INTERNAL SYMMETRIES OF NON-ABELIAN GAUGE FIELD CONFIGURATIONS

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ABSTRACT

Which gauge transformations are symmetries (in the sense of Schwarz, and Forgács and Manton) of a given gauge field configuration?

First, in topologically non-trivial gauge theories, there may be an obstruction for implementing gauge transformations on the fields; next, even those which can be implemented may fail to be symmetries.

For a test particle in such a background field, those gauge transformations which are symmetries generate ordinarily conserved Noether currents - one of which is the usual electric current.

This sheds a new light on the problem of "global color" in monopole theory and explains why no conserved electric charge can be defined in general in the non-Abelian Aharonov-Bohm experiment proposed by Wu and Yang.

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1. INTRODUCTION

In non-Abelian gauge theories, the word "symmetry" has two meanings: on the one hand, it means a transformation which changes the Lagrangean to an equivalent one. This is what we call a symmetry of the theory. On the other hand, this same word is used to refer to a transformation which leaves a specific field configuration invariant. It is in this sense that we talk, for example, of spherically symmetric monopoles, etc.

Space-time symmetries of gauge field configurations have been studied extensively in the literature [1-7]. The aim of this paper is to carry out a similar analysis for internal transformations. More exactly, we are concerned with the question: which gauge transformation are symmetries for a given non-Abelian gauge field configuration?

This problem is closely related to that of "global color" which arose recently in monopole theory [8,9]: The first step in defining a symmetry of a given field configuration is, in fact, the implementation of this transformation. The argument used for monopoles shows however, that, in topologically non-trivial situations, a topological obstruction may prevent us from doing so.

Next, an implementable transformation may fail to be a symmetry. Those which are symmetries form a subgroup H of the full gauge group. Since H acts trivially on space-time we shall call H an internal symmetry group. That H may be actually smaller then G has first been advocated, at a conceptual level, by Piecher [7].

Let us consider, for example, a monopole (A_j, Φ) [10-12] created in a Grand Unified Gauge Theory (GUT) [13] when the original gauge group $\tilde{G}$ is spontaneously broken to a subgroup G by the vacuum expectation values of the Higgs field Φ. G – the so-called residual "symmetry" group – is the gauge group for the new (spontaneously broken) theory. The actual symmetry group of the monopole configuration is however not G. To see this consider two monopole states in a given topological sector labelled by the the "Higgs charge" $[\Phi] \in \pi_2(\tilde{G}/G)$. Semiclassically, the path integral which expresses the transition amplitude between the two states, splits

$$K_{[\Phi]} = \sum_{\substack{\text{classical} \\ \text{solutions}}} \exp\{\frac{i}{\hbar} S(A_j^{cl}, \Phi^{cl})\} \tilde{K}_{cl} \quad (1.1)$$

where $(A_j{}^{cl},\Phi^{cl})$ is a classical solution to the field equations and $\tilde{K}_{cl}$ denotes the reduced propagator;

$$S(A_j^{cl},\Phi^{cl}) = -\int_{t_1}^{t_2}\int_{R^3}\{\tfrac{1}{4}(F_{ij}^{cl},F^{cl\,ij}) + \tfrac{1}{2}(D_i\Phi,D^i\Phi) + V(\Phi)\}d^3x\,dt \quad (1.2)$$

is the classical action for the configuration $(A_j{}^{cl},\Phi^{cl})$.

(1.1) shows clearly,that the actual symmetry group is not merely G, the stability group of the Higgs field alone, rather $H \subset G$, determined by the whole classical field configuration. This has first been noticed in the study of dyonic excitations of a monopole [14-16].

To identify the associated conserved quantities, observe that, for a test particle moving in our non-Abelian background, the internal symmetry group H of the given configuration becomes a symmetry for the particle Lagrangean. So,by the Noether theorem [6,17], we have a conserved current associated to each generator. In particular, we can get conserved electric charge. So internal symmetries generate "electric" electric charge just like rotations generate angular momentum!

This sheds a new light on the role of internal symmetries: while the total YM current is merely covariantly conserved, those components parallel to internal symmetry generators are already ordinarily conserved.

The main application of our theory is to the "color problem" [8,9] in monopole theory. We show first that a subgroup K of G is implementable if and only if the standard transition function [10-12,18] $h(t) = \exp 4\pi Q t$, $0 \le t \le 1$ - where Q is the "non-Abelian charge" of Goddard,Nuyte and Olive [18] - is homotopic to a loop in

$$Z_G(K) = \{g \in G \mid gk = kg,\ \forall k \in K\}, \quad (1.3)$$

the centralizer of K in G. In particular,G itself is implementable iff $h(t)$ is homotopic to a loop in $Z(G)$, the centre of G [19]. This condition is expressed in terms of the non-Abelian charge as

$$(1.4) \qquad \exp 4\pi z(Q) = 1,$$

where $z: \mathfrak{g} \longrightarrow Z(\mathfrak{g})$ is the projection onto the centre $Z(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G. (1.4) is a constraint on the Higgs charge (see Section 6).

Next we show that K is a symmetry if and only if it is a subgroup of

$$(1.5) \qquad Z_G(Q) = \{g \in G \mid g^{-1}Qg = Q\},$$

the centralizer of the non-Abelian charge Q. The whole G is a symmetry if and only if Q belongs to the centre.

From a mathematical viewpoint, implementability and symmetry are thus very different notions. Are they physically different? Observe first that K is or is not implementable simultaneously for all monopoles in a chosen topological sector. However, in each topological sector, there is only one stable monopole [12,20,21]. On the other hand, the main contrubution to the path integral (1.1) comes from the neighbourhood of this stable monopole which has the least energy. We show below that, for the unique stable monopole of a given topological sector, G is implementable exactly when G is an internal symmetry (for a subgroup K of G the situation is more complicated).

A second illustration is provided by the non-Abelian Aharonov-Bohm experiment, proposed by Wu and Yang to test the existence of gauge fields [22,23]. No topological obstruction arises in this case for implementing $SU(2)$ - gauge transformations. There is however an ambiguity: $SU(2)$ admits two inequivalent implementations. Even worse, for a given field configuration, none of the implementations is a symmetry in general. This explains why the electric charge of a nucleon moving in such a background field is not conserved in general [22,23].

2. IMPLEMENTABILITY OF GAUGE TRANSFORMATIONS

Let G denote a compact and connected Lie group and let us consider a gauge theory with gauge group G over (possibly a portion of) space-time M. Let us choose a covering of M by contractible open sets V_α. In each V_α the Yang-Mills field is given by a gauge potential $A_\mu{}^\alpha$, which satisfy, with the transition functions $h_{\alpha\beta}$: $V_\alpha \cap V_\beta \longrightarrow G$, the consistency relation

$$(2.1) \quad A_\mu{}^\alpha(x) = (h_{\alpha\beta})^{-1}(x) A_\mu{}^\beta(x) h_{\alpha\beta}(x) + (h_{\alpha\beta})^{-1}(x)\, \partial_\mu h_{\alpha\beta}(x)$$

for all $x \in V_\alpha \cap V_\beta$.

Similarly, a matter field Φ is specified by giving, in each V_α, a local representative Φ^α which transforms according to a unitary representation $\Phi \longrightarrow g\cdot\Phi$ of G. The Φ^α's satisfy the consistency relation

$$(2.2) \quad \Phi^\alpha(x) = h_{\alpha\beta}(x)\cdot\Phi^\beta(x)$$

Let K be a group and consider a fixed field configuration (A_μ, Φ). Let us assume that

(i) K acts on M, $x \longrightarrow k\cdot x$;

(ii) in each V_α a G-valued function $\tau_k{}^\alpha$ is associated to each $k \in K$ such that

$$(2.3) \quad \tau^\alpha_{k_1 k_2}(x) = \tau^\alpha_{k_1}(k_2\cdot x)\,\tau^\alpha_{k_2}(x)$$

and which satisfy the consistency condition

$$(2.4) \quad \tau^\alpha_k(x) = (h_{\alpha\beta}(k\cdot x))^{-1}\tau^\beta_k(x)\; h_{\alpha\beta}(x)$$

(2.3) - (2.4) imply that <u>K is implementable</u>, i.e. for each $x \in V_\alpha$

$$(2.5) \quad (k\cdot A_\mu)^\alpha(x) = \tau^\alpha_k(x) k^\nu_\mu(x) A^\alpha_\nu(x)[\tau^\alpha_k(x)]^{-1} - \partial_\mu \tau^\alpha_k(x)[\tau^\alpha_k(x)]^{-1}$$

where $\left(k_\mu{}^\nu(x)\right)$ is the matrix of the linear map on $T_xM \longrightarrow T_{k\cdot x}M$ induced by $x \longrightarrow k\cdot x$, and

$$(2.6)\qquad (k\cdot\Phi)^{\alpha}(x) = \tau_k^{\alpha}(x)\cdot\Phi^{\alpha}(x)$$

are well-defined, i.e. have the correct transformation rules (2.1)-(2.2), and $k \longrightarrow k\cdot A_\mu$ (respectively $k \longrightarrow k\cdot\Phi$) is a group action.

(iii) Following Schwarz [1] K is called a symmetry group for the configuration (A_μ, Φ) with respect to this implementation if, furthermore,

$$(2.7)\qquad A_\mu^{\alpha}(k\cdot x) = (k\cdot A)_\mu^{\alpha}(x)$$

and

$$(2.8)\qquad \Phi^{\alpha}(k\cdot x) = (k\cdot\Phi)^{\alpha}(x)$$

We want to apply this general definition to a subgroup K of G, acting trivially on M: $x \longrightarrow k\cdot x = x, \forall k \in K$.

Notice that the conditions above are trivially satisfied by $\tau_k{}^{\alpha}(x) = 1\ \forall \alpha, k, x$. This is however a trivial action. To have a sensible theory, some regularity conditon has to be imposed. In this paper we consider a very strong one: we shall require that, for each x, $\tau_{(.)}{}^{\alpha}(x)$ is the restriction to K of an automorphism of G. [8,9,19]. So $K\subset G$ implementable means now the existence of a family of G-automorphisms $\tau^{\alpha}(x)$ such that

$$(2.3')\qquad \tau_{k_1 k_2}^{\alpha}(x) = \tau_{k_1}^{\alpha}(x)\ \tau_{k_2}^{\alpha}(x)$$

satisfying the consistency condition

$$(2.4')\qquad \tau_k^{\alpha}(x) = (h_{\alpha\beta}(x))^{-1}\tau_k^{\beta}(x)\ h_{\alpha\beta}(x)$$

Such an action is an internal symmetry if, additionally,

$$(2.7')\qquad A_\mu^{\alpha}(x) = \tau_k^{\alpha}(x)\ A_\mu^{\alpha}(x)[\tau_k^{\alpha}(x)]^{-1} - \partial_\mu\tau_k^{\alpha}(x)[\tau_k^{\alpha}(x)]^{-1}$$

and

$$(2.8')\qquad \Phi^{\alpha}(x) = \tau_k^{\alpha}(x)\cdot\Phi^{\alpha}(x)$$

We study first implementabiliy. Following the pattern in monopole theory, we show that a <u>topological obstruction</u> may prevent us from implementing K [8,9,19].

Indeed, let us assume that K is implementable, and let $x_0 \in M$ be an arbitrary reference point. There is no loss of generality in assuming $\tau_k{}^{\alpha}(x_0) = k$ since this can always be achieved by replacing $\tau_{(.)}{}^{\alpha}(x)$ by $\tau_{(.)}{}^{\alpha}(x).[\tau_{(.)}{}^{\alpha}(x_0)]^{-1}$. $\tau^{\alpha}(x)$ belongs then, for each x, to $(\mathrm{Aut}\ G)_0$, the connected component of the group of automorphisms of G. $(\mathrm{Aut}\ G)_0$ is known however to consist of inner automorphisms for any compact and connected G [24,25]. It follows that, for each $x \in V_\alpha$, there exists an $h_\alpha(x) \in G$ such that

$$\tau_k^{\alpha}(x) = h_\alpha(x)\ k\ h_\alpha^{-1}(x)$$

The h_α's can be chosen to be smooth since the V_α's are contractible by assumption. The h_α define hence a gauge transformation in each V_α. In the new gauge (we still denote it by α) the action (2.5) - (2.6) of k becomes <u>rigid</u>, i.e. position-independent:

$$(2.9) \qquad (k\cdot A_\mu)^{\alpha}(x) = kA_\mu{}^{\alpha}(x)k^{-1}$$

and

$$(2.10) \qquad (k\cdot\Phi)^{\alpha}(x) = k\cdot\Phi^{\alpha}(x).$$

The consistency condition (2.4) requires now

$$(2.11) \qquad k^{-1}h_{\alpha\beta}(x)k = h_{\alpha\beta}(x),\ \forall k \in K,\ x \in V_\alpha \cap V_\beta$$

where the $h_{\alpha\beta}$ are the <u>new</u> transition functions between the rigid gauges in V_α and V_β. By reversing the argument we see that, by (2.11), <u>K is implementable</u> if and only if there exist gauges such that all transition functions $h_{\alpha\beta}(x)$ take their values in

$$(2.12) \qquad Z_G(K) = \{g \in G \mid g^{-1}kg = k,\ \forall k \in K\},$$

the <u>centralizer of K in G.</u> In particular, <u>G itself</u> is implementable if and only if all transition functions belong to Z(G), the <u>centre of G.</u>

3. INTERNAL SYMMETRIES

Let K be a connected Lie group with Lie algebra $\mathfrak{k}$, and assume that K is implementable. Let its internal action be given by a family τ_α. We can work infinitesimally: set

$$(3.1)\qquad \omega^\alpha_\kappa(x) = \frac{d}{dt}\Big|_{t=0} \tau^\alpha_{(\exp -t\kappa)}(x)\ ,\ \kappa \in \mathfrak{k}\ , x \in V_\alpha.$$

The infinitesimal action of $\kappa \in \mathfrak{k}$ corresponding to the considered internal action of $K \subset G$ is given by

$$(3.2)\qquad (\kappa \cdot A_\mu) = D_\mu \omega_\kappa$$

$$(3.3)\qquad (\kappa \cdot \Phi) = \omega_\kappa \cdot \Phi$$

(to keep the notation simple, we dropped the indice α.)

(2.4') implies that $\omega_\kappa(x)$ is a "Higgs" field of the adjoint type. The property (2.3) requires now

$$(3.4)\qquad \omega_{[\kappa_1,\kappa_2]}(x) = [\omega_{\kappa_1}(x), \omega_{\kappa_2}(x)],\ \forall \kappa_1,\kappa_2 \in \mathfrak{k},\ x \in M.$$

so, taking into account our regularity- and normalization conditions, $\kappa \dashrightarrow \omega_\kappa(x)$ is, for each x, the restriction to $\mathfrak{k}$ of a Lie algebra automorphism $\mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying $\omega_\kappa(x_0) = \kappa$.

By (3.3) and (3.4), if the action of K is an internal symmetry, then

$$(3.5)\qquad D_\mu \omega_\kappa = 0$$

and

$$(3.6)\qquad \omega_\kappa \cdot \Phi = 0.$$

Conversely, any normalized solution of (3.5)-(3.6) provides us with an internal action of $k=\exp -2\pi\kappa$. Indeed, (3.5) is solved by parallel transport,

$$(3.7)\qquad \omega_\kappa(x) = g(x)\kappa\, g^{-1}(x)$$

where g(x) is the non-integrable phase factor

$$(3.8) \qquad g(x) = \mathcal{P}(\exp -\int_{x_0}^{x} A_\mu \, dx^\mu)$$

(3.8) is in general path-dependent. Let $\kappa \in \mathfrak{k}$ such that (3.7) is nevertheless path-independent. Let us assume (3.6) is also satisfied (this is automatic if $D_\mu \Phi = 0$, since in this case $\Phi(x) = g(x)\Phi_0 g(x)^{-1}$ and so

$$\omega_\kappa(x)\cdot\Phi(x) = g(x)\kappa g(x)^{-1}g(x)\Phi_0 g(x)^{-1} = \kappa\cdot\Phi_0 = 0$$

since $\kappa \in \mathfrak{k} \subset \mathfrak{g}$).

$k = \exp -2\pi\kappa$ is now implementable:

$$(3.9) \qquad \tau_k(x) = \exp(-\omega_\kappa(x))$$

admits, as one proves easily, the properties (2.3') - (2.4').The corresponding action of k on the fields is plainly a symmetry: it satisfies (2.7') and (2.8').

Those $\eta \in \mathfrak{g}$ for which (3.7) is path-independent and also (3.6) is satisfied generate a connected subgroup H of G. (By (3.7) the group-property (3.4) is now automatic.) Some differential geometry shows that H is in fact the centralizer of the holonomy algebra of the Yang-Mills potential [7,25].

H is the <u>maximal internal symmetry group</u> of the YMH configuration we consider: the argument above shows plainly that any subgroup K of G which is a symmetry group is necessarily a subgroup of H.

4. CONSERVED CHARGES

If we have symmetries, we expect to find conservation laws. What are the conserved quantities generated by internal symmetries? Charges! To see this, consider a test particle ψ moving in a background YMH field (A_μ, Φ). For the sake of simplicity we consider only a spin-1/2 Dirac particle, with Lagrangean :

(4.1) $$-\mathcal{L} = \bar{\psi}(\gamma^\mu D_\mu + c\Phi + m)\psi$$

where c is a group-independent constant, and ψ is assumed to transform according to a unitary representation U of G. ψ is just another matter field, so, as explained in Section 2, in each V_α it is described by a local representative ψ^α which transforms according to U. The consistency condition (2.2) becomes now

(4.2) $$\psi^\alpha(x) = U(h_{\alpha\beta}))\psi^\beta(x), \quad x \in V_\alpha \cap V_\beta.$$

Let us assume that K is a connected group of internal symmetries for the given background YMH configuration (A_μ, Φ). Let K be implemented by $\tau^\alpha(x) \in \mathrm{Aut}G|K$ (restriction of an automorphism of G to K). According to (2.6), K acts on ψ as

(4.3) $$(k\cdot\psi)^\alpha(x) = U(\tau^\alpha{}_k(x))\psi^\alpha(x);$$

the infinitesimal action reads

(4.4) $$(\kappa\cdot\psi)^\alpha(x) = (\omega_\kappa(x))\cdot\psi^\alpha(x).$$

The point is that the implementation (4.3) leaves invariant the Lagrangean (4.1). This follows by straightforward calculation. Hence the internal symmetry of the background field becomes a Noether symmetry for the test particle. This is furthermore an internal symmetry, since the action on space-time is trivial. Consequently, for each generator $\kappa \in \mathfrak{K}$ the current

(4.5) $$j^{\mu}_{\kappa} = \frac{\delta \mathcal{L}}{\delta(\partial^{\mu}\psi)}(\kappa\cdot\psi) = \bar{\psi}\,\gamma^{\mu}\omega_{\kappa}\cdot\psi\ .$$

is <u>ordinarily conserved</u> [17]:

(4.6) $$\partial_{\mu} j_{\kappa}{}^{\mu} = 0\ .$$

Let us consider the non-Abelian current

(4.7) $$J_{a}{}^{\mu} = \bar{\psi}\,\gamma^{\mu}\tau_{a}\cdot\psi\ ,\ a = 1,\ldots,\dim\ .$$

where the τ_a's are a basis of the Lie algebra.

The gauge-invariance of the Lagrangean (4.1) implies that (4.7) is <u>covariantly</u> conserved:

(4.8) $$D_{\mu}J^{\mu} = 0.$$

However, the ω_{κ}- component

(4.9) $$j_{\kappa}{}^{\mu} = \mathrm{Tr}(\omega_{\kappa}J^{\mu})$$

is already <u>ordinarily conserved</u> since $D_{\mu}\omega_{\kappa} = 0$ by assumption. It is straightforward to verify that (4.9) is just (4.5), as anticipated by the notation.

Interestingly, the formula (4.9) has already been proposed to define <u>conserved electric charge</u> [26]. Now we understand its origin : it is the (ordinarily) conserved current associated to an internal symmetry generator. This sheds a new light on the role of internal symmetries.

It is instructive to pursue this direction. Let us assume in fact that $D_{\mu}\Phi = 0$ and so the YM field satisfies the vacuum field equations

(4.10) $$D_{\mu}F^{\mu\nu} = 0\ ,\quad D_{\mu}(\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma})/2 = 0\cdot$$

and identify the <u>electromagnetic field</u> as the ω_{κ}-component of $F_{\mu\nu}$:

(4.11) $$\mathcal{F}_{\mu\nu}(x) = (1/e)\ \mathrm{Tr}(F_{\mu\nu}\ \omega_{\kappa}/|\omega_{\kappa}|)$$

where e is a coupling constant. (4.10) implies that $\mathcal{F}_{\mu\nu}$ satisfies the vacuum Maxwell equations

(4.12) $$\partial_\mu \mathcal{F}^{\mu\nu} = 0 , \quad \partial_\mu(\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma})/2 = 0.$$

Let us define the (semiclassical) electric charge operator by

(4.13) $$Q_{em}(x) = e\omega_\kappa(x)/|\omega_\kappa(x)|.$$

The electric charge of any particle in the theory is an eigenvalue of (4.13). As demonstrated in [27,28], these eigenvalues are quantized if and only if κ generates a U(1) (rather then merely a torus-) subgroup of G. If so, all electric charges are integer multiples of

(4.14) $$q_{min} = e/|\kappa_0|,$$

where κ_0 is a "minimal" U(1) generator (i.e. such that $\exp 2\pi\kappa t = 1$ the first time for t=1) parallel to κ.

Let us assume that ψ is an eigenstate of Q_{em} with eigenvalue nq_{min}. The particle's electric charge is hence

(4.15) $$q = \int_{R^3} j^0_\kappa(x)d^3x = \int_{R^3} \bar{\psi}\, \gamma^0 \omega_\kappa \cdot \psi = nq_{min} \int_{R^3} \bar{\psi}\, \psi = nq_{min}$$

as expected. If the background field is that of monopole, we have further properties (see Section 7).

5. ASYMPTOTIC PROPERTIES OF MONOPOLE CONFIGURATIONS

The principal application of the general theory outlined in the preceeding sections is to non-Abelian monopoles. Here we resume briefly those properties we need in the sequel. (For reviews see, e.g., [10-12]).

Let us consider a YMH theory with a compact, connected and simply connected (and hence semisimple) "unifying" gauge group $\tilde{G}$. At some energy scale ($O(10^{14})$ GeV) the $\tilde{G}$-symmetry is spontaneously broken to a subgroup G of $\tilde{G}$ by the v.e.v. of the Higgs field Φ. Consequently, the asymptotic values of the Higgs field provide us with a map

$$(5.1) \qquad \Phi : S^2 \dashrightarrow \tilde{G}\cdot\Phi_0 \cong \tilde{G}/G.$$

Magnetic monopoles are everywhere-regular, static, finite-energy, purely- magnetic solutions to the YMH equations, satisfying (5.1) and the "finite-energy" condition

$$(5.2) \qquad D_\mu\Phi = 0 \text{ on } S^2.$$

The map (5.1) provides us with the fundamental topological invariant

$$(5.3) \qquad [\Phi] \in \pi_2(\tilde{G}/G)$$

we call the Higgs charge.

The injective homomorphism $\partial : \pi_2(\tilde{G}/G) \dashrightarrow \pi_1(G)$ is now an isomorphism since $\tilde{G}$ is assumed to be simply connected.

In a previous paper [29] we studied the Higgs charge in some detail. We have shown that, for any compact and connected Lie group G,

$$(5.4) \qquad \pi_1(G) = \pi_1(G)_{free} + \pi_1(G_{ss})$$

(direct sum). Here $\pi_1(G)_{free} \cong Z^p$, where p is the dimension of the centre $Z(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G, and G_{ss} is the subgroup of G generated by the derived algebra $[\mathfrak{g},\mathfrak{g}]$. G_{ss} is semisimple, so $\pi_1(G_{ss})$ is a finite Abelian group.

The isomorphism $\pi_1(G)_{free} \cong Z^p$ is established explicitly as follows: let $\Gamma = \{ \xi \in \mathcal{G} \mid \exp 2\pi\xi = 1 \}$ denote the unit lattice of G, and consider the image $z(\Gamma)$ of Γ under the projection map $z: \mathcal{G} \to Z(\mathcal{G})$. $z(\Gamma)$ is a p-dimensional lattice in $z(\Gamma)$, and, as we have shown in [29],

$$(5.5) \qquad \rho([\gamma]) = \frac{1}{2\pi} \int_\gamma z\,(g^{-1}dg) \quad \in Z(\mathcal{G}),$$

where γ is a loop in G, is an isomorphism between $\pi_1(G)_{free}$ and $z(\Gamma)$. If $\zeta_1, \ldots, \zeta_p$ is a Z-basis for the lattice $z(\Gamma)$, then

$$(5.6) \qquad \rho([\gamma]) = \sum_j^p m_j \zeta_j ;$$

$[\gamma] \to (m_1, \ldots, m_p)$ is the aformentioned isomorphism.

It is a known fact that any loop in G is homotopic to one of the form $\gamma(t) = \exp 2\pi\xi t$, $\xi \in \mathcal{G}$. The image of such a loop is simply

$$(5.7) \qquad \rho(\gamma) = z(\xi).$$

(5.2) implies that on S^2 the YMH equations decouple and we are left with a pure G-valued Yang-Mills theory. On S^2 the field equation is simply

$$(5.8) \qquad D_j F^{jk} = 0$$

The general solution of (5.7) has been found by Goddard, Nuyts and Olive [18]: let us cover S^2 with the contractible open sets $V_1 = S^2 \setminus \{\text{south pole}\}$ and $V_2 = S^2 \setminus \{\text{north pole}\}$. There exist gauges over $V_{1,2}$ - the so-called U-gauge- such that $\Phi = \Phi_0$ and the solution of (5.8) is

$$(5.9) \qquad A^{1,2}_\theta = 0\ , \quad A^{1,2}_\varphi = \pm Q(1 \mp \cos\theta)$$

Q - the non-Abelian charge- is a constant vector in the Lie algebra. Q can be chosen, with no loss of generality, in any given Cartan subalgebra. To have a well-defined theory, Q must be quantized:

(5.10) $\exp 4\pi Q = 1$.

A loop in $\delta[\Phi]$ representing the Higgs charge is then expressed as

(5.11) $h(t) = \exp 4\pi Q t \;, 0 \leq t \leq 1.$

(5.11) is in fact the transition function between the U-gauges over V_1 and V_2. By (5.7)

(5.12) $\rho(\Phi) \; (= \rho(\delta[\Phi]) \;) = 2z(Q).$

Let us decompose Q as

(5.13) $Q = z(Q) + Q'$

where Q' belongs to the derived algebra. The result of Brand and Neri [20] tells us that the stability of the monopole depends only on Q':the monopole is stable if and only if, for any root α of the semisimple Lie algebra $[\mathfrak{g},\mathfrak{g}]$,

(5.14) $2\alpha(Q') = 0$ or 1 for any root α of $[\mathfrak{g},\mathfrak{g}]$,

cf. [21].In each topological sector there exists hence exactly one stable monopole [12].

6. THE PROBLEM OF GLOBAL COLOR FOR MONOPOLES

Let us now consider a non-Abelian monopole (A_j, Φ), and let G denote the little group of the Higgs field at infinity. Let K be a subgroup of G. According to the general theory of Section 2, K is implementable if and only if, in V_α $(\alpha = 1,2)$, there exist G-automorphisms $\tau^\alpha(x)$ which satisfy the consistency condition (2.4') with the transition function (5.11). Both V_1 and V_2 are contractible, so we can go to rigid gauges so that the consistency condition reads:

$$(6.1) \qquad k\, h(x) = h(x) k, \quad \forall k \in K, x \in V_1 \cap V_2.$$

where h is the transition function for the new (rigid) gauges.

The homotopy class of the transition function is however independent of the choice of a gauge, so (6.1) holds if and only if any transition function - in particular (5.11) - is homotopic to one in

$$(6.2) \qquad Z_G(K) = \{g \in G \mid gkg^{-1} = k, \forall k \in K\},$$

the centralizer in G of K. The full "residual" group G is implementable if and only if (5.11) is homotopic to a loop in the centre of G [19].

Requiring the implementability of a subgroup K is a topological constraint on the Higgs charge. Indeed, (5.11) homotopic to a curve in $Z_G(K)$ means exactly that

$$(6.3) \qquad \delta[\Phi] = [h(t)] \in \mathrm{Im}\, i_*,$$

where i_* is the homomorphism $i_*: \pi_1(Z_G(K)) \longrightarrow \pi_1(G)$ induced by the inclusion map $i : Z_G(K) \hookrightarrow G$. (Such a condition has been encountered before in the study of the fate of Grand Unified monopoles under subsequent symmetry breakings [33,34]).

To translate (6.3) to more down-to-earth terms, let us study first the case K = G. $i_*\pi_1(Z(G))$ lies in the free part, so

(i) if G is implementable, $\delta[\Phi] \in \pi_1(G)_{free}$.

This implies at once that if $\pi_1(G)$ is <u>finite</u> (as it happens in some GUTs - see Section 8 -) then G is <u>never implementable</u> for topologically non-trivial Higgs fields.

Let $\zeta_1,\ldots,\zeta_p$ be a Z-basis of $z(\Gamma)$. For each j = 1,...,p there exists a least positive integer M_j such that $\exp 2\pi\zeta_j M_j = 1$ [29]. The loops

(6.4) $\gamma_j(t) = \exp 2\pi\zeta_j M_j t, \; j = 1,\ldots,p$

generate $\pi_1(Z(G))$ -and thus also its image under i_*. $[\gamma_j] \in \pi_1(G)_{free} \simeq Z^p$ has "quantum" numbers $(0,\ldots,M_j,\ldots,0)$. The parameter space of Im i_* consists hence of integer combinations of these p-tuples. (6.3) means thus that

(ii) $[\Phi] \simeq (m_1,\ldots,m_p)$ must satisfy

(6.5) $m_j = n_j M_j$ for some integer $n_j, j = 1,\ldots,p$.

Conversely, (i) and (ii) imply (6.3).

The physically most interesting situation is when $Z(\mathfrak{g})$ is 1-dimensional. In this case (6.5) is simply

(6.6) $m = n.M,$

where M labels the homotopy class of the central $U(1)$.

The condition of implementability has a nice expression in terms of the non-Abelian charge Q. Indeed, if $Z(\mathfrak{g}) \neq 0$, (6.3) is equivalent to

(6.7) $\exp 4\pi Q' t,\; 0 \leq t \leq 1,$ is <u>contractible</u> in G_{ss};

and

(6.8) $\exp 4\pi\, z(Q) = 1.$

First, (6.7) is exactly (i) above. On the other hand, (5.11) homotopic to a curve $\gamma(t)$ in Z(G) means that (5.11) and $\gamma(t)$ have the same image under ρ. But a $\gamma(t)$ in Z(G) is homotopic to a loop of the form $\gamma(t) = \exp 2\pi\zeta t$, with $\zeta \in Z(\mathfrak{g})$, whose image under ρ is ζ itself. Hence $\rho(\gamma(t))$ = (by (5.7)) = $2z(Q) = \zeta$. However, $\exp 2\pi\zeta = 1$, proving (6.8).

Conversely, if (6.7) and (6.8) are satisfied, then (5.11) is homotopic to $\gamma(t) = \exp 4\pi z(Q)t \subset Z(G)$ since they have the same image under ρ.

If $\mathfrak{g}$ has no centre, $Z(G)$ is a discrete subgroup of G and thus $\pi_1(G)$ is finite, so that the constraint (6.3) is violated.

Similar, although slightly more complicated, results hold for a general K. Let us assume, for simplicity, that $\pi_1(G)$ is free, Z^p. (This happens, for example, if Φ is in the adjoint representation). $\rho[i_*(\pi_1(Z_G(K))]$ is a sublattice in $z(\Gamma)$, so it is generated by elements $\xi_j \in Z(\mathfrak{g}), j=1,\ldots,r < p$. There is no loss of generality in assuming that each ξ_j is parallel to a suitable ζ_j, $\xi_j = c_j\zeta_j$. The coefficient c_j here is an integer, since the ζ_k's form a Z-basis in $z(\Gamma)$. Denote L_j the least common multiple $L_j = [c_j, M_j]$, $j=1,\ldots,r$ with M_j as above, and let

$$M = \left[\frac{L_1}{c_1},\ldots,\frac{L_r}{c_r}\right]$$

be the least common multiple of the L_j/c_j's. K implementable means now the <u>quantization condition</u>

$$\text{(6.9)} \qquad \exp 4\pi M\, z(Q) = 1.$$

Alternatively, the implementability condition (6.3) is also expressed as

$$\text{(6.10)} \qquad m_j = \begin{cases} c_j n_j & \text{for some integer } n_j,\ j=1,\ldots,r \\ 0 & \text{for } j=r+1,\ldots,p. \end{cases}$$

For $K = G$ $c_j = M_j$ so $M = 1$ and (6.9) reduces to (6.8)

Having settled the problem of implementablity, let us ask if K is a <u>symmetry group</u>. Using the infinitesimal approach of *Section 3 we see that this happens if and only if* (3.7) is path-independent for each generator κ of K (since $D_\mu\Phi = 0$ and thus (3.6) is automatically satisfied). This is however a gauge-invariant condition so we can work in the U-gauge (5.8), where $\omega_\kappa = \kappa$, in V_1 <u>and</u> in V_2 so pathindependence means simply

$$(6.11) \qquad \kappa \in \mathfrak{K} = Z_{\mathfrak{g}}(Q) = \{ \eta \in \mathfrak{g} \mid [\eta, Q] = 0 \}.$$

We conclude that any symmetry group K must belong to the centralizer of Q in G. Notice that $Z(\mathfrak{g})$ is always in (6.11)

In particular, the whole of G is a symmetry with respect to the internal action defined by (3.2)-(3.3) if and only if Q is in the centre of the Lie algebra.

From a mathematical viewpoint, to be a symmetry is thus a much stronger condition then to be merely implementable. What is the physical difference between the two requirements?

Consider first the case K= G. G is simultaneously implementable or not implementable for an entire topological sector. Let us assume [Φ] satisfies (6.7) and (6.8) and thus G is implementable for all monopoles in this homotopy class. In particular, [Φ] belongs to the free part of $\pi_2(\widetilde{G}/G)$ .However, there is exactly one stable monopole in this homotopy sector, namely the one with Q'= 0. But this implies that Q = z(Q) is in the centre -so, for the unique stable monopole, symmetry and implementability are the same. For the other (unstable) monopoles the two statements are different.

The main contribution to the path integral (1.1) comes however from the neighbourhood of the stable solution, and thus, semiclassically, implementability and symmetry are essentialy the same.

The general situation when K $\subsetneq$ G is more complicated and the conclusion is different. Again, the full topological sector is simultaneously implementable or not. The non-Abelian charge of the unique stable monopole of our homotopy class may however not belong to $Z_{\mathfrak{g}}(\mathfrak{K})$, and thus K may fail to be a symmetry for the stable monopole. If, on the other hand, we choose Q in $Z_{\mathfrak{g}}(\mathfrak{K})$, K is a symmetry - but the corresponding monopole is generally unstable (see Section 8 for examples).

7. CONSERVED CHARGES AND ELECTROMAGNETIC PROPERTIES IN THE FIELD OF A MONOPOLE

Let us consider now a spin 1/2 Dirac field ψ coupled to a background monopole field (A_j,Φ). As explained in Section 4, to any symmetry generator η – i.e., to any η which commutes with the non-Abelian charge vector Q – is associated a conserved current. In the U-gauge this current is simply

$$(7.1)\qquad j_\eta^\mu = e\bar{\psi}\gamma^\mu\eta\psi.$$

In particular, a generator ζ of the centre is an internal symmetry direction for all monopoles created when the symmetry is spontaneously broken to G. In other words, ζ is an admissible electromagnetic direction for all monopoles in the theory. (This is the choice made in [29]—the generalization of the standard approach [27] valid when Φ is in the adjoint representation and $Z(\mathfrak{g})$ is 1-dimensional).

For a fixed monopole configuration however, we have *slightly more freedom: any vector which commutes with the* non-Abelian charge is admissible.

Monopoles carry also a magnetic charge. This is defined by the flux integral

$$(7.2)\qquad g = \frac{1}{4\pi e}\int_{S^2} \mathcal{F}_{\mu\nu}\ ,$$

where the electromagnetic field $\mathcal{F}_{\mu\nu}$ is defined by (4.11). In the U-gauge (7.2) is calculated at once:

$$(7.3)\qquad g = \frac{1}{e|\eta|}\,\mathrm{Tr}\,(Q\eta)$$

Observe, that the magnetic charge is quantized: indeed, $Q = (n/2)Q_0$ for some integer n, where Q_0 is a minimal U(1)-generator parallel to Q. Consequently g is an integer multiple of

$$(7.4)\qquad g_{min} = \frac{1}{2e|\eta_0|}\,\mathrm{Tr}\,(Q_0\eta_0)\ ,$$

where η_0 is a minimal $U(1)$-generator parallel to η.

The comparision of (7.4) with (4.14) shows now that the electic-respectively magnetic charges satisfy the generalized Dirac condition

$$(7.5) \qquad 2q_{min}g = \frac{Tr(2Q\ \eta_0)}{|\eta_0|^2}.$$

Notice that the value of (7.5) depends in general on Q and not only on the Higgs charge. In other words, it is not a topological invariant. If, however, η is in the centre, $\eta \in Z(\mathfrak{h})$, then the r.h.s. of (7.5) satisfies

$$(7.6) \qquad Tr(2Q\zeta) = Tr(2z(Q)\zeta) = Tr(\rho(\Phi)\zeta),$$

so (7.5) becomes rather

$$(7.7) \qquad 2q_{min}g = \frac{Tr(\rho(\Phi)\eta_0)}{|\eta_0|^2}$$

which is already a topological invariant: it depends only on $\rho(\Phi)$, the free part of the Higgs charge cf. [29].

Let us consider the particular case when $\pi_1(G) = \pi_1(G)_{free} \cong Z$. Let $[\Phi] \cong m$. (7.7) is simply

$$(7.8) \qquad 2q_{min}g = m/M,$$

where the integer M labels the homotopy class of the central $U(1)$.

On the other hand, G implementable means now that $m = n.M$ (cf. (6.6)). We conclude that, in this special case, G is implementable exactly when the generalized Dirac condition (7.7) reduces to the original (integer) Dirac condition. (If $Z(\mathfrak{h})$ is not one-dimensional, this conclusion is however false, see the $SO(10)$-example below).

8. EXAMPLE: GRAND UNIFIED MONOPOLES

As a first illustration, we consider monopoles in the $\tilde{G} = SU(5)$ GUT [13,30]. Following the general pattern, let us assume Φ is in the <u>24</u> (adjoint) representation; the choice

$$(8.1) \qquad \Phi_0 = \nu i \,\mathrm{diag}\,(2,2,2,-3,-3)$$

yields the little group

$$(8.2) \qquad G = S[U(3)\times U(2)] = [SU(3)_C \times SU(2)_W \times U(1)_Y]/Z_6$$

$Z(\mathfrak{g})$ is generated by (8.1) itself, and $\pi_1(G) \cong Z$. The "quantum number" $[\Phi] \cong m$ is calculated by

$$(8.3) \qquad m = \mathrm{Tr}_3(\rho(\Phi))/i = 2\,\mathrm{Tr}Q/i.$$

(trace on the upper 3x3 block, cf. [28,29]). The generating loop $\exp 2\pi\xi_0 t = \exp(2\pi\Phi_0/\nu t)$ of the centre of G has quantum number $M = 6$, so, according to (6.6), <u>G is implementable</u> if and only if <u>m is an integer multiple of 6, m = 6n.</u> This is seen alternatively from (6.8), observing that $z(Q) = (m/6)M$.

G contains the <u>color</u> subgroup

$$(8.4) \qquad SU(3)_c = \left[\begin{array}{c|c} A & \\ \hline & \mathbb{1}_2 \end{array}\right], \quad A \in SU(3)$$

whose centralizer is

$$(8.5) \qquad Z_G(SU(3)_c) = U(2)_{WS} = \left[\begin{array}{c|c} (\det B)^{-1}\mathbb{1}_3 & \\ \hline & B \end{array}\right], \quad B \in U(2)$$

$\pi_1(U(2)_{WS}) \cong Z$ is generated, e.g., by

$$(8.6) \qquad \gamma(t) = \exp 2\pi i \left[\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & & & -3 & \\ & & & & 0 \end{array}\right] t, \qquad 0 \le t \le 1,$$

whose homotopy class is labelled by $c = 3$. Hence, by (6.9), $SU(3)_C$ is implementable if and only if $\underline{m = 3n}$. (Alternatively, this follows from (6.9) noting that $M = 2$ now).

Similarly, consider

$$(8.7) \qquad SU(2)_W = \left[\begin{array}{c|c} 1_3 & \\ \hline & B \end{array}\right], \quad B \in SU(2)$$

the subgroup of <u>weak interactions</u>. $Z_G(SU(2)_W)$ is just

$$(8.8) \qquad U(3) = \left[\begin{array}{c|c} A & \\ \hline & (\det A)^{-1/2}\, 1_2 \end{array}\right], \quad A \in U(3)$$

$\pi_1(U(3)) \cong \mathbb{Z}$ is generated, e.g., by

$$(8.9) \qquad \gamma(t) = \exp 2\pi i t \operatorname{diag}(-2,0,0,1,1)$$

whose class in $\pi_1(G)$ is $c = 2$. Thus $SU(2)_W$ is implementable if and only if $\underline{m=2n}$.

Furthermore, <u>G is an internal symmetry group only for the stable charge-6 monopole</u> [32] given by

$$(8.10) \qquad Q = \zeta/2 = i \operatorname{diag}(1,1,1,-3/2,-3/2).$$

$SU(3)_C$ is a symmetry if and only if $Q \in U(2)_{WS}$. This is realized by <u>two different charge-3 monopoles:</u>

$$(8.11) \qquad Q_1 = (i/2) \operatorname{diag}(1,1,1,0,-3),$$

and

$$(8.12) \qquad Q_2 = (i/2) \operatorname{diag}(1,1,1,-1,-2).$$

<u>Only (8.12)</u> satisfies the BN condition (5.14) and is thus <u>stable</u> [32]. $SU(3)_C$ is hence a symmetry group simultaneously for a stable and an unstable monopole.

Similarly, $SU(2)_W$ is a symmetry if an only if $Q \in U(3)$ in (8.8). This condition is met by <u>two charge-2 monopoles</u>:

$$(8.13) \qquad Q_1 = (i/2) \operatorname{diag}(2,0,0,-1,-1),$$

and

(8.14) $Q_2 = (i/2)\ \text{diag}\ (1,1,0,-1,-1)$.

Both monopoles are hence $SU(2)_W$-symmetric, but only (8.14) is stable.

For the "elementary" monopole $2Q = i\ \text{diag}\ (1,0,0,0,-1)$, so the maximal symmetry group is

(8.15) $$H = \begin{bmatrix} U(1) & & & \\ & U(2)_c & & \\ & & U(1) & \\ & & & \cdot \end{bmatrix}.$$

At much lower energies (O(100 GeV)) the symmetry is further broken to $G = U(3)$ ($= [SU(3_c \times U(1)_{em}]/Z_3$) by a Higgs $\underline{5}$. $\pi_1(U(3)) \simeq Z$, and the quantum number m is still calculated by (8.2) [29].Z(u(3)) is generated by

(8.16) $Q_{em} = i\ \text{diag}\ (1,1,1,-3,0)$,

(a minimal generator). (8.16) is the usual choice for the electromagnetic direction.

The central U(1) has quantum number 3,so G=U(3) is implmentable for the U(3)-monopole if and only if $\underline{m = 3n}$ [19]. This is seen alternatively from (6.6) since $2z(Q) = m.Q_{em}/3$ in this case.

The color subgroup $SU(3)_c$ belongs to U(3);its centralizer in U(3) is

(8.17) $Z_{U(3)}(SU(3)_c) = U(1)_{em} = U(1)_{centre}$.

so $SU(3)_c$ is implementable if and only if $\underline{m = 3n}$ (alternatively, in (6.9) M = 1).

<u>G= U(3) is an internal symmetry</u> iff $Q \in Z(U(3))$, i.e. iff $Q = (m/2)Q_{em}$. But this is simultaneously the centralizer for $SU(3)_c$, so they are simultaneously symmetries or not.

The <u>charge-3</u> monopole given by $2Q = Q_{em}$ is <u>stable</u> by the BN condition,and is thus <u>U(3)-symmetric</u>.

The "<u>elementary</u>" S(U(3xU(2)) monopole survives the "phase transition" S(U(3)xU(2)) --> U(3) [34].The maximal symmetry group (8.15) is reduced however to

$$(8.18) \qquad H' = U(2)_C \times U(1)_{em}$$

since $U(2)_{WS}$ is broken to $U(1)_{em}$ in this process.

Let us consider the SU(5) $\underline{5}$-plet

$$(8.19) \qquad \psi = (d_R, d_B, d_G, e^-, \nu_e)_L .$$

where R,B,G refer to the quark colors. The internal symmetry group (8.18) is generated by

$$\sigma_0 = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -2 & \\ & & & & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & & & & \\ & & i & & \\ & i & & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & & & & \\ & & -1 & & \\ & 1 & & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

(8.20)

$$\sigma_3 = \begin{pmatrix} 0 & & & & \\ & i & & & \\ & & -i & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \qquad Q_{em} = \begin{pmatrix} i & & & & \\ & i & & & \\ & & i & & \\ & & & -3i & \\ & & & & 0 \end{pmatrix}.$$

All these generators are internal symmetries for ψ - considered as a test particle in the field of an SU(5)-GUT monopole.

Q_{em} is the standard choice for the electromagnetic direction. The corresponding electric charge is quantized in units of

$$(8.21) \qquad q_{min} = e/2\sqrt{3} = q/3.$$

The electromagnetic current is thus expressed as

$$(8.22) \qquad j^{\mu}_{em} = qi\{\tfrac{1}{3}(\bar{d}_R\gamma^{\mu}d_R + \bar{d}_B\gamma^{\mu}d_B + \bar{d}_G\gamma^{\mu}d_G) - \bar{e}\,\gamma^{\mu}e\}.$$

(8.22) is conserved in <u>all</u> background monopol fields, not only for the elementary one. The other four currents, (8.23a) $j_0^{\mu} = ci[(\bar{d}_B\gamma^{\mu}d_B + \bar{d}_G\gamma^{\mu}d_G) - 2\bar{e}\,\gamma^{\mu}e]$,

(8.23b) $j_1^{\mu} = ci[\,\bar{d}_B\gamma^{\mu}d_G + \bar{d}_G\gamma^{\mu}d_B\,]$,

(8.23c) $j_2^{\mu} = c\,[-\bar{d}_B\gamma^{\mu}d_G + \bar{d}_G\gamma^{\mu}d_B]$,

(8.23d) $j_3^{\mu} = ci[\,\bar{d}_B\gamma^{\mu}d_B - \bar{d}_G\gamma^{\mu}d_G]$,

(where $c = e/\sqrt{2}$), are however conserved only for the elementary monopole.

The corresponding "magnetic" charges - defined as the flux integral of the corresponding "electromagnetic" fields are

(8.24) $g^{em} = 1/2e,\ g_8 = 1/6c,\ g^1 = g^2 = g^3 = 0$,

so the generalized Dirac conditions read

(8.25a) $2q^{em}{}_{min}g^{em} = 1/3$,

(8.25b) $2q^{8}{}_{min}g^{8} = 1/3$,

(8.25c) $2q^{j}{}_{min}g^{j} = 0\ , j = 1,2,3$.

As a second example, consider $\underline{\tilde{G} = \mathrm{Spin}10}$ (the double covering of SO(10)) broken to

(8.26) $G = [\mathrm{Spin}6\times\mathrm{Spin}4]/\mathbb{Z}_2$

by a Higgs $\underline{54}$ (10x10 symmetric matrices) with basepoint

(8.27) $\Phi_0 = \mathrm{diag}\,(2,2,2,2,2,2,-3,-3,-3,-3)$

[31,35]. $\pi_1(G) \simeq \mathbb{Z}_2$. The Lie algebra $\mathfrak{G} = so(6)\times so(4)$ has trivial centre so G is <u>never</u> implementable.

Let us consider the (stable) elementary monopole given by

(8.28) $Q = (J_{56} - J_{78})/2$

where the J_{ab} are the usual rotation generators (anti-symmeric, imaginary, 10x10 matrices, $(J_{ab})_{ij} = -i(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})$).

The only vectors in $\mathfrak{g}$ which commute with Q are the multiples of Q, so the maximal symmetry algebra is the one generated by Q. This is the only choice of electromagnetic direction.

Electric charge is quantized in units

(8.29) $q_{min} = e/|Q| = e/2,$

The magnetic charge is

(8.30) $g = |Q|/2e = 1/e,$

so the original Dirac condition is satisfied. Implementability and integer Dirac condition are hence different in this case.

9. THE NON-ABELIAN AHARONOV-BOHM EXPERIMENT

Another tricky example is provided by the non-Abelian Aharonov-Bohm experiment proposed by Wu and Yang in their celebrated paper on the non-integrable phase factor [22]. They suggest in fact to set up an SU(2)-gauge field confined to a cylinder. If a nucleon beam is scattered around this flux line, a non-trivial interference would prove the existence of Yang-Mills fields.

It is not difficult to show [23,36] that there exists a gauge- analogous to the U-gauge (5.9) for monopoles - where the gauge field with $F_{ij} = 0$ in $M = R^3 \setminus \{\text{cylinder}\}$ is simply

$$(9.1) \qquad A_r = 0, \; A_\theta = 0, \; A_\rho = \alpha\sigma_3/i.$$

α here is a real parameter, defined modulo integers.

Let us try to implement $G=SU(2)$ by an $AutG_0$ -valued "Higgs" field $\tau_{(.)}(x)$ on M. As explained in Section 3, we can gauge any such $\tau_g(x)$ to identically g simultaneously in $V_1 = \{(r,\theta,\rho) \mid \epsilon \le \theta < \pi+\epsilon\}$ and $V_2 = \{(r,\theta,\rho) \mid \pi-\epsilon < \theta \le 2\pi\}$ since both V_1 and V_2 are contractible. The price to pay for this is that we introduce a transition function h - which is now just a constant element of SU(2). Consistency requires now

$$(9.2) \qquad g.h = h.g,$$

i.e. h must be in the centre of $SU(2)$. So we have two solutions: $h = 1$ or $h = (-1)$. We conclude that, although there is no obstruction to implement G=SU(2), there is an ambiguity. In the U-gauge (9.1) the two implementations are found explicitly as

either

$$(9.3) \qquad \tau^1_g(x) = g$$

or

$$\tau^2_g(x) = \begin{bmatrix} \exp i\rho/2 & 0 \\ 0 & \exp -i\rho/2 \end{bmatrix} g \begin{bmatrix} \exp -i\rho/2 & \\ 0 & \exp i\rho/2 \end{bmatrix} =$$

(9.4)

$$= \begin{bmatrix} g_{11} & \exp i\rho/2 g_{12} \\ \exp -i\rho g_{21} & g_{22} \end{bmatrix}$$

where $x = (r, \theta, \rho)$ and $g = (g_{ij})$ a matrix.

Are these implementations symmetries? The corresponding local expressions read

$$(9.5) \qquad \omega^1_\eta = \eta \, , \; \eta \in su(2),$$

and

$$(9.6) \quad \omega^2_\eta = \begin{bmatrix} \eta_{11} & \exp i\rho\eta_{12} \\ \exp -i\rho\eta_{21} & \eta_{22} \end{bmatrix} , \; \eta \in su(2).$$

To be an internal symmetry, ω_η must be covariantly constant. However,

$$(9.7) \qquad D\omega^1_\eta(x) = \frac{\alpha}{i} [\sigma_3, \eta] = \frac{1}{i} \begin{bmatrix} 0 & 2\alpha\eta_{12} \\ 2\alpha\eta_{21} & 0 \end{bmatrix}$$

and

$$(9.8) \qquad D\omega^2_\eta(x) = \frac{1}{i} \begin{bmatrix} 0 & (2\alpha-1)\eta_{12} \\ -(2\alpha-1)\eta_{21} & 0 \end{bmatrix}$$

respectively. We conclude that
either

<u>(i) 2α is neither 0 nor 1</u>, and then $\eta_{12} = \eta_{21} = 0$ so the only symmetry-direction is the one given by the field (9.1) itself;

or

<u>(ii) $2\alpha=0$ or $2\alpha=1$</u>. The whole SU(2) is then a symmetry. However, for $\alpha=0$ only the implementation (9.7), for $\alpha=1$ only the implementation (9.8) is a symmetry.

These, at first sight rather abstract, statements have however a fundamental importance. Indeed,the nucleons in the generalized Aharonov-Bohm experiment can be viewed as test particles moving in a background YM vacuum [23]. How can we tell which of the nucleons is a proton, which is a neutron? As emphasised by Yang and Mills in the very first paper on gauge theory [37],this can be done only by measuring the electric charge.The particle's charge alone is conserved however only if the background field has symmetries. Indeed,the cases (i) and (ii) are exactly those when the nucleon's charge is conserved.In other cases - for $\alpha = 1/2$ e.g. -protons can be turned to neutrons [22,23].

REMARKS

Using the same technique as for monopoles, one can show [38] that, for an SU(2)-instanton, G = SU(2) is never implementable.The physical consequences of this fact are not entirely clear however.

The theory outlined in this paper admits a nice fibre-bundle interpretation. This is explained in a companion paper [25].

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