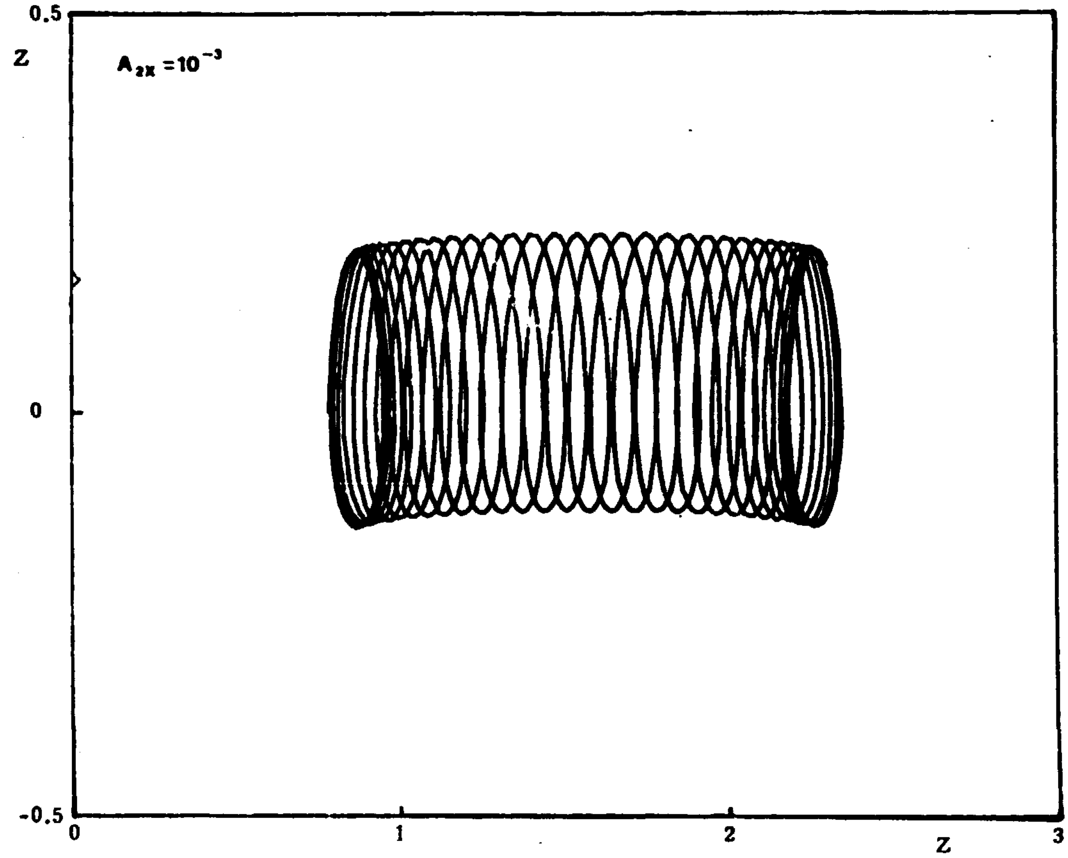

FIG. 13